Course Name- Nanophotonics, Plasmonics and Metamaterials

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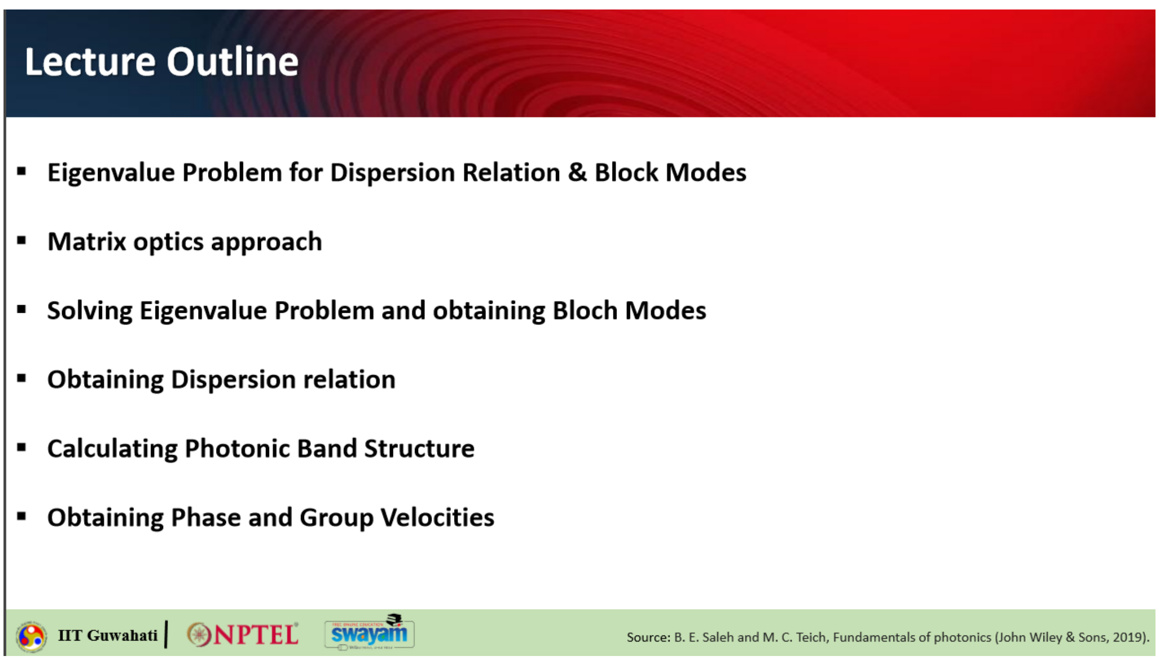
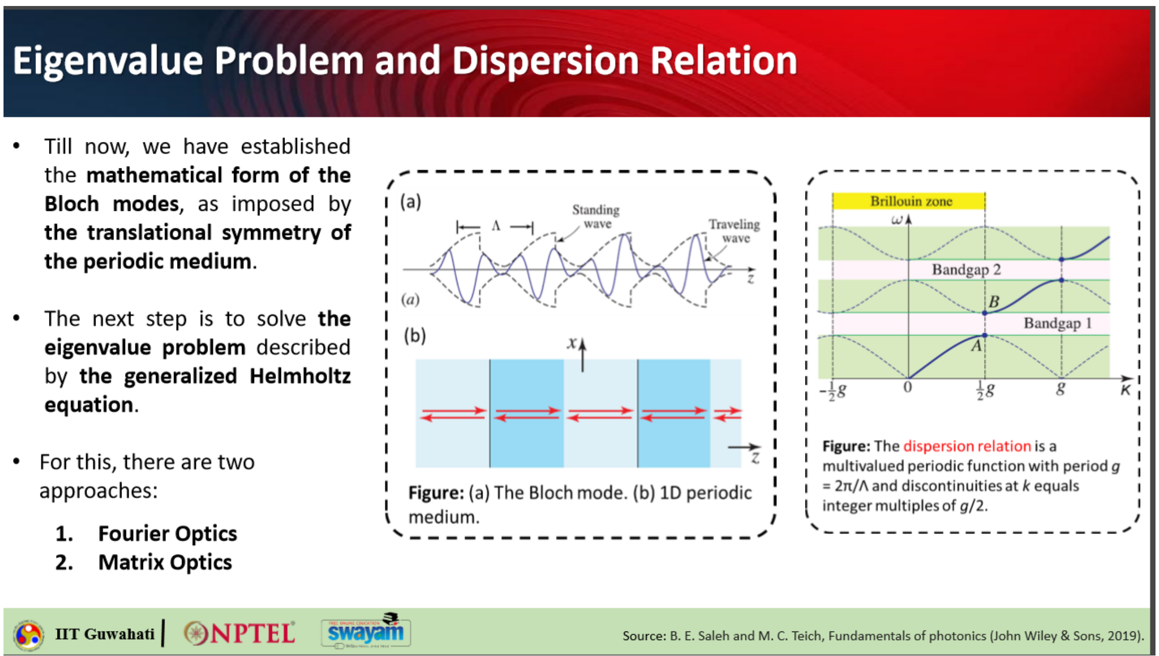
Week-04

Lecture -12

Hello students, welcome to lecture 12 of the online course on Nanophotonics, Plasmonics and Metamaterials. In this lecture we will be covering dispersion relation and photonic band structure. So, here is the lecture outline. So, we will introduce the eigenvalue problem on dispersion relation and bloch modes. And we will discuss the matrix optics approach for solving you know eigenvalue problem and obtaining the bloch modes. We will also see how to obtain the dispersion relation calculating photonic band structure and obtaining phase and group velocities.



So, eigenvalue problem and dispersion relation. If you remember from the previous lecture that till now we have established the mathematical form of the bloch modes as imposed by the translational symmetry of the periodic medium. So, we were considering periodic medium where the refractive index periodically alters and these are the bloch modes, ok. And this is the typical 1D periodic medium we have discussed in the previous lecture. We have also seen that you know for this 1D periodic medium we are able to find dispersion relation. Dispersion relation in brief we can say it is basically omega k relation and in this particular case we were also able to see that certain you know frequencies were not allowed to propagate. So, those actually gave you band gap 1, band gap 2 and so on. So, our objective here is to know how do we go to this particular dispersion relation starting from a 1D periodic medium. So, our objective here would be to solve the eigenvalue problem described by the generalized Helmholtz equation for this 1D periodic system.



So, for this there are two approaches one is Fourier optics another is matrix optics. So, the first approach let us have a quick look that is called Fourier optics. Now this approach is based on expanding the periodic function say eta of z of the medium, ok or you can say $\boldsymbol{\eta}(\mathbf{z})$ eta is the impedance or you can also talk in terms of refractive index, ok. And the periodic function $\boldsymbol{p_K}(\mathbf{z})$ of the bloch mode. So, you have to expand this in Fourier series, ok.

Eigenvalue Problem and Dispersion Relation

- **Approach 1: Fourier Optics**
  - This approach based on expanding the periodic function $\boldsymbol{\eta}(\mathbf{z})$ of the medium and the periodic function $\boldsymbol{p_K}(\mathbf{z})$ of the Bloch mode in Fourier series.
  - Then converting the Helmholtz differential equation into a set of algebraic equation cast in the form of a matrix eigenvalue problem — **which are solved numerically**.
- **Approach 2: Matrix Optics**
  - This approach is applicable to **layered (piecewise homogeneous) media with planar boundaries**.
  - Instead of solving the Helmholtz equation, we make direct use of the laws of propagation and reflection/refraction at boundaries, which are known consequences of Maxwell's equations.
  - We then use the matrix methods developed for multilayer media.
  - This **Matrix Optics** approach leads to a 2 × 2 matrix eigenvalue problem ***from which the dispersion relation and the Bloch modes are determined***.

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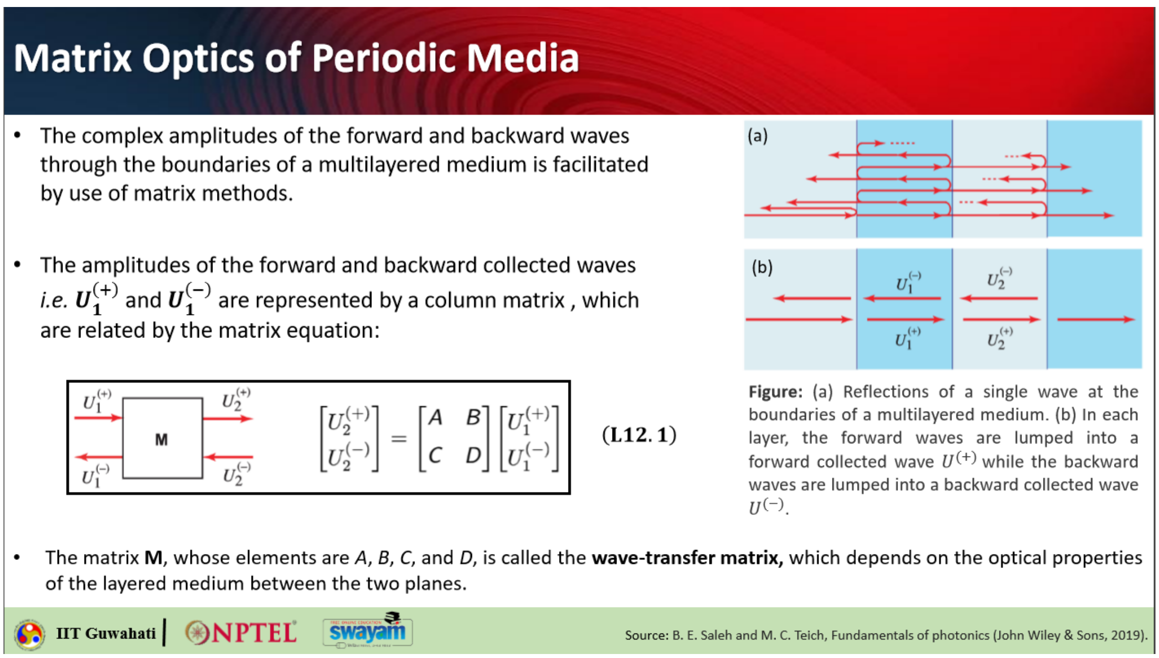
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

And then you convert the Helmholtz differential equation into a set of algebraic equation cast in the form of matrix eigenvalue problem and this you have to solve numerically. So, that's what is known as the Fourier optics method. The other method is called matrix optics. So, in this case this particular method is applicable to layered which are basically piecewise homogeneous like the previous example you have taken periodic alteration of refractive index a dielectric 1 2 1 2 1 2 and so on, ok. And if you have those kind of layered media with planar boundaries you are able to use matrix optics method. Now, in this particular method instead of solving the Helmholtz equation we make direct use of the laws of propagation reflection and refraction that is more or less you know the transfer matrix formulation that you have studied couple of lectures back. So, you can use those you know laws of propagation, reflection, refraction at the boundaries, ok, which are basically the known of the Maxwell's equation. And then you can use the matrix methods developed for multilayer media, right. So, when you apply matrix optics method finally, you will get a 2 by 2 matrix eigenvalue problem from which you can obtain the dispersion relation and the bloch modes. So, we will be mainly covering this particular matrix optics approach in this lecture and this course of course.

