

Digital Signal Processing
Prof. S.C. Dutta Roy
Department of Electrical Engineering
Indian Institute of Technology, Delhi
Lecture – 37
Example of IIR Design (Contd.....)
Digital to Digital Frequency Transformation

This is the 37th lecture. In the last lecture we started working on an example but we could not complete. So now we will continue working on this example and also introduce something new, called digital-to-digital frequency transformations. The results are available in the book but the method of derivation is new.

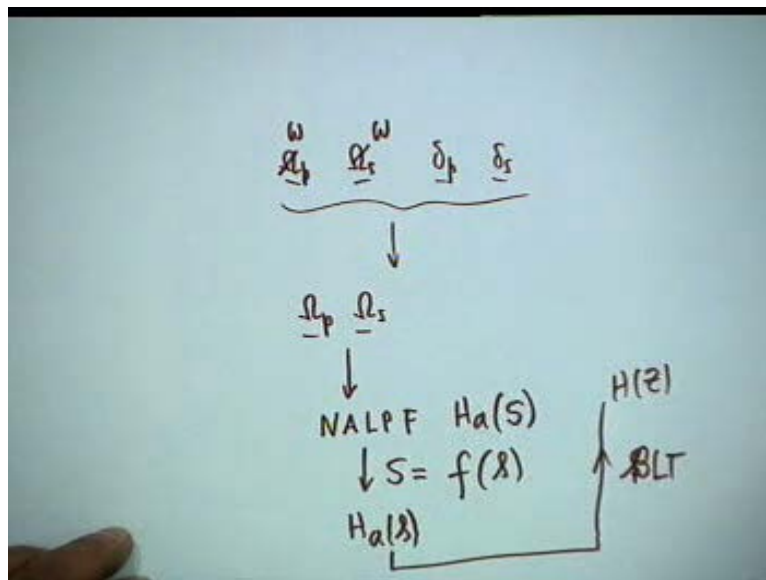
In the previous lecture, we used an example of an IIR design and worked it out completely for Butterworth case using Bilinear Transformation, and Butterworth case using Impulse Invariant Transformation. With regard to Chebyshev design with bilinear transformation and impulse invariance, we brought out many points; not all of them are available in the book. We also took another problem of low-pass design and we went to the design of other types of filters with the specifications ω_p , ω_s , δ_p and δ_s .

The first thing you do is convert actual frequencies to ω 's. Then you convert them to Ω 's; the tolerances remain the same. In the process, you could assume $T = 2$ because the final result is not affected by T if you use Bilinear Transformation. If you use impulse invariance, it does and that is why it gives rise to aliasing. But in BLT, the final result $H(z)$ is independent of T . So even if T is given, you can normalize this to 2 seconds and work it out. The advantage is that you do not have to handle large numbers, as the example of design illustrated.

The disadvantage is that, if you assume $T = 2$ then the numbers you handle are small and they will usually be fractions. Therefore use a large enough number of digits after the decimal point. After the design is complete, you of course have to subject $H(e^{j\omega})$ to a MATLAB Program, draw the magnitude characteristic and phase characteristic and judge whether it is satisfactory or not.

For the design we worked out for the Band Stop filter, we did not use $T = 2$ but we used the given T . So, you find the Normalized Analog Low-pass Filter, that is $H_a(S)$, normalized in the sense that Ω_p is normalized to 1 radian per second and the magnitude is normalized to have a maximum value of unity. Then you transform the normalized low-pass to other types. That is, replace S by the appropriate expression in the variable s . Next, you apply the Bilinear Transformation (BLT) to get $H(z)$. Now we shall find an alternative route to this when we discuss digital-to-digital transformation directly, rather than going via analog. Let us first complete this example.