So, let's look into matrix optics approach. The complex amplitude of the forward and backward waves through the boundaries of multilayered medium is facilitated by the use of matrix method something like this. So, here you can take an example of multilayered media, medium 1 2 3 4 and so on. So, just to make you understand how complex the system could be. So, you start with one particular wave that is partially getting reflected, some part is getting refracted or transmitted.

Now this light when it is in this particular medium, when it hits this particular interface between this medium and this medium, some part of this light is getting partially reflected, remaining is getting transmitted. And again, this transmitted light when it encounters this particular interface, some part is getting reflected back, some is getting transmitted. Now what happens to this reflection? This basically a backward propagating light or wave. So, here again at this interface it has got two options, one is to transmit, one is to reflect and so on. So, this is how things happen now in a multilayered medium.

So, you start with the single wave, but because of these boundaries you end up with getting you know numerous transmitted and reflected light beams. Now in part b this particular figure what is shown is that in each layer the forward moving waves can be named as plus, ok. And backward ones can be denoted using this minus symbol. So, when you are in say medium 1 you can say that it has got a you know all the forward moving waves can be summed up together that can be called as $\boldsymbol{U}_1^{(+)}$ and all the backward propagating waves means all these reflections they can be summed up together or collected together and you can call them as $\boldsymbol{U}_1^{(-)}$. Same in layer 2 you can have $\boldsymbol{U}_2^{(+)}$and $\boldsymbol{U}_2^{(-)}$.



Here you can have this is the incident medium and this is the final transmitted medium. So, we are just these are the two you know layers that actually form this multilayered system in this case. Now there is a way this $\boldsymbol{U}_1^{(+)}$and $\boldsymbol{U}_1^{(-)}$ are kind of correlated. So, the amplitudes of this forward and backward collected waves they can be represented using a column matrix. So, if you actually consider this particular layer by a matrix M.

So, what you see here is that there is incoming wave, there is a backward propagating wave and there is a forward and again backward propagating wave. So, you can actually represent this interface using this matrix M. So, this is the column matrix. So, let's correlate the coefficients. So, you can have on the right side you have $\boldsymbol{U}_2^{(+)}$, $\boldsymbol{U}_2^{(-)}$ and this M matrix can have four elements A B C D that are correlating this amplitude to the amplitudes on the left hand side that is $\boldsymbol{U}_1^{(+)}$ and $\boldsymbol{U}_1^{(-)}$.

So, the matrix M whose elements are A B C D this is called the wave transfer matrix and it depends between depends on the optical properties of the layered medium between the two planes. Now, how do you apply this for a periodic media? So, in a periodic media you can actually see that this a unit cell is basically repeated right. So, if you have two alternating dielectric material say one is having high and another is having relatively low refractive index. So, the periodic media will be something like high low high low high low and so on. So, you can actually represent each unit cell using one matrix and then you can repeat it like this ok.

So, this is the wave transfer matrix representation of a periodic medium right. So, you can see that you know the periodicity is basically capital lambda. So, here you can say this is m lambda and this is m plus 1 lambda. What will be this one? This is m minus 1 lambda like from here to here it is the period ok. That is given by capital lambda.

So, between one period to the other there is a matrix that is correlating the parameters of $\boldsymbol{U}_m^{(+)}$, $\boldsymbol{U}_{\mathrm{m}}^{(-)}$ to $\boldsymbol{U}_{m+1}^{(+)}$ and $\boldsymbol{U}_{\mathrm{m}-1}^{(-)}$. So, you are basically correlating the forward propagating waves and the backward propagating waves. So, as you can see here a 1D periodic medium comprises of this identical segments like M O ok. They are called unit cells. They are repeated along one direction in this case we have considered Z is the direction of periodicity ok and they are separated by period capital lambda.

Matrix Optics of Periodic Media

- A 1-D periodic medium comprises these identical segments $\mathbf{M}_o$.
- Identical segments are called **unit cells**.
- That are repeated periodically along **one direction** (z axis) and separated by the period **Λ.**
- Unit cells contain repetition of **lossless dielectric layers or partially reflective mirrors**
- Forming a symmetric system represented by a generic wave-transfer matrix.

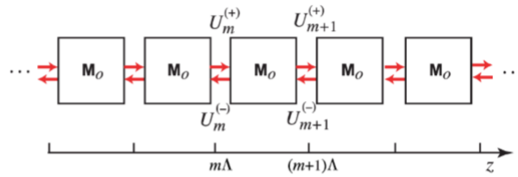
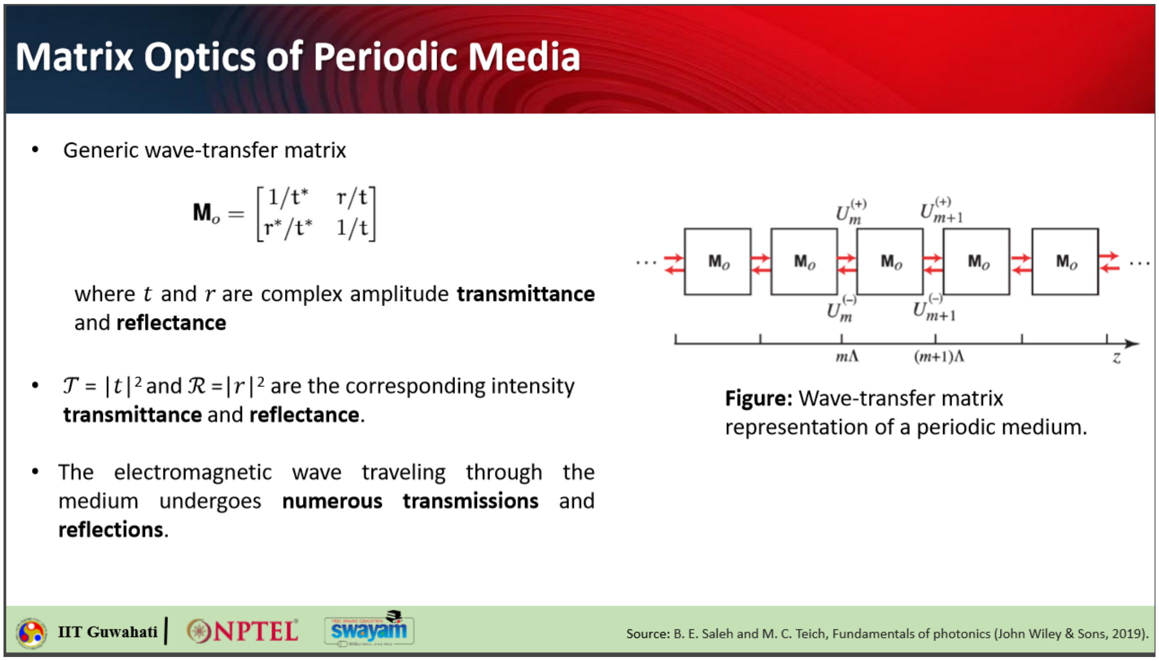


Figure: Wave-transfer matrix representation of a periodic medium.

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

And unit cells contain repetition of lossless dielectric layers or you can say these are partially reflective mirrors. Why they are called partially reflective mirrors? If you remember that any interface wherever there is a difference between the refractive indices ok there will be light reflection. This can also be now discussed in terms of impedance mismatch. So, if you take the impedance of the two different layers across the interface you will see that there is some difference in the or there is a mismatch in the impedance and that is why you will get some reflection of the incident wave from that interface. Now forming a symmetric system generated by a generic wave transfer matrix.



So, you can actually make a generic wave transfer matrix which looks like this. What are the elements here? 1 by t conjugate, r over t, r conjugate over t conjugate and 1 over t. So, this is how you are able to write a generic wave transfer matrix where r and t are basically the reflectance and transmittance. This you have already seen from the Fresnel equation you know what is reflectance and transmittance. So, if you want to calculate what is or you can this these are basically amplitude transmittance.

So, small t can be called as transmission coefficient or you can call them as amplitude transmittance, small r can be called as reflection coefficient or amplitude reflectance. So, correspondingly you can find out what is the intensity transmittance and intensity reflectance. So, this is how you can calculate ok. t equals modulus t square, r equals modulus small r square. So, the electromagnetic wave traveling through the medium they will undergo numerous transmissions and reflections that we have seen in the previous slide and that will actually give you one particular you know forward and one particular backward moving wave at each plane ok.

And the transfer matrix this particular matrix method not transfer matrix this is called matrix optics method this can be used to determine the block modes. So, let's assume $\boldsymbol{U}_{\boldsymbol{m}}^{(\pm)}$ as the complex amplitudes. So, plus 1 correspond to the forward and minus corresponds to the backward wave at any initial position z equals m lambda. So, here this particular one. So, what is m? m is the number of the unit cell ok.