(Refer Slide Time: 02:48 - 17: 13 min)



The digital filter specifications were converted to analog filter specs, and we had two pass bands. We also converted db to fractions with passband magnitudes between 1 and 0.966. The stopband tolerance was worked out as 0.0178. The stopband had to be modified by increasing Ω_{s2} to Ω'_{s2} . Before aiming at Chebyshev, we have to go to the Normalized Analog Low-pass Filter which has $\Omega_p = 1$ but you have to find out the edge of the stop band, that is Ω_s of the normalized low-pass filter. From our earlier discussion, it should be amply clear that $\Omega_{p2} - \Omega_{p1}$ is not the bandwidth of the band reject filter. It does not have a physical significance but it is simply the difference between two pass band edges. This difference divided by stop bandwidth gives Ω_s . So

Ω_s is given by $(\Omega_{p2} - \Omega_{p1})/(\Omega'_{s2} - \Omega_{s1})$. And if you substitute the numerical values, then Ω_s comes out as 3.418. You should use the value Ω_{s2}' and not the original one. Now N_c has to be found out which will be $N_c \geq \cosh^{-1} \sqrt{[(1/0.178)^2 - 1]/[(1/0.966)^2 - 1]}/\cosh^{-1} 3.418$. Again $\cosh^{-1} y$ is calculated by taking $\ln(y + \sqrt{y^2 - 1})$. And this comes as 3.178, so $N_c = 4$. The number is neither close to 4 nor close to 3, so we can confidently say $N_c = 4$. And now the actual Ω_{s2} realized will go beyond Ω_{s2}' because we are using a larger value than 3.178. Similarly, Ω_{s1} would be smaller than what was specified but Ω_{p1} and Ω_{p2} shall remain intact.

(Refer Slide Time: 09:10 - 12:11 min)

The image shows a whiteboard with the following handwritten calculations:

$$\Omega_s = \frac{B = \Omega_{p2} - \Omega_{p1}}{\Omega'_{s2} - \Omega_{s1}}$$

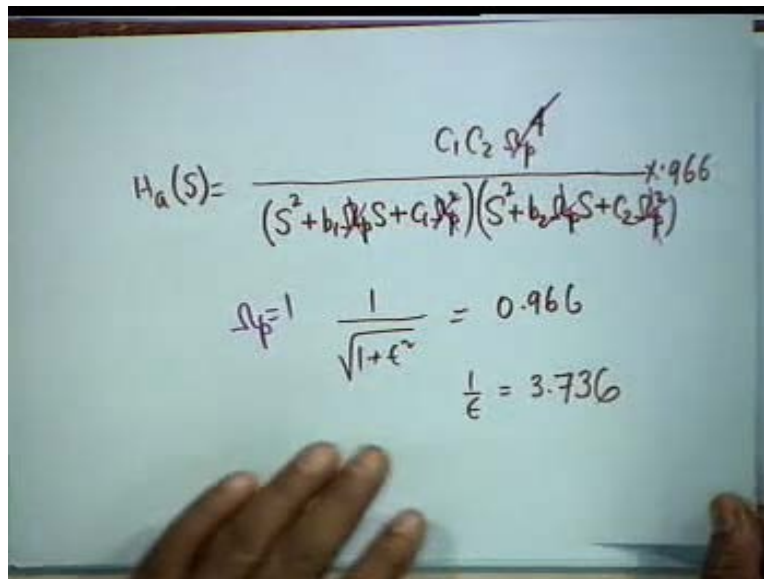
$$= 3.418$$

$$N_c \geq \frac{\cosh^{-1} \sqrt{\frac{(1/0.178)^2 - 1}{(1/0.966)^2 - 1}}}{\cosh^{-1} 3.418} = 3.178$$

$$N_c = 4$$

So, now $H_a(s)$ can be written down. Because the order is 4, we do require $1/\sqrt{1 + \varepsilon^2}$ which is 0.966. We require $1/\varepsilon$ and it has to be calculated. We did not calculate ε and $1/\varepsilon$. You require them. You require $1/\varepsilon$ to calculate y_4 . $1/\varepsilon$ comes to 3.736. So, in the denominator there shall be two quadratics that is $S^2 + b_1\Omega_p S + c_1\Omega_p^2$ and the other factor would be $S^2 + b_2\Omega_p S + c_2\Omega_p^2$ and in the numerator we shall have $c_1 c_2 \Omega_p^4$ multiplied by 0.966. There is a simplifying feature here because $\Omega_p = 1$.