Matrix Optics of Periodic Media

- These multiple transmissions and reflections add up to one forward and one backward wave at every plane.
- The matrix method is then used to determine **the Bloch modes.**
- Let's assume, $\left\{U_m^{(\pm)}\right\}$ are the **complex amplitudes** of the forward and backward waves at the initial position $z = m\Lambda$ of unit cell m.
- The amplitudes elsewhere within the cell can be determined by straightforward application of the appropriate wave transfer matrices, **as discussed already in previous lectures.**

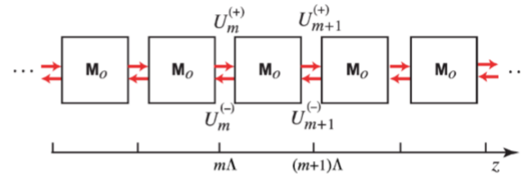


Figure: Wave-transfer matrix representation of a periodic medium.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

So, the amplitudes elsewhere within the cell can be determined by a straight forward application of the appropriate wave transfer matrices as discussed in the previous lecture. So, we have seen this already that you know if you know at one position you can add that phase and you can get the amplitude at any other location. Now the dynamics of the amplitude varies from one cell to another which is described by the recurrence relation.That means the amplitude $\boldsymbol{U}_{\mathrm{m}}^{(+)}$ and $\boldsymbol{U}_{\mathrm{m}}^{(-)}$ they will vary from one cell to another, but in a repeated pattern. So, what is the pattern? That is kind of like this.

Matrix Optics of Periodic Media

- The dynamics of the amplitudes $\left\{U_m^{(\pm)}\right\}$ varies from one cell to the next, described by the **recurrence relations**

$$\begin{bmatrix} U_{m+1}^{(+)} \\ U_{m+1}^{(-)} \end{bmatrix} = \mathbf{M}_o \begin{bmatrix} U_m^{(+)} \\ U_m^{(-)} \end{bmatrix}$$

- These relations are used to compute the complex amplitudes at a particular cell if the amplitudes at the previous cell are known.

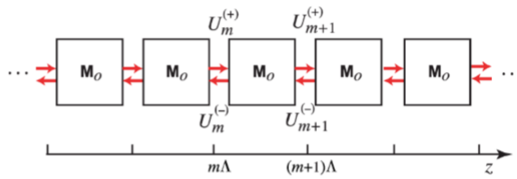

Figure: Wave-transfer matrix representation of a periodic medium.

IIT Guwahati | NPTEL swayam Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

So, this is the initial amplitude and you multiply it by this particular unit cell matrix that is M_0, you will get the next set of amplitude of the forward and backward moving wave. So, this relations are used to compute the complex amplitude at any particular cell if the amplitude of the previous cell are known. Make sense? These are the amplitude of the previous cell, this is the cell matrix. So, when you multiply this you get the amplitude of the current cell. Now let us see how do you obtain eigenvalue problem and block modes from this.

Eigenvalue Problem and Bloch Modes

- By definition, the modes of the periodic medium are self reproducing waves, for which

$$\begin{bmatrix} U_{m+1}^{(+)} \\ U_{m+1}^{(-)} \end{bmatrix} = e^{-j\Phi} \begin{bmatrix} U_m^{(+)} \\ U_m^{(-)} \end{bmatrix}, \quad m = 1, 2, \ldots;$$

  the magnitude of the forward and backward waves remains unchanged after transmission through a distance Λ (in this case a unit cell).

- Only, the phases are altered by a common shift Φ, called the **Bloch phase**.

- Therefore, the corresponding **Bloch wavenumber** is $K = \Phi/\Lambda$

  as $\Phi = K\Lambda$

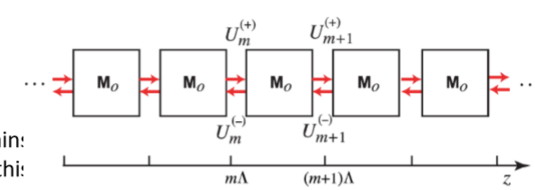

Figure: Wave-transfer matrix representation of a periodic medium.

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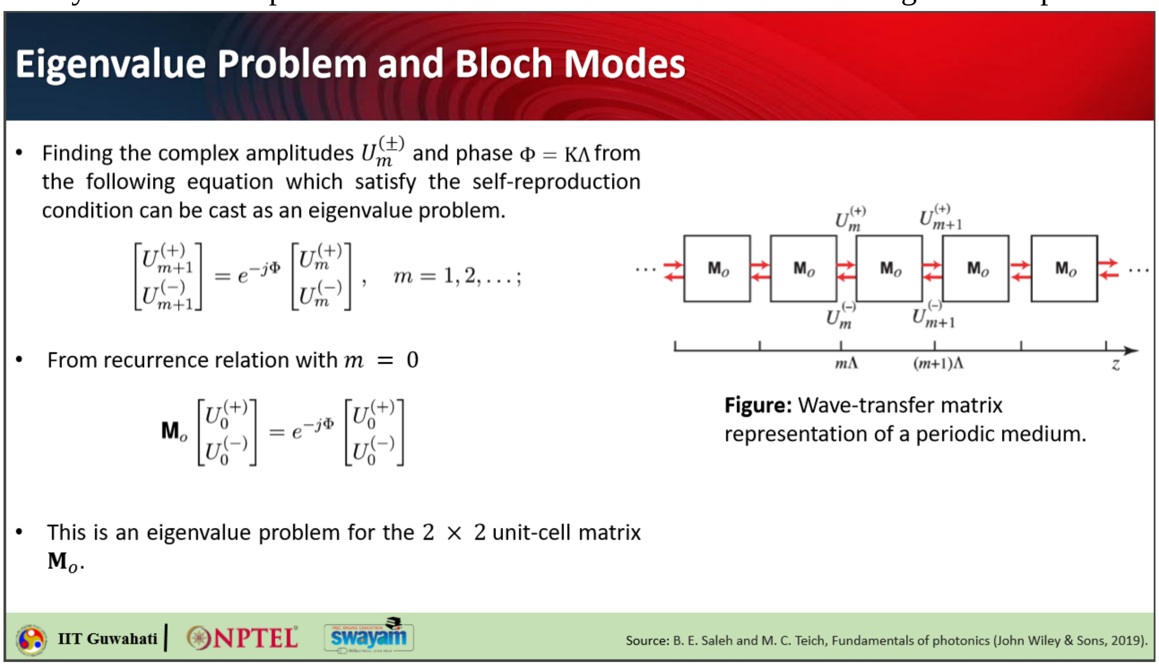
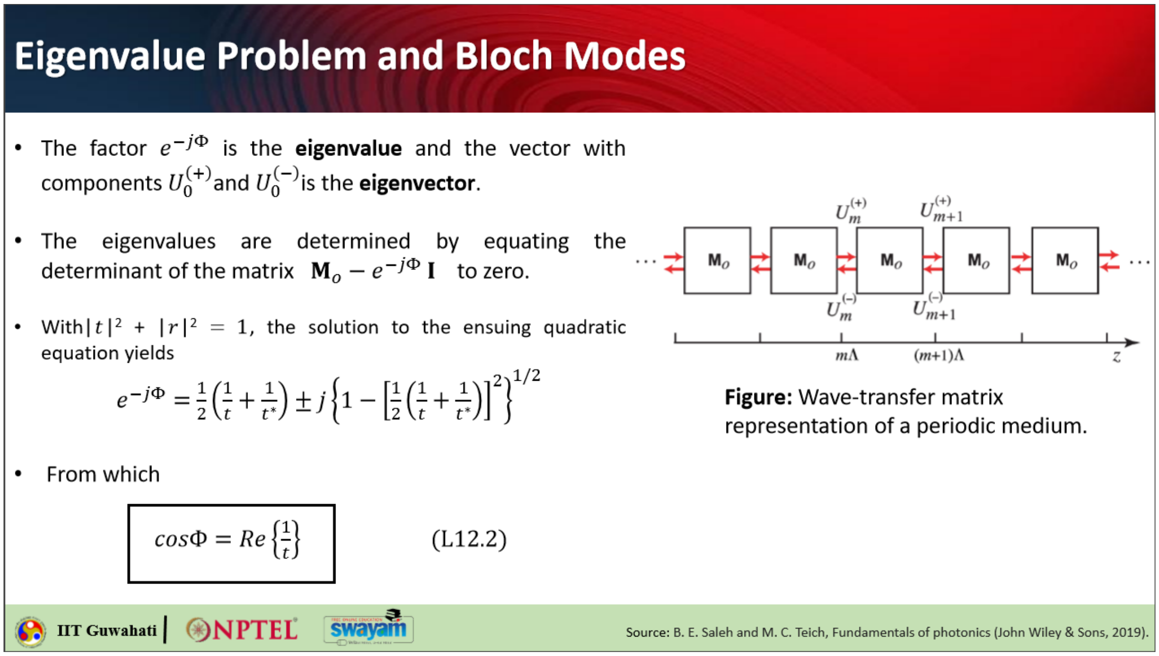
So, by definition the modes of the periodic medium are self reproducing and why so because they actually maintain a particular phase relation. So, you can say that if you take the amplitude of the forward and backward moving waves for mth cell where m is 1 or 2 or 3 or so …. anything. In that case if you multiply this by e to the power minus j phi that is the amount of phase accumulated while crossing this unit cell you can actually get the amplitude of the forward and backward propagating waves of the next unit cell ok. So, here what is important that you know this phi is basically the phase accumulated over the distance of the period and the period is nothing, but capital lambda. So, there is a name to this phase. So, this phase are basically altered by a common shift phi and this phi is called as block phase ok. So, there is a corresponding block wave number which is defined as K capital K that is given by phi over capital lambda. So, obviously what is that then phi which is block phase phi turns out to be K capital lambda ok. So, this is nothing, but block phase. So, finding the complex amplitudes that is Um plus minus and the phase phi which is defined as K capital lambda from the following equation which satisfy the self reproduction condition can be cast as an eigenvalue problem.