(Refer Slide Time: 12: 4 – 14:07 min)



The image shows a whiteboard with handwritten mathematical equations. The first equation is a transfer function $H_a(s)$ with a numerator $c_1 c_2 \omega_p^4$ and a denominator consisting of two quadratic terms: $(s^2 + b_1 \omega_p s + \omega_p^2)$ and $(s^2 + b_2 \omega_p s + \omega_p^2)$. A multiplier $\times 0.966$ is written to the right of the denominator. Below this, two equations are written: $\omega_p = 1$ followed by $\frac{1}{\sqrt{1+\epsilon^2}} = 0.966$, and $\frac{1}{\epsilon} = 3.736$.

We now have to calculate b_1 , c_1 , b_2 , and c_2 . The first step is to calculate y_4 where $y_4 = (1/2)[(\sqrt{(1+1/\epsilon^2)} + (1/\epsilon))^{1/4} + (\sqrt{(1+1/\epsilon^2)} + (1/\epsilon))^{-1/4}]$ and this calculates out as 0.528896 (note that I go up to fifth decimal place). b_1 is $2 \times y_4 \sin(\pi/8)$ and that calculates to 0.40479, c_1 is $y_4^2 + \cos^2 \pi/8$ and this calculates to 1.1332.

(Refer Slide Time: 14:12 - 16: 11 min)

The whiteboard shows the following calculations:

$$y_4 = \frac{1}{2} \left[\left(\left(1 + \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \right)^{\frac{1}{4}} + \left(\right)^{-\frac{1}{4}} \right]$$
$$= 0.528896$$
$$y_4^2 = 0.27973$$
$$b_1 = 2y_4 \sin \frac{\pi}{8} = 0.40479$$
$$c_1 = y_4^2 + \cos^2 \frac{\pi}{8} = 1.1332$$

Then we have to calculate b_2 and c_2 . b_2 is $2 \times y_4 \sin 3\pi/8$ and the value is 0.9773 while c_2 is $y_4^2 + \cos^2 3\pi/8$ and this is computed as 0.4261. Therefore $H_a(S) = 1.133 \times 0.462 \times 0.966 / [(S^2 + 0.405S + 1.133)(S^2 + 0.977S + 0.426)]$.

(Refer Slide Time: 16:17 - 18:28 min)

The whiteboard shows the following calculations:

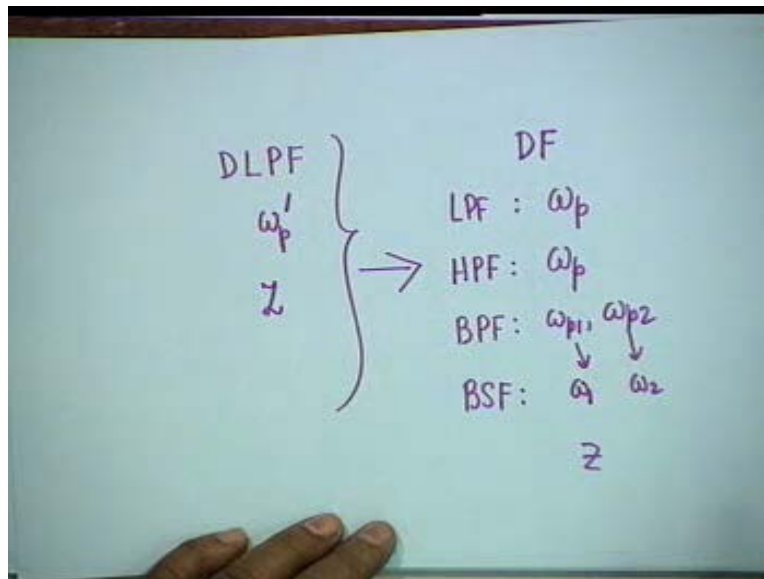
$$b_2 = 2y_4 \sin \frac{3\pi}{8} = 0.9773$$
$$c_2 = y_4^2 + \cos^2 \frac{3\pi}{8} = 0.4261$$
$$H_a(S) = \frac{1.133 \times 0.426 \times 0.966}{(S^2 + 0.405S + 1.133)(S^2 + 0.977S + 0.426)}$$
$$S = \frac{759.4568}{s^2 + (293.765)^2}$$

The next step is to put $S = 759.456s/(s^2 + \Omega_0^2)$; Ω_0^2 , if you recall, came out as $(293.765)^2$ so put this and get $S = 759.456s/(s^2 + (293.765)^2)$ and then finally you substitute $s = (2/T)$ (now you have to bring in T because you did not discard T at the beginning) $(1 - z^{-1})/(1 + z^{-1})$ and that will give you $H(z)$.

The next topic for discussion is to design digital filters of any kind starting from another digital filter. Here also, you start from low-pass filter. That is, given a digital low-pass filter of some pass band edge ω_p' you have to design another digital filter of the same or some other kind. It could be another low-pass filter whose pass band edge is ω_p , or a high pass filter with a pass band starting at ω_p or a band pass filter or a band stop filter with corresponding ω_{p1} and ω_{p2} . And in order that we do not have to write p again and again we shall simplify these to ω_1 and ω_2 .

When we write ω_1 and ω_2 , it should be implied that we are referring to the pass band edges and this is true about band stop case also. This digital low-pass filter is specified not only in terms of ω_p' but also ω_s' , δp and δs . It has been carefully designed so that by transformation only the pass band changes; the stop band change is automatic, so the stop band need not be considered. Given an arbitrary digital low-pass filter with a pass band at ω_p' , our purpose is to go from here to another by an appropriate transformation. Obviously the transformation will be from one z domain to another z domain. Let the lowpass z domain be denoted by Z and the transformed filter z -domain by z . So we are seeking a relationship between Z and z . All pass filters once again come to the rescue. You will see that this transformation is an all pass function, i.e. Z^{-1} is equal to an all pass function in z^{-1} .

(Refer Slide Time: 19:33 - 22:53 min)

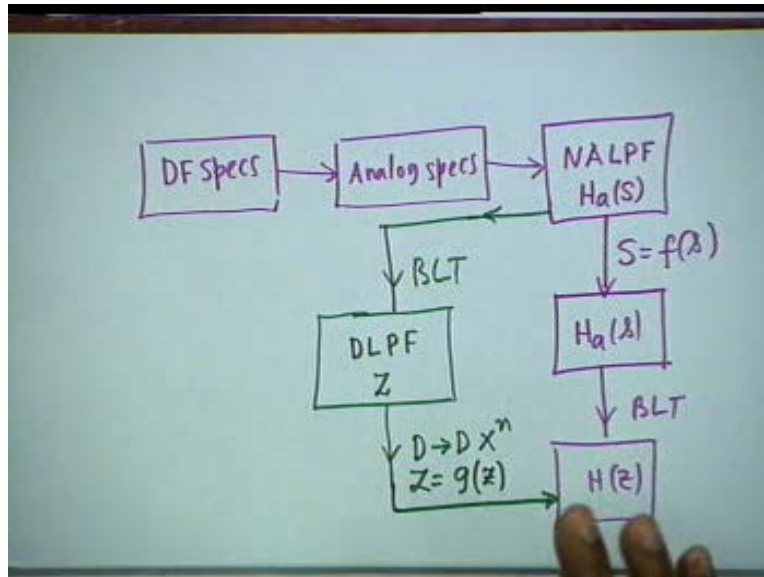


The above diagram precisely illustrates what we wish to do. Let us first review what we did in the previous procedure of transforming an analog filter. From digital filter specs, the procedure we have followed so far is through an analog filter of the same type. For digital filter specs of a band stop type, we first transform the frequency so that there is an analog band stop. And then from analog band stop we found out the required transition ratio of a normalized analog low pass filter. So you get a normalized analog low-pass filter, that is $H_a(S)$, and then we have put S equal to an appropriate function of s to transform it to a band stop filter. Finally, we have used the BLT to get $H(z)$. This has been our design flow diagram. Now, suppose instead of following this route, we use a BLT on the normalized analog low-pass filter to get a digital low-pass filter with the variable Z . This BLT would be $S = (2/T) (1 - Z^{-1}) / (1 + Z^{-1})$. So we get a digital low-pass filter in the Z domain and then look for a digital-to-digital transformation that is $Z = g(z)$ to arrive at $H(z)$ of the desired type of filter.