So, let us see how it looks like. So, if you take this particular problem where you know you already seen this equation that this is the phase relationship between the amplitude of the next cell and the previous cell and here if you put m equals 0 you get $\boldsymbol{U}_0^{(+)}$, $\boldsymbol{U}_0^{(-)}$ ok and what you will have here basically $\boldsymbol{U}_1^{(+)}$and $\boldsymbol{U}_1^{(-)}$ right and $\boldsymbol{U}_1^{(+)}$and $\boldsymbol{U}_1^{(-)}$ you can go back and from here you can get that $\boldsymbol{U}_1^{(+)}$and $\boldsymbol{U}_1^{(-)}$ can be written as M_0 $\boldsymbol{U}_0^{(+)}$and $\boldsymbol{U}_0^{(-)}$ right. So, if you bring this equation here. So, on the left hand side you get M_0 $\boldsymbol{U}_0^{(+)}$, $\boldsymbol{U}_0^{(-)}$

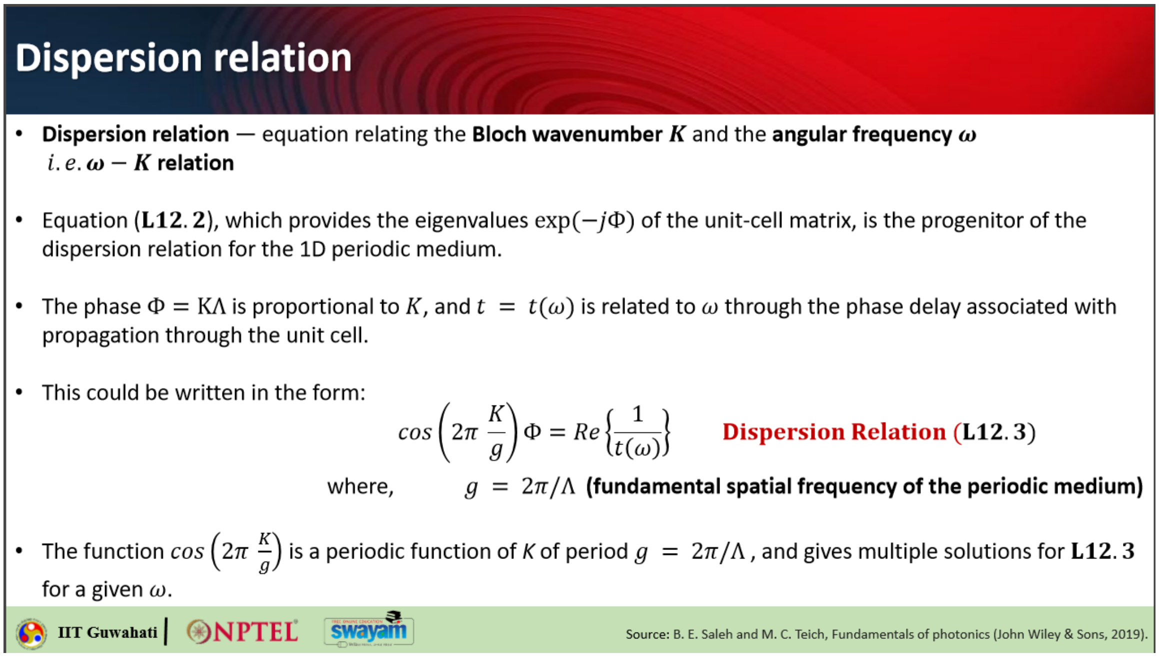
is equal to $e^{-j\Phi}\begin{bmatrix} U_0^{(+)} \\ U_0^{(-)} \end{bmatrix}$. So, this is an eigenvalue problem of this 2 by 2-unit cell matrix M_0 right. So, here if you look into this particular equation this is your eigenvalue.



So, the factor e to the power minus J phi is the eigenvalue ok and the vector with components $\boldsymbol{U}_0^{(+)}$ and $\boldsymbol{U}_0^{(-)}$ are basically the eigenvector right. So, how do you obtain the eigenvalues? The eigenvalues are basically obtained by equating the determinant of the matrix that is $\mathbf{M}_o - e^{-j\Phi}$ times identity matrix ($\mathbf{I}$). If you take this determinant and equate it to 0 you will get the values at which you will get those solutions are basically the eigenvalues ok. Now we already know that you know the reflect these are non-absorbing material. So, amplitude of transmission amplitude transmission coefficient square plus you know square of the amplitude reflection coefficient square is equal to 1.

In that case you can actually find out the values which is e to the power minus J phi ok can be given as this quantity. So, you have this transmission coefficient also its conjugate and this is the value that you obtain. And from this you can write that you know if you separate it out to the real and imaginary part on both sides you can find that cos Φ can be written as real of 1 over the amplitude transmission coefficient. So, real of 1 over t.

So, keep this equation in mind. So, now let us try to obtain what is the dispersion relationship. Always remember dispersion relationship is basically the relationship between the bloch wave number K and the angular frequency omega that is we are looking for omega k relationship ok. So, the previous equation that you have seen this one this equation. So, this equation provides the eigenvalues which is exponential minus J phi of the unit cell matrix. And this is basically the progenerator or the source of the dispersion relation for the 1D periodic medium.



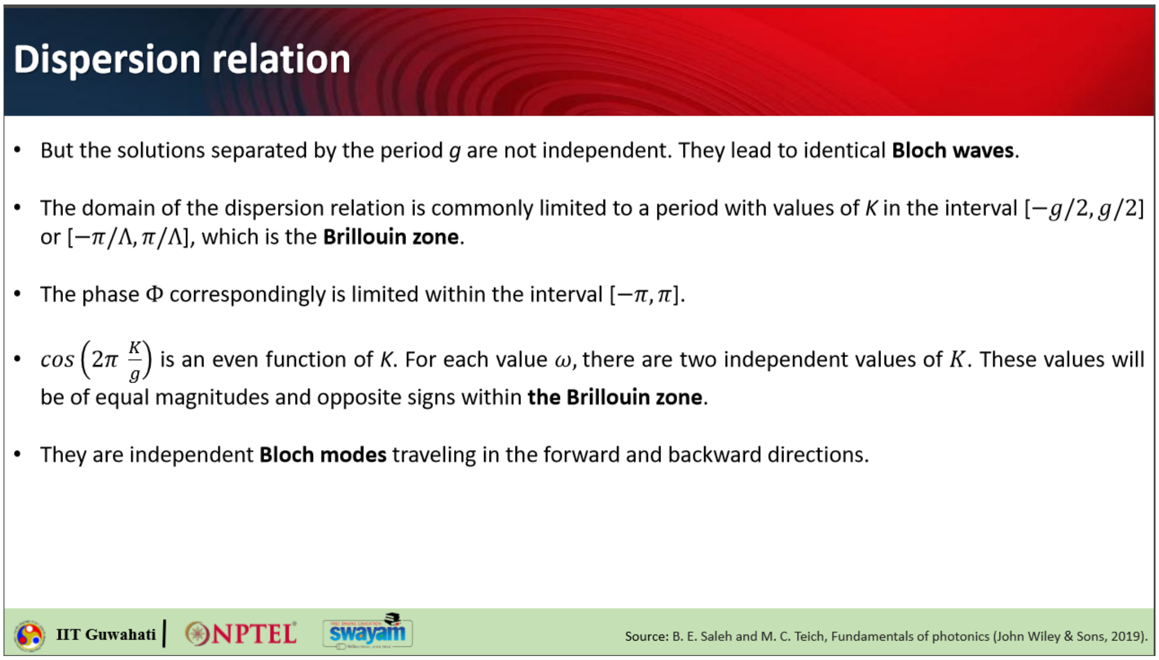
So, how it works? So, we already know that phi that is the phase can be given as capital K that is the bloch wave number times the period. So, phi phase is basically proportional to K ok and t the transmission is also associated with frequency at different different wavelength of frequency will have different transmission. So, t can be written as t omega right. So, these two are related through the phase delay associated with the propagation through the unit cell. So, you can actually write that you know $\cos\left(2\pi \frac{K}{g}\right) \Phi$ is nothing but $\text{Re}\left\{\frac{1}{t(\omega)}\right\}$ which is a function of omega.

So, this one directly correlates your K and omega and hence it can be named as dispersion relation. Now, the question arises what is g here? g is basically the fundamental spatial frequency of the periodic medium. So, what is the period? Period is capital lambda. So, g will be $\frac{2\pi}{\Lambda}$ right. So, this particular function that you see here on the left side $\cos\left(2\pi \frac{K}{g}\right)$ *is* nothing but a periodic function of this K which has got a period of g ok.

So, g is nothing, but $\frac{2\pi}{\Lambda}$ and this gives multiple solutions for you know this equation for any given omega ok. And that is how you are able to obtain that dispersion relation which is typically shown in as in the photonic band diagram. Now, but the solutions separated by the period g they are not independent. They basically lead to identical block modes.

So, the domain of the dispersion relation is typically limited with the values of K ranging from interval of $[-g/2, g/2]$.

That means, it is basically ranging from $[-\pi/\Lambda, \pi/\Lambda]$ which is nothing, but the Brillouin zone. So, that is where the concept of Brillouin zone comes on. And that allows your phase ok phi to be limited to an interval of minus pi to pi. So, once you know the phase variation from minus pi to pi you are basically covering the entire 2 pi right.



So, after that it is just a repetition. So, there is no point computing those ok. So, this range this interval can give you the interval of phase starting from minus pi to pi. Also, we need to keep in mind that this cos function is an even function of K. So, for each value of omega there are 2 possible values of K ok 2 independent values and they could be equal in magnitude, but opposite in sign within the same Brillouin zone ok or within the Brillouin zone. So, Brillouin zone is from minus pi to pi in terms of phase or you can say it is from $-g/2$ to $g/2$ in terms of K.

So, this actually gives us that they are independent bloch waves. So, one solution is for you know forward propagating wave another solution is for the backward waves. So, dispersion relation gives you the photonic band structure. So, dispersion relation will also tell you the multiple spectral bands which can be typically classified into 2 regions or 2 regimes.

So, one is propagation regime. So, spectral band within which capital K that is the block wave number is real those are the propagating modes. So, in those cases the real part of 1 over t which is a function of omega ok that is less than 1 and these bands can be numbered as 1, 2 and so on starting from the lowest, make sense. And there could be other cases where in some spectral bands this K is complex. That means, they correspond to evanescent waves ok. It means these waves will get rapidly attenuated and they cannot propagate within that periodic medium.

Photonic Band Structure

- Dispersion relation demonstrates multiple spectral bands which are classified into two regimes:

  - **Propagation regime**. Spectral bands within which K is real — correspond to the **propagating modes**.

    Here, $\left|Re\left\{\frac{1}{t(\omega)}\right\}\right| \leq 1,$ these frequency bands are numbered, 1, 2, . . . , starting with the lowest.

  - **Photonic-bandgap regime**. Spectral bands within which K is complex — correspond to evanescent waves. These waves are rapidly attenuated.

    Here, $\left|Re\left\{\frac{1}{t(\omega)}\right\}\right| > 1,$ these bands behaves as the stop bands of the diffraction grating.

    Also called **photonic bandgaps (PBG)** or **forbidden gaps** since no existing propagating modes.