If we have done the procedure correctly then obviously we should get the same result because Bilinear Transformation and $S = f(s)$ are one-to-one transformations; given one, you can go to the other. And therefore if we follow either route, we should get the same result. Let us see what this transformation $Z = g(z)$ is. Note that when we go from normalized analog low-pass filter to a digital low-pass filter, the latter is not assured to be normalized, i.e. ω_p' will not in general be

unity. Therefore we seek not only a digital-to-digital transformation but from an arbitrary digital low-pass filter. It is arbitrary because it is not normalized.

(Refer Slide Time: 23:00 - 26: 56)



First, consider the problem of transforming a digital low-pass filter of cutoff frequency ω_p' to another low-pass filter with cutoff frequency ω_p . This is a trivial thing in analog domain. What you have to do is to replace S by s divided by the new cutoff frequency. From a digital low-pass filter to digital low-pass filter with another cutoff frequency is not trivial; it has to be worked out. The reason is that relationship between Z and z is not linear, like that of S and s .

Let us work it out systematically. If we had transformed these two low-pass digital filters to corresponding analog low-pass filters, the corresponding frequency Ω_p' would have been $(2/T)$ tangent of $\omega_p'/2$. And similarly, for the new filter, it would have been $\Omega_p = (2/T)$ tangent of $\omega_p/2$. Let the analog frequency variable of the first case be S' . It is any arbitrary low-pass digital filter which when transformed to the analog domain may not give a pass band edge Ω_p' equal to 1. Hence we use a prime to denote the complex frequency variable. Then replacing S' by S'/Ω_p' , obviously I shall get a Normalized Analog Low-pass Filter. The pass band edge will be now 1 radian/second and this is what we call S . If Ω_p' is one, then the two are identical. Let the

complex frequency variable in the new transformed analog low-pass filter be s ; then replacing s by s/Ω_p would give a normalized analog low-pass filter with cutoff frequency of 1 radian/sec, i.e. $s/\Omega_p = S$. these two S 's are identical; hence $S'/\Omega_p' = s/\Omega_p$.

(Refer Slide Time: 28:08 - 31:52 min)

Handwritten mathematical derivations on a whiteboard:

Left side (Passband Edge):

$$\text{LPDF}$$

$$\omega_p'$$

$$\Omega_p' = \frac{2}{T} \tan \frac{\omega_p'}{2}$$

$$\frac{S'}{\Omega_p'} = S$$

Right side (LPDF):

$$\text{LPDF}$$

$$\omega_p$$

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2}$$

$$\frac{s}{\Omega_p}$$

Final result:

$$S = \frac{s}{\Omega_p} = \frac{S'}{\Omega_p'}$$

So, $S' = (\Omega_p'/\Omega_p)s$. In terms of Bilinear Transformation, this relation translates to $(1 - z^{-1})/(1 + z^{-1}) = \Omega_p'/\Omega_p(1 - z^{-1})/(1 + z^{-1})$. I get a relationship between Z^{-1} and z^{-1} . Now replace Ω_p' by $(2/T)\tan \omega_p'/2$ and Ω_p by $(2/T)\tan \omega_p/2$ and do a little algebra. The final result can be put in the form $Z^{-1} = (z^{-1} - \alpha)/(1 - \alpha z^{-1})$ where $\alpha = \sin((\omega_p' - \omega_p)/2)/\sin((\omega_p' + \omega_p)/2)$.

Notice that the transformation is an all pass function. All pass seems to appear everywhere in digital signal processing.

(Refer Slide Time: 31:59 - 34:46 min)

$$S' = \frac{\Omega_p'}{s} s$$

$$\frac{1 - z^{-1}}{1 + z^{-1}} = \frac{\tan \frac{\omega_p'}{2}}{\tan \frac{\omega_p}{2}} \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} ; \alpha = \frac{\sin \frac{\omega_p' - \omega_p}{2}}{\sin \frac{\omega_p' + \omega_p}{2}}$$

$\alpha = \frac{k-1}{k+1}$

So this is the low-pass to low-pass transformation. It is not a linear transformation but it is again a Bilinear Transformation, because it is a linear polynomial in z^{-1} divided by another linear polynomial in z^{-1} , but it is an all pass function. Could alpha be negative? Yes, because ω_p' can be $< \omega_p$. Now the procedure should be clear. It is a very simple concept; we simply appeal to the analog transformation and that gives us the result.

We next want to go to a digital high pass filter; once again the basic digital LPF cutoff is ω_p' and the passband edge of the HPF is ω_p . So S' , the corresponding analog filter complex frequency variable, divided by Ω_p' , which is equal to S , should be replaced by Ω_p/s ; this is the analog to analog transformation, low-pass to high pass. And therefore what you have is that S' should be equal to $\Omega_p \Omega_p'/s$ which means that $(1 - Z^{-1})/(1 + Z^{-1})$ should be equal to $\tan(\omega_p/2) \tan(\omega_p'/2) (1 + z^{-1})/(1 - z^{-1})$.

In deriving this transformation, we assumed that $T = 2$. In the low-pass to low-pass, the question did not arise because it was a ratio but now it is a product and therefore the question is important. You can carry $(2/T)$ through but finally you will get the same result.

(Refer Slide Time: 35:37 - 38:30)

DLPF ω_p'

DHPF ω_p

$\frac{S'}{\Omega_p'} = S$

$\frac{\Omega_p}{\delta} = R$ (T=2)

$S' = \frac{\Omega_p \Omega_p'}{\delta} = \tan \frac{\omega_p'}{2} \tan \frac{\omega_p}{2}$

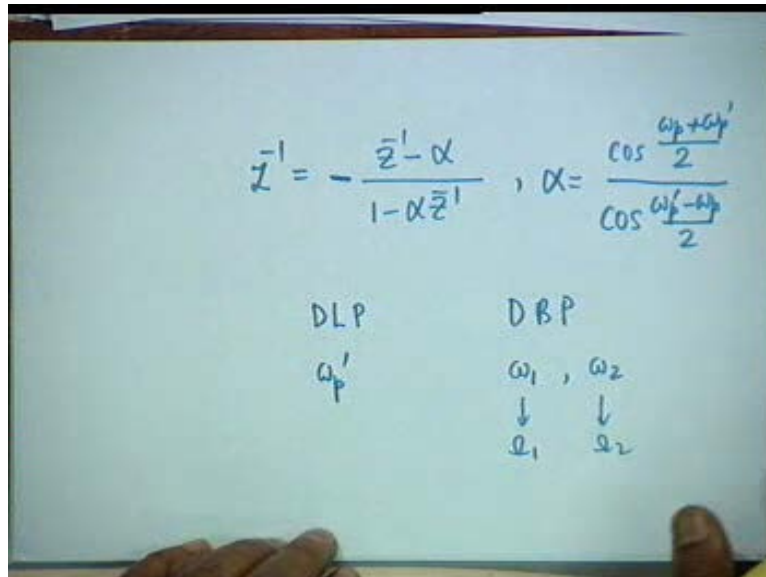
$\frac{1-z^{-1}}{1+z^{-1}} = R \frac{1+e^{-1}}{1-e^{-1}}$

The final result comes as $Z^{-1} = -\frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$, the same transformation as we got from low-pass to

low-pass but with a negative sign. Here α is a ratio of cosines. Precisely this is $\alpha = \cos[(\omega_p + \omega_p')/2] / \cos[(\omega_p' - \omega_p)/2]$. These transformations were given by A.G. Constantinides, a Professor at the Imperial College of Science and Technology, London in his Ph.D. thesis. But the derivation was more complicated than what has been done here.

Next we go from digital low-pass to digital band pass and in band pass, there are two pass band edges ω_1 and ω_2 . For reasons mentioned earlier, let us not write ω_{p1} and ω_{p2} again and again so we shall take them as ω_1 and ω_2 . The corresponding analog frequencies are Ω_1 and Ω_2 . Once again, I assume that $T = 2$ so that I have to handle only tangent functions.

(Refer Slide Time: 38:40 - 41:27 min)



Therefore the procedure is S'/Ω_p' , which is S , should now be replaced by $(s^2 + \Omega_1 \Omega_2)/[(\Omega_2 - \Omega_1)s]$. Recall that $\Omega_0^2 = \Omega_1 \Omega_2$. Now put $S' = (1 - Z^{-1})/(1 + Z^{-1})$, $\Omega_p' = \tan(\omega_p'/2)$, $\Omega_1 = \tan(\omega_1/2)$, $\Omega_2 = \tan(\omega_2/2)$ and $s = (1 - z^{-1})/(1 + z^{-1})$. After a little involved algebraic and trigonometric manipulation, you get the final result as $Z^{-1} = -\frac{z^{-2} - (2\alpha k / (k+1))z^{-1} + (k-1)/(k+1)}{[(k-1)/(k+1)z^{-2} - (2\alpha k / (k+1))z^{-1} + 1]}$.