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

So, in this case if you see they will give you ignore this particular sign it's only modulus of real 1 over t and that will come out to be greater than 1. And these bands behave as stop bands of that diffraction grating. So, they are also called as photonic band gap PVG or forbidden band gap since no existing propagation mode are possible in this particular case. So, now let us look into the calculation of photonic band structure by taking an example of periodic stack of partially reflective mirrors. So, here is a stack of periodic stack you should say or partially reflective mirror and the wave travelling along the axis of the periodic stack is in the direction of z.

Photonic Band Structure

Example L12. 1. Periodic Stack of Partially Reflective Mirrors

- The dispersion relation for a wave traveling along the axis of a periodic stack of identical partially reflective lossless mirrors separated by Λ :

  Power reflectance $|r|^2$
  Intensity transmittance $|t|^2=1-|r|^2$

- **Matrix Optics** approach can be used to derive explicit expressions for elements of the scattering matrix of a composite system in terms of elements of the scattering matrices of the constituent systems.

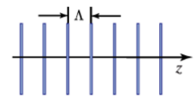
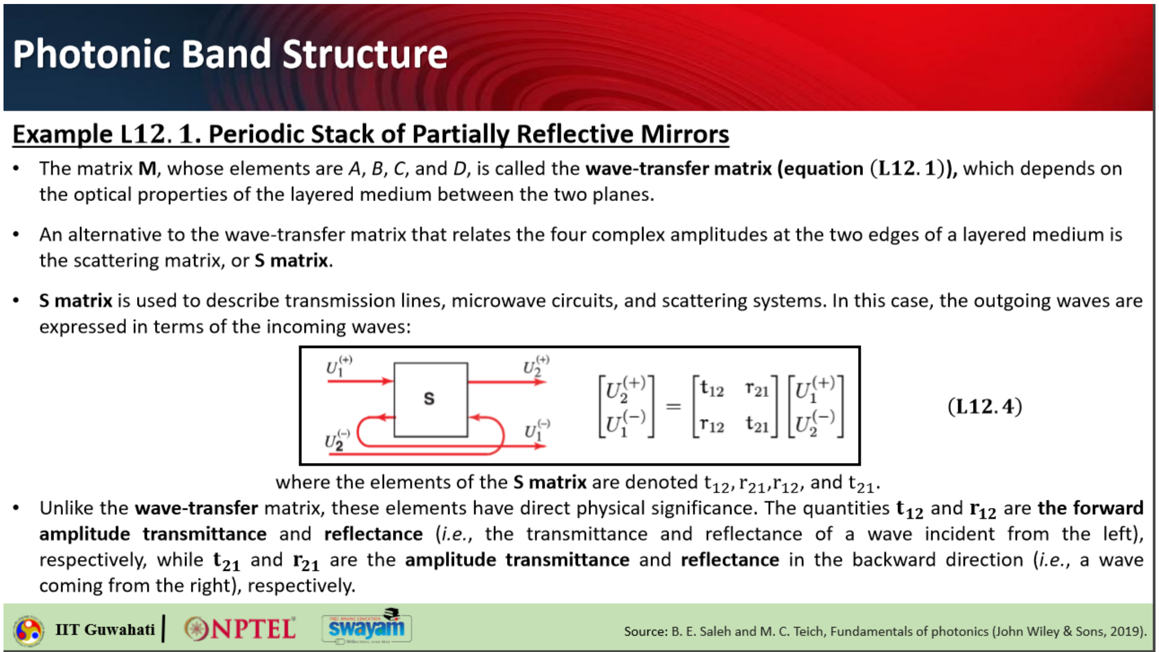


Figure: Wave traveling along the axis of a periodic stack of identical partially reflective lossless mirrors.

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

What is the period here? Capital lambda. Now, let us see how we actually characterize this. So, the dispersion relation for a wave travelling along the axis of the periodic stack of identical that is very important identical partially reflective lossless mirrors which are separated by capital lambda. So, in this case the power reflectance is modulus of r square and intensity transmittance is nothing but what is not reflected is getting transmitted because these are non absorbing case. So, you can say modulus of t square that is transmittance is nothing, but 1 minus modulus of small r square. Now, let us use the matrix optics approach to derive explicit expression for elements of the scattering matrix of the composite system in terms of the elements of the scattering matrix of the constituent system that is we will take the elements of the unit cell and we will try to get the matrix elements for the overall system.

So, the matrix M whose elements are say A B C D you can call them as wave transfer matrix which we have seen in this particular equation. So, they depend on the optical properties of the layered media between the 2 planes, right. So, we have already seen this particular case that you can obtain those equations or those elements from Fresnel reflection coefficients. An alternative to the wave transfer matrix that relates the 4 complex amplitude of the at the 2 edges of layered medium is a scattering matrix S matrix. So, you can also have S matrix and S matrix are more popularly used in describing transmission lines microwave circuits and scattering systems ok.



So, in this case the outgoing waves are basically expressed in terms of incoming waves something like this. So, S matrix is used to describe transmission lines microwave circuits and scattering systems. So, in this case the outgoing waves are basically expressed in terms of the incoming waves. So, here is a schematic representation of S matrix. So, you see that you have the incoming wave $\boldsymbol{U}_1^{(+)}$ and you have one outgoing wave that is $\boldsymbol{U}_1^{(+)}$.

Now in this case this is a reflection, but you are actually trying to represent it in terms of outgoing wave because the reflection is also outgoing. So, you put it on the right side. So, you call it as $\boldsymbol{U}_1^{(-)}$ and the reflection from the other side becomes kind of incoming wave. So, you can actually take that $\boldsymbol{U}_1^{(-)}$ as a incoming one. So, in that case you can simply see that what are the 2 outgoing waves from this particular system that is $\boldsymbol{U}_2^{(+)}$and $\boldsymbol{U}_1^{(-)}$.

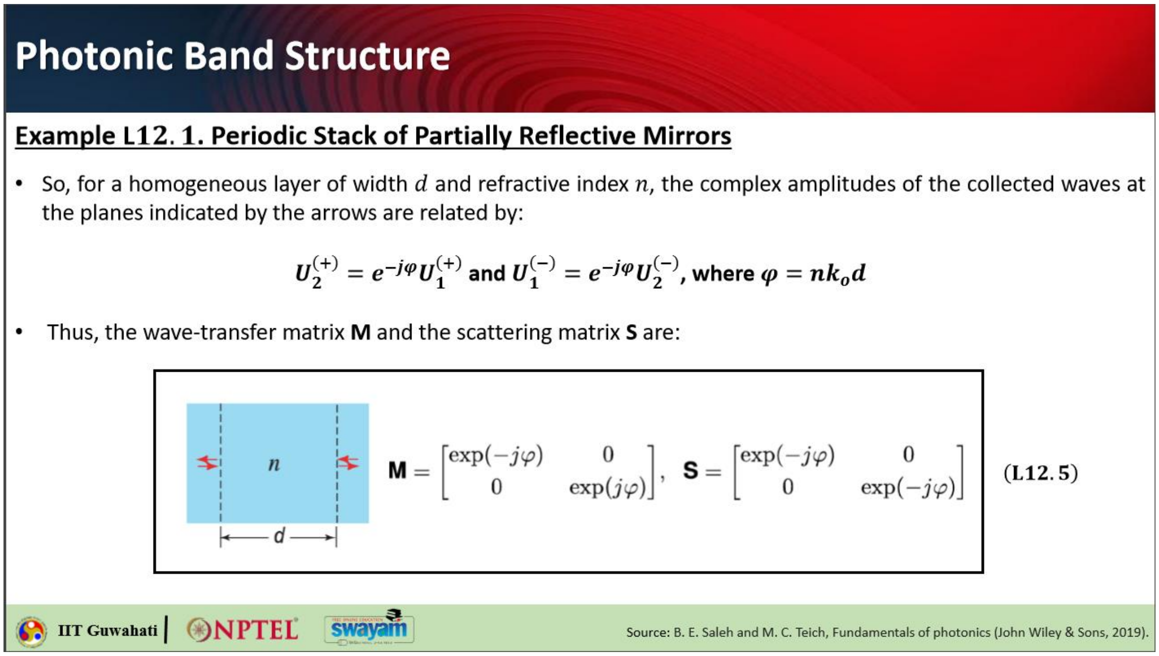
So, $\boldsymbol{U}_2^{(+)}$and $\boldsymbol{U}_1^{(-)}$ are the outgoing and what are the incoming $\boldsymbol{U}_1^{(+)}$and $\boldsymbol{U}_2^{(-)}$. So, $\boldsymbol{U}_1^{(+)}$and $\boldsymbol{U}_2^{(-)}$ and you are trying to correlate this outgoing set of waves with the incoming set of waves. So, what are the coefficients? So, $\boldsymbol{U}_2^{(+)}$ as you know $\boldsymbol{U}_2^{(+)}$ will be nothing, but $\boldsymbol{U}_1^{(+)}$ times the transmission that is t_{12}. So, t_{12} times $\boldsymbol{U}_1^{(+)}$ also it will have another component coming from this one. So, whatever is this wave whatever is getting reflected that will also contribute to $\boldsymbol{U}_2^{(+)}$.

So, you can have this is 2 this is 1. So, you can this reflection coefficient will be called

r_{21}. So, you will have r_{21} times $\boldsymbol{U}_2^{(-)}$. Is it clear? So, you will have $\boldsymbol{U}_2^{(+)}$ that is given as t_{12} times $\boldsymbol{U}_1^{(+)}$. So, t_{12} times $\boldsymbol{U}_1^{(+)}$ this one. So, there is also one contribution coming from this one some part of it will get reflected and add up to this outgoing wave.

That will be r_{21} times $\boldsymbol{U}_2^{(-)}$. The other one also you can easily make it. So, this one $\boldsymbol{U}_1^{(-)}$ is nothing, but r_{12} $\boldsymbol{U}_1^{(+)}$. So, whatever is incidenting some part is getting reflected. So, that reflection is this one r_{12} $\boldsymbol{U}_1^{(+)}$ and then whatever you are putting here some part is getting transmitted and that also comes back as $\boldsymbol{U}_1^{(-)}$. So, that is t 21 times $\boldsymbol{U}_2^{(-)}$. So, this equation $\boldsymbol{U}_1^{(-)}$ is nothing, but r_{12} $\boldsymbol{U}_1^{(+)}$ plus t_{21} $\boldsymbol{U}_2^{(-)}$. Clear? So, unlike the wave transfer matrix this elements here in scattering matrix they have direct physical significance. Something like you know if you take r_{12} and r_{21} they are basically the forward amplitude transmittance and reflection. That is, they are basically the transmittance and reflection coefficient of the wave incident from the left side. On the other hand if you see t_{21} and t1 t_{21} and r_{21} they are basically amplitude transmittance and reflectance in the backward direction that is for wave that is coming from the right side.So, it is easy to you know correlate physically what is happening in the case of S matrix.

Now, for a homogeneous layer of width d. So, this interface we have seen that what happens with the incoming and outgoing incoming and outgoing or you can say what is falling getting reflected and so on. So, for a homogeneous layer of width d and refractive index n that is shown here the complex amplitudes of the collected waves at the planes indicated by the arrow. So, if you are looking about the complex amplitude at this particular planes. So, you can call this as U1 ok. So, this is U1 forward one will be $\boldsymbol{U}_1^{(+)}$ backward one will be $\boldsymbol{U}_1^{(-)}$. this will be U2 ok, the amplitudes here will be U2 the forward one will be $\boldsymbol{U}_2^{(+)}$ and the reverse one will be $\boldsymbol{U}_2^{(-)}$.



So, how they are related? They are propagating or they are travelling this particular distance. So, they will add up a phase. So, $\boldsymbol{U}_2^{(+)}$ will be simply $\boldsymbol{U}_1^{(+)}$ times $\boldsymbol{e}^{-\boldsymbol{j\varphi}}$. What is phi? It will be n k naught small k naught and d ok, n is a refractive index k naught is the free space wave factor or wave number and d is the thickness of that layer.

So, that is similarly you can also correlate what is $\boldsymbol{U}_1^{(-)}$ and $\boldsymbol{U}_2^{(-)}$. So, that is how you can obtain the wave transfer matrix as well as scattering matrix for this particular case. So, if you see that wave transfer matrix M will look like $\exp(-j\varphi)$, 0, 0, $\exp(j\varphi)$. Whereas, the scattering matrix because scattering matrix will try to represent all outgoing in terms of incoming not left and right ok. So, it will look like $\exp(-j\varphi)$, 0, 0, $\exp(-j\varphi)$ ok. So, that's the only difference between the wave matrix where wave transfer matrix and scattering matrix.

Photonic Band Structure

Example L12. 1. Periodic Stack of Partially Reflective Mirrors

- Now, consider a wave transmitted through a system described by an **S** matrix with elements t_{12}, r_{21}, r_{12}, and t_{21}, followed by another system with **S** matrix elements t_{23}, r_{32}, r_{23}, and t_{32}.
- By multiplying the two associated **M** matrices, and then converting the result to an **S** matrix, the following formulas for the overall forward transmittance and reflectance can be derived:

$$t_{13} = \frac{t_{12}t_{23}}{1 + r_{21}r_{23}} \qquad \text{(L12.6)}$$

$$r_{13} = r_{12} + \frac{t_{12}t_{21}r_{23}}{1 - r_{21}r_{23}} \qquad \text{(L12.7)}$$

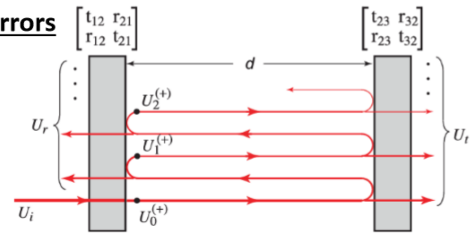


Figure: Transmission of a plane wave through a cascade of two separated systems.

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{1}{t_{21}} \begin{bmatrix} t_{12}t_{21} - r_{12}r_{21} & r_{21} \\ -r_{12} & 1 \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} t_{12} & r_{21} \\ r_{12} & t_{21} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} AD - BC & B \\ -C & 1 \end{bmatrix}.$$

Relation between M & S matrix

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

So, now let us consider a wave transmitted through a system which is described by S matrix which has got this elements t $_{12}$, t $_{21}$, r $_{12}$, r $_{21}$. So, it these are easy to handle because we already know this transmission and reflection coefficient from the Fresnel equation. So, let's assume that you know this system has got two such separate systems ok. And these are the S matrix for these two separate systems.

So, by multiplying the two associated M matrix. So, you can convert this into M matrix this one into M matrix you can multiply the M matrix and then convert it back to the scattering matrices ok. And you will be able to obtain the overall transmittance and reflection. So, overall transmittance in this case will be t_{13} which is given by $\frac{t_{12}t_{23}}{1+r_{21}r_{23}}$. So, this is how you will be analyzing the multilayer system ok.

You can also find out what is the reflection coefficient for this overall system ok. So, one important thing is that the relationship between M and S matrix in this case. So, as I mentioned M matrix are having four elements a b c d and they are not directly the reflection and transmission coefficient whereas, the S matrix are directly the reflection and transmission coefficient.So, sometimes it is easy to deal with S matrices. Now in this particular case the transmission of a plane wave through a cascade of two separate systems that we have seen which are separated by a distance of d ok.

Photonic Band Structure

Example L12. 1. Periodic Stack of Partially Reflective Mirrors

- If the two cascaded systems are mediated by propagation through a homogeneous medium, then the overall transmittance and reflectance can be derived by using the wave-transfer matrix:

$$t_{13} = \frac{t_{12}t_{23}\exp(-j\varphi)}{1 - r_{21}r_{23}\exp(-j2\varphi)} \qquad \text{(L12.8)}$$

$$r_{13} = r_{12} + \frac{t_{12}t_{21}r_{23}\exp(-j2\varphi)}{1 - r_{21}r_{23}\exp(-j2\varphi)} \qquad \text{(L12.9)}$$

where, the phase $\varphi = nk_o d$. d is the propagation distance and n is the refractive index of the medium.

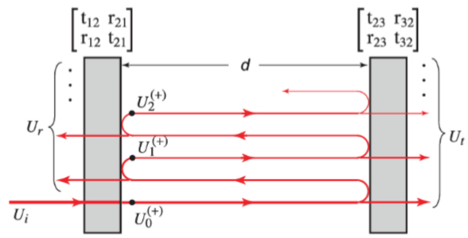


Figure: Transmission of a plane wave through a cascade of two separated systems.

IIT Guwahati | NPTEL swayam Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

Now we have assumed that if the two cascaded systems are mediated by propagation through a homogeneous medium it means the medium in between is a homogeneous medium then the overall transmittance and reflectance will also have this extra factor adding up that is $\exp(-j\varphi)$ and φ is nothing but n k_0 d ok.

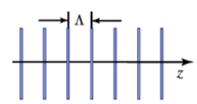
Photonic Band Structure

Example L12. 1. Periodic Stack of Partially Reflective Mirrors

Therefore, for periodic stack of identical partially reflective lossless mirrors, using the results obtained from L12.8 & L12.9 , we get:

$$\cos\left(2\pi \frac{K}{g}\right) = \frac{1}{|t|}\cos\left(\frac{\omega}{\omega_{\mathcal{B}}}\right) \qquad \text{(L12.10)}$$

where $g = 2\pi/\Lambda$, and $\omega_{\mathcal{B}} = c\pi/\Lambda$



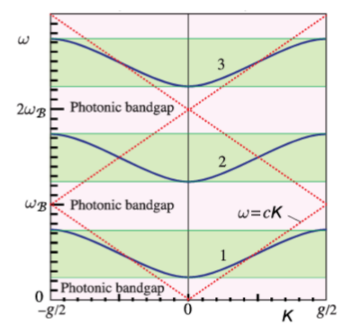


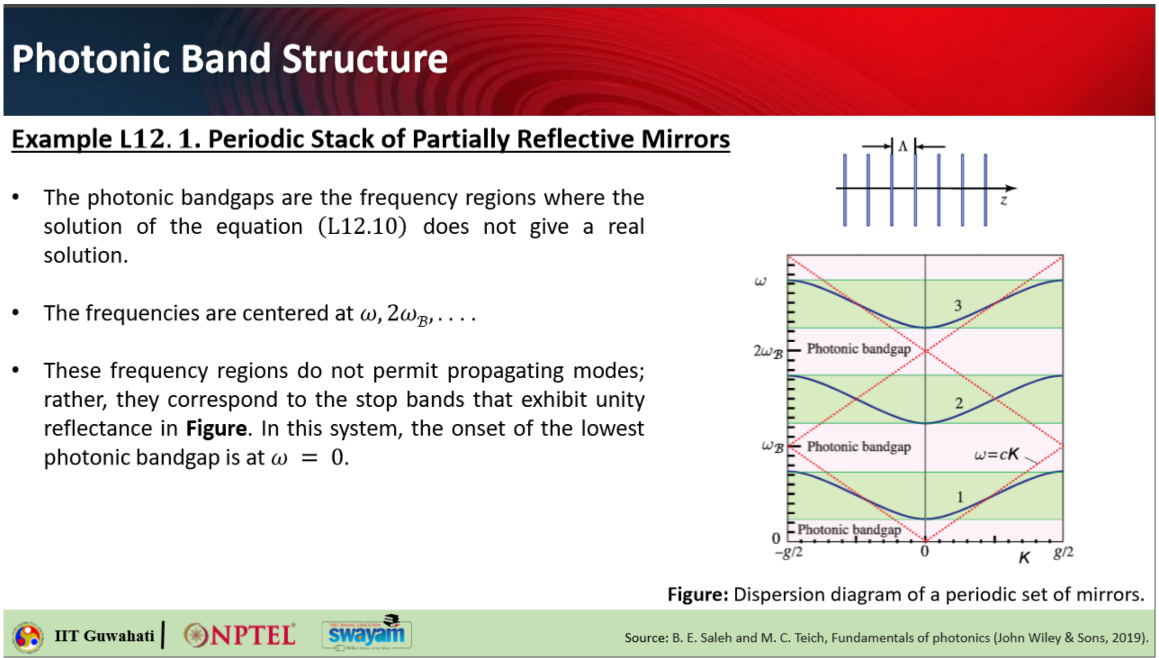
Figure: Dispersion diagram of a periodic set of mirrors, each with intensity transmittance $|t|^2 = 0.5$, separated by a distance Λ. Here, $\omega_{\mathcal{B}} = \pi c/\Lambda$ and $g = 2\pi/\Lambda$. The dotted straight lines represent propagation in a homogeneous medium for which $\omega/K = \omega_{\mathcal{B}}(g/2) = c$.