(Refer Slide Time: 41:34 - 45:32)

$$\frac{S'}{s_p'} = S = \frac{s^2 + \Omega_1 \Omega_2}{(\Omega_2 - \Omega_1) s}$$

$$S' = \frac{\Omega_p' (s^2 + \Omega_1 \Omega_2)}{(\Omega_2 - \Omega_1) s} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right)^2$$

$$\frac{1 - z^{-1}}{1 + z^{-1}} = \frac{\tan(\frac{\omega_p'}{2})}{k'} \frac{s^2 + \tan(\frac{\omega_1}{2}) \tan(\frac{\omega_2}{2})}{(\tan(\frac{\omega_2}{2}) - \tan(\frac{\omega_1}{2})) \frac{1 - z^{-1}}{1 + z^{-1}}}$$

Please note the negative sign; once again, this is also an all pass function of second order. Here, $\alpha = \cos [(\omega_2 + \omega_1)/2] / \cos [(\omega_2 - \omega_1)/2]$. You can show that this is cosine of the center frequency of the digital band pass filter. Also k here is $\cot[(\omega_2 - \omega_1)/2] \tan(\omega_p'/2)$.

(Refer Slide Time: 45:45 - 48:02 min)

$$z^{-1} = - \frac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$$

$$\alpha = \frac{\cos \frac{\omega_2 + \omega_1}{2}}{\cos \frac{\omega_2 - \omega_1}{2}} = \cos \omega_0$$

$$k = \cot \frac{\omega_2 - \omega_1}{2} \tan \frac{\omega_p'}{2}$$

Let us now consider transformation of a digital low-pass filter to digital band stop filter. Once again the band stop is also defined by its pass band edges ω_2 and ω_1 . If you follow the same steps, you get the final result as $Z = \frac{z^{-1} - ((2\alpha / (k+1)) z^{-2} + ((1-k) / (1+k)))}{[(1-k) / (1+k)] z^{-2} - ((2\alpha / (k+1)) z^{-1} + 1]}$, where α is the same as in the bandpass case, and can be shown to be equal to $\cos \omega_0$, where ω_0 is the null frequency i.e. the center frequency of the stopband. Also $k = \tan(\omega_p'/2) \tan((\omega_2 - \omega_1)/2)$. Note that this is also an all pass function of the second order. Note carefully the difference between bandpass and bandstop transformation functions, as well as the differences between low-pass and high-pass transformations. It is best to tabulate them.

(Refer Slide Time: 48:05)

The whiteboard shows the following derivation:

DLPF ω_p'	DBSF ω_2, ω_1
$Z = \frac{z^{-1} - \frac{2\alpha}{k+1} z^{-2} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{k+1} z^{-1} + 1}$	
$\alpha = \frac{\cos \frac{\omega_2 + \omega_1}{2}}{\cos \frac{\omega_2 - \omega_1}{2}} = \cos \omega_0$	
$k = \tan \frac{\omega_2 - \omega_1}{2} \tan \frac{\omega_p'}{2}$	

As compared to the derivations by Constantinides, which is quoted in all text books, the derivation given here is conceptually much simpler and the algebra and trigonometry can be handled with not too much of difficulty. This new method appeared in the following paper by the author: S.C.Dutta Roy, "A simple derivation of the spectral transformations of IIR filters", IEEE Transactions on Education, vol.48, issue 2, pp. 274-278 May 2005.

In the next lecture, we shall consider a very simple example, that of a first order Chebyshev low-pass filter. First order Chebyshev is the same as first order Butterworth in analog but not necessarily in digital. So we will start from a first order analog low-pass filter and then transform it to digital low-pass. From digital low-pass, we shall go to other kinds of filters just as an example. Then we shall change the topic.