IIT Guwahati | NPTEL swayam Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

So, that way the equations will also get slightly modified ok. So, as I mentioned here the phase phi is nothing, but n k naught d and d is the propagation distance n is nothing but the refractive index of this particular medium inside. So, with that what we learnt is that we understood the overall reflectance and transmittance and for this periodic stack of identical partially reflective lossless mirrors. So, using the equations this and this you can obtain what is the dispersion relation or you can find out that $\cos\left(2\pi\frac{K}{g}\right)$ is nothing, but $\frac{1}{|t|}\cos\left(\frac{\omega}{\omega_{\mathcal{B}}}\right)$.

So, here a new term omega B has come. So, omega b. So, g you already know g is $2\pi/\Lambda$ that is the special frequency spatial space related. So, spatial frequency and that you have $\omega_{\mathcal{B}}$, which is $\frac{c\pi}{\Lambda}$. So, this is particularly a plot of the dispersion relation for a set of periodic mirrors. So, here certain values have been assumed like modulus t square has been taken as 0.5 and they have been considered to have a separation of capital lambda. $\omega_{\mathcal{B}} = c\pi/\Lambda$, g is $2\pi/\Lambda$ those are all fine. So, only important thing is the value of t is already assumed here and you can see this red dotted straight lines they are basically the approximation of a homogeneous medium. So, if you assume the entire medium to be homogeneous in which omega by K equals c or you can carefully work this out and see that omega by K will be omega B times g by 2 that also comes out to be c. So, you will have this straight lines.

So, these are basically the homogeneous medium approximation. So, what it tells you that you know this graph tells you that because of the periodicity how much the dispersion relation deviates from the homogeneous medium and in homogeneous medium you see there is no band cap also. So, all the bands are allowed all the bands are allowed means all the frequencies have some k vector. It means at all frequencies you have solution for waves which has got real propagation constants, ok. But here in this case you can see it starts with a band gap then there is some band which is allowed then again there is a band gap then again there is some band where the propagation is allowed then again there is a band gap and so on ok. Now, yeah this is what I have already discussed that here the photonic band gaps there is no real solution.



So, you do not have anything and all the band gap frequencies are basically centered around omega omega or you say $\omega_{\mathcal{B}}$, $2\omega_{\mathcal{B}}$ and so on $3\omega_{\mathcal{B}}$. So, the frequencies they this particular frequencies they do not permit any propagating mode rather in that case if the wave is not allowed to propagate inside the periodic medium what will happen? In terms of reflectance you will see that they have unity reflectance and this is a particular system where you also see that you know the lowest photonic band gap is at omega equals 0.

So, if you take a real example with some values like n 1 equals 1. 5 and n 2 equals 3.5 and keep the thickness of the 2 layer similar. So, this is one layer this is another layer and then you are repeating this unit cell. So, this is the period ok period of the unit cell. So, if you take this and you try to calculate the dispersion relation you will see that you know the photonic band gaps have center frequencies at omega B here also $\omega_{\mathcal{B}}$ and its multiples like $\omega_{\mathcal{B}}$, $2\omega_{\mathcal{B}}$, $3\omega_{\mathcal{B}}$ and so on. And they occur at either the brilliant zone center that is K equals 0 or at the edges that is K equals plus minus g by 2.

Photonic Band Structure

Example L12. 2. Alternating Dielectric Layers

- An example with $n_1 = 1.5$ and $n_2 = 3.5$, and $d_1 = d_2$.
- The photonic bandgaps have center frequencies ω_B and its multiples, and occur at either **the Brillouin zone's center** ($K = 0$) or its edge ($K = g/2$).

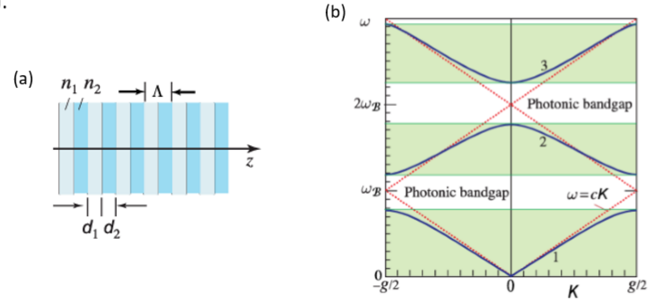


Figure: (a) Dispersion diagram of an alternating-layer periodic dielectric medium with $n_1 = 1.5$ and $n_2 = 3.5$, and $d_1 = d_2$. (b) The photonic bandgap.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

So, this is the range of the brilliant zone. So, K value starts from you know minus g by 2 to g by 2 as we discussed before. So, here you see that initially all frequencies are permitted at omega B you have a particular band gap again at 2 omega B you have a band gap and so on ok. And this is how it deviates from the homogeneous medium approximation. So, these are the values associated with this particular band gap. Now in this setup of partially reflective mirrors the frequency region surrounding to $\omega = 0$ does not fall in the band gap it has got some solution.

Photonic Band Structure

Example L12. 2. Alternating Dielectric Layers

- In this setup of partially reflective mirrors, the frequency region surrounding $\omega = 0$ allows propagating modes instead of a forbidden gap.
- Dielectric materials with lower contrast have bandgaps of smaller width but bandgaps exist no matter how small the contrast.
- The **red dotted straight lines** demonstrate propagation in a homogeneous medium with mean refractive index of n_1 and n_2.

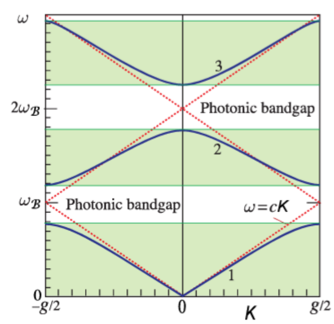

Figure: The photonic bandgap of an alternating-layer periodic dielectric medium with $n_1 = 1.5$ and $n_2 = 3.5$, and $d_1 = d_2$.

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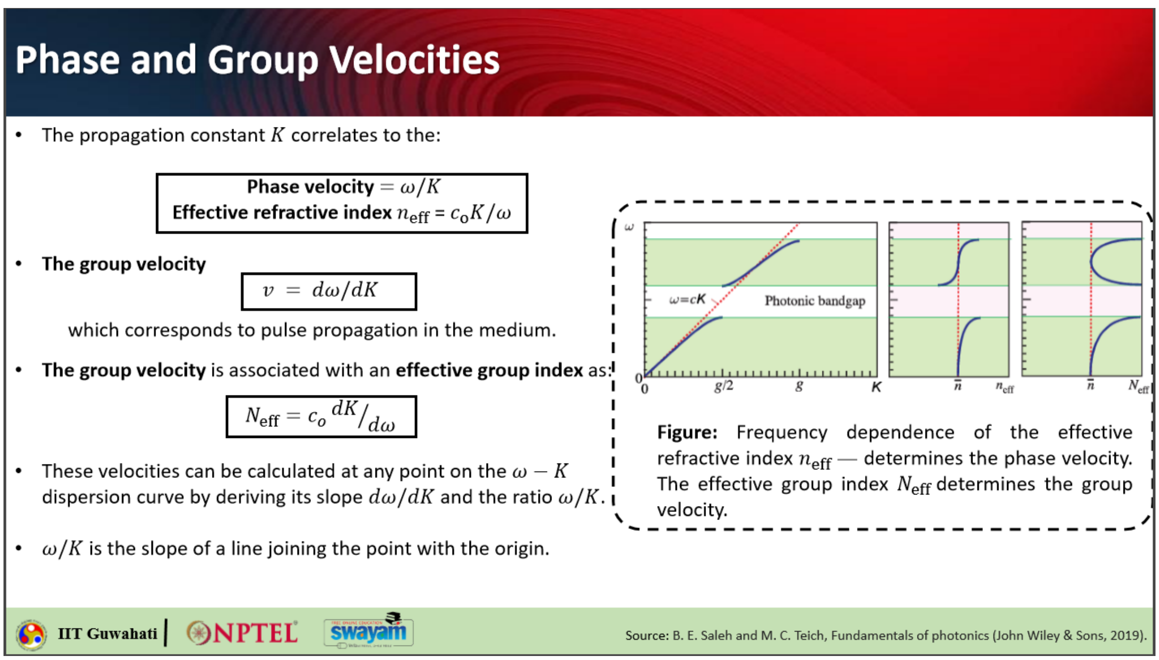
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics (John Wiley & Sons, 2019).

So, it's good in this case ah there are some propagating modes possible here ok. Now dielectric materials with lower contrast they will have band gaps of smaller width.

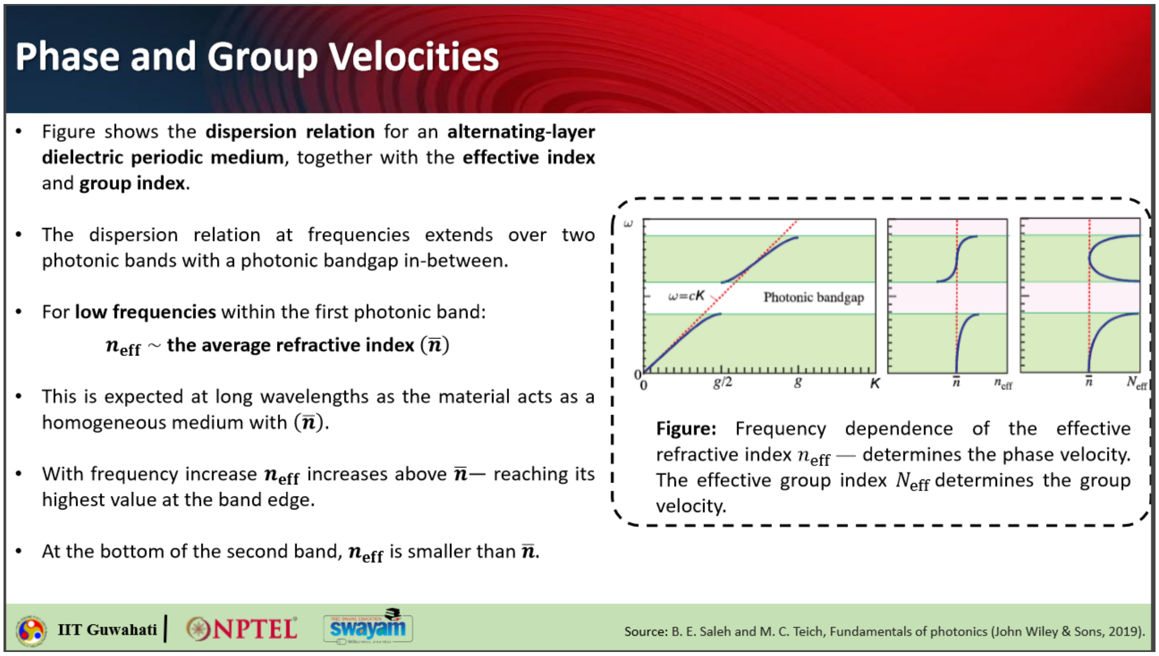
Now here you see the contrast is really good. So, n_1 is 1.5, n_2 is 3.5. Now if you take 2 materials, where the difference between n_1 and n_2 like you can say deltan is less this band gap will also become very narrow ok. So, if you want the larger band gap you choose 2 materials which have higher contrast between them ok. And this red straight lines we already ah mentioned that this is how light would have behaved if we have a homogeneous medium with refractive index of the mean of n_1 and n_2 fine. So, from this you also can derive the information about the phase and group velocities.

So, the propagation constant capital K it correlates to the phase velocity as well. So, phase velocity will be omega over capital K. So, once you know the phase velocity you can also find out what is the effective refractive index that is small n effective ok that is c naught over the phase velocity. So, you will get c naught capital K over omega fine. So, this is we are taking only up to this one.

So, here you can see that clearly see what is the photonic band gap ok. And this is the plot of effective refractive index that is $n_{\text{eff}} = c_0 K/\omega$ ok. So, that is basically the effective refractive index. You can also find out what is the group velocity that is $v = d\omega/dK$ ok which corresponds to the pulse propagation in the medium. So, any pulse will have a you know frequency spread ok means it will not be monochromatic it will have certain frequencies.



So, you should calculate the group velocity in that case. So, group velocity should be obtained by $d\omega/dK$. So, accordingly you can also find out what is the effective index seen by that group or you can call it effective group index that is also defined as ah capital N effective and that can be given as c_0 over this v. So, you get $N_{\text{eff}} = c_o dK/d\omega$ ok. Now, these velocities can be calculated at any point of the dispersion relation curve by deriving the slope $d\omega/dK$ and you can also take the ratio of ω by K. So, as shown here you can also you can calculate what is small n effective, what is your capital N effective, what is n bar n bar this is the mean value ok, mean refractive index.

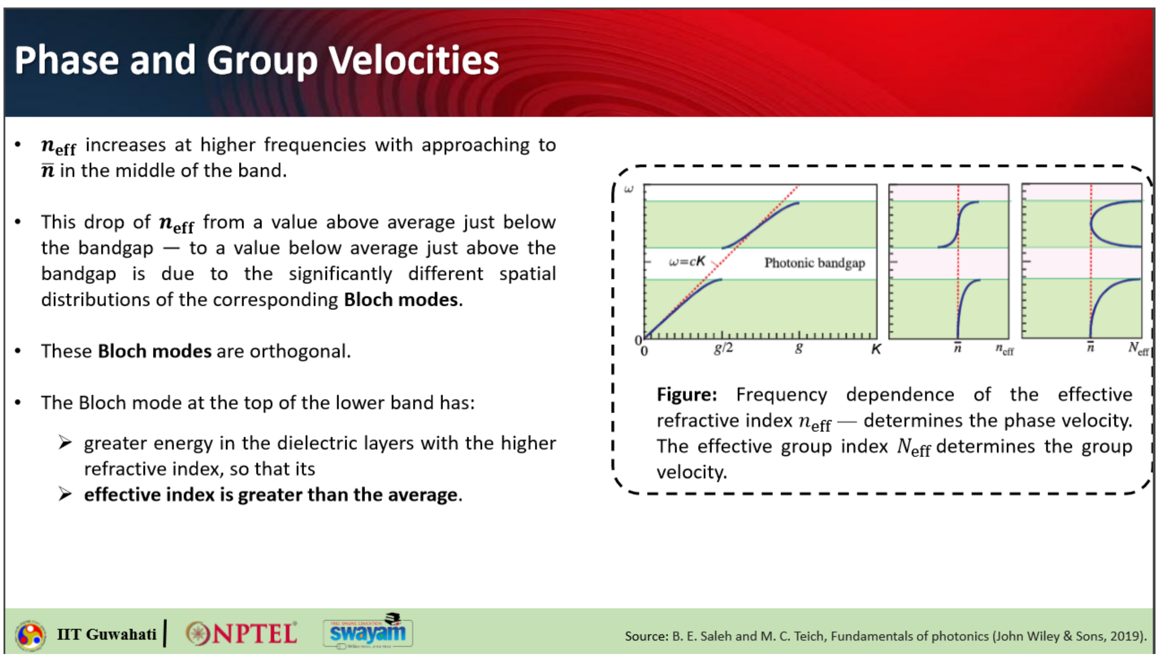


So, the figure here the first one shows the dispersion relation of a long rotating layer of dielectric medium and this two shows the effective index of one particular frequency and this is the group index ok. So, here what you see you are able to clearly see two frequency bands where the propagation is possible and there is a definite photonic band gap ok. And for lower frequencies within the first photonic band you can see that this is how the effective index is. So, you can say that n effective is very close to the average effective index.

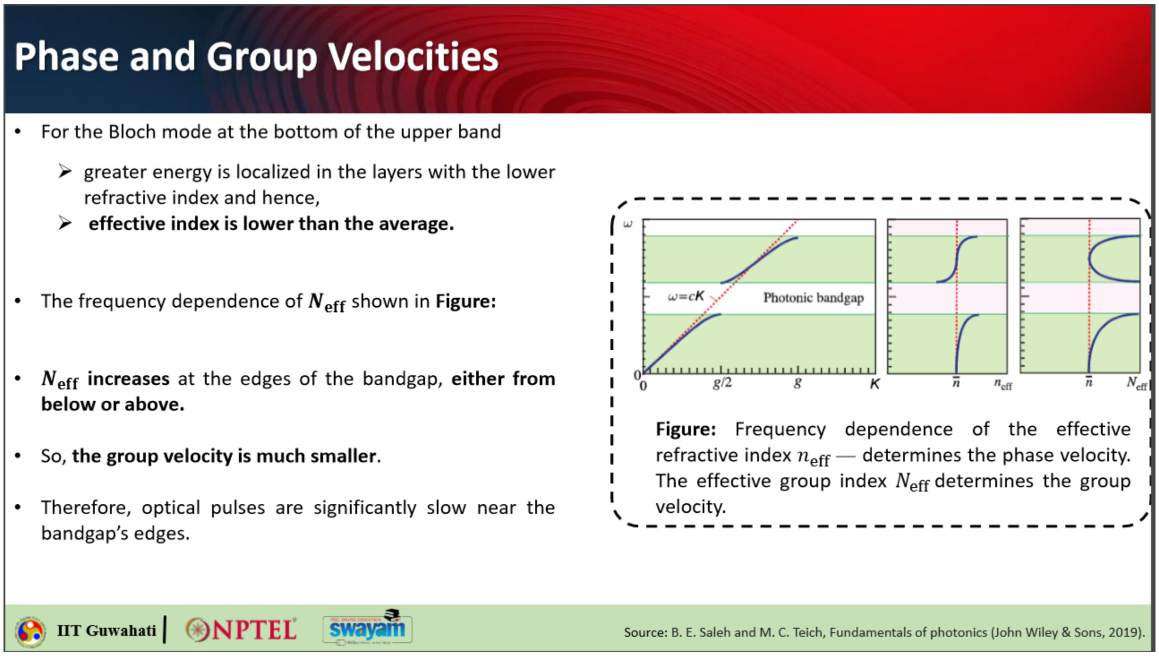
So, initially this dotted line and this blue line they are very much overlapping ok. So, here also you see they are very much overlapping, but as you keep on you know going further with omega or you can say with wavelength it is expected that at longer wavelength the material becomes homogeneous. So, K is reducing means wavelength will be increasing. So, this is the case where you see more homogenized picture of your periodic medium, but as K is increasing your wavelength is basically reducing. So, you will be able to see the definite structures and that is where your you will be deviating from the line this particular red dotted line which represents a homogenized medium ok.

So, here also you can see with frequency increase. So, this way the frequency is increasing ok this way the frequency is increasing. So, it is better to correlate with frequency and wavelength and this is the spatial period ok. So, you can correlate with the frequency here that at lower frequency wavelengths are high. So, you are seeing much homogeneous picture whereas, when you go for higher frequency you have lower wavelength you start deviating from the mean refractive index ok that is the crux of this particular thing. And at the second at the bottom of the second band that is here you will see that you know the n effective is much smaller than the mean refractive index.

So, that is how it works that you know with the frequency increase initially n effective goes way above the mean value, but then suddenly it encounters a band gap and after the band gap at the bottom of the second band you will see that the n effective starts from a value which is much slower or much smaller than the mean refractive index. So, n effective increases at higher frequencies and with approaching to n bar which is the mean value at the middle of the band, understood. Now this drop of n effective from a value above average which is just below the band gap to a value which is below average just above the band gap is due to the significantly different spatial distribution of the corresponding block modes. So, there is a band gap because of which the bloch modes which are propagating here and here are significantly different. So, we do not expect them to have you know similar kind of feature and that is why there is a drastic change in this effective refractive index as well.



And this bloch modes are orthogonal to each other. So, there is no similarity basically between this bloch modes. Now, the bloch modes at the top of the lower band has greater energy in the dielectric layers with higher refractive index. So, that the effective index is basically greater than the average. And if you look into the block modes at the bottom of the upper band it will be reverse. It means in that case greater energy is localized in the layers which are having lower refractive index and that is why the overall effective index is lower than the average.



So, these are like two different mode configuration for the two different bloch modes which is present here and here fine. Lastly let us also look into the frequency dependence of the capital N effective which is the group effective index. And you see the group effective index increases at the edges of the band gap either from below or above ok. In both cases it is behaving the same way and that means the group velocity is much smaller. So, when index is larger the group velocity is smaller.

It means when you are approaching a band gap you will see that the waves are much slower. So, the optical pulses are significantly slow near band gaps edge. So, that way you can actually make different devices based on this particular concept. So, with that we will stop here today.

Thank you. Any questions you can drop an email to this particular email address and we will see you in the next lecture. Bye. Thank you.