**Course Name- Nanophotonics, Plasmonics and Metamaterials**

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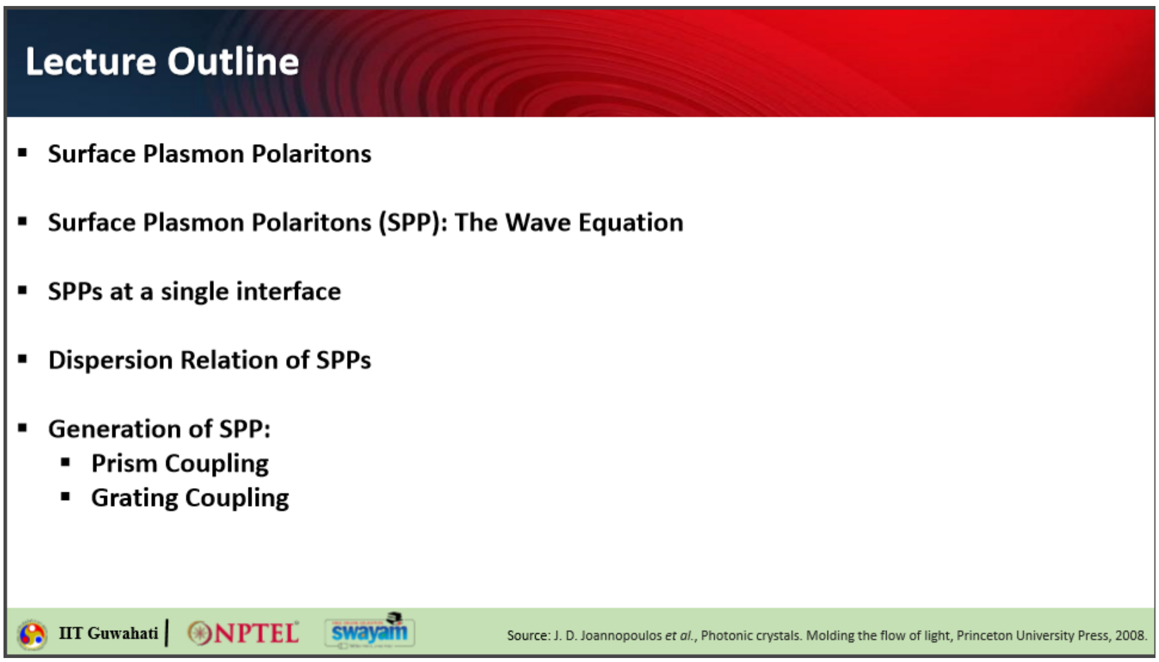
**Institute Name- Indian Institute of Technology Guwahati**

**Week-06**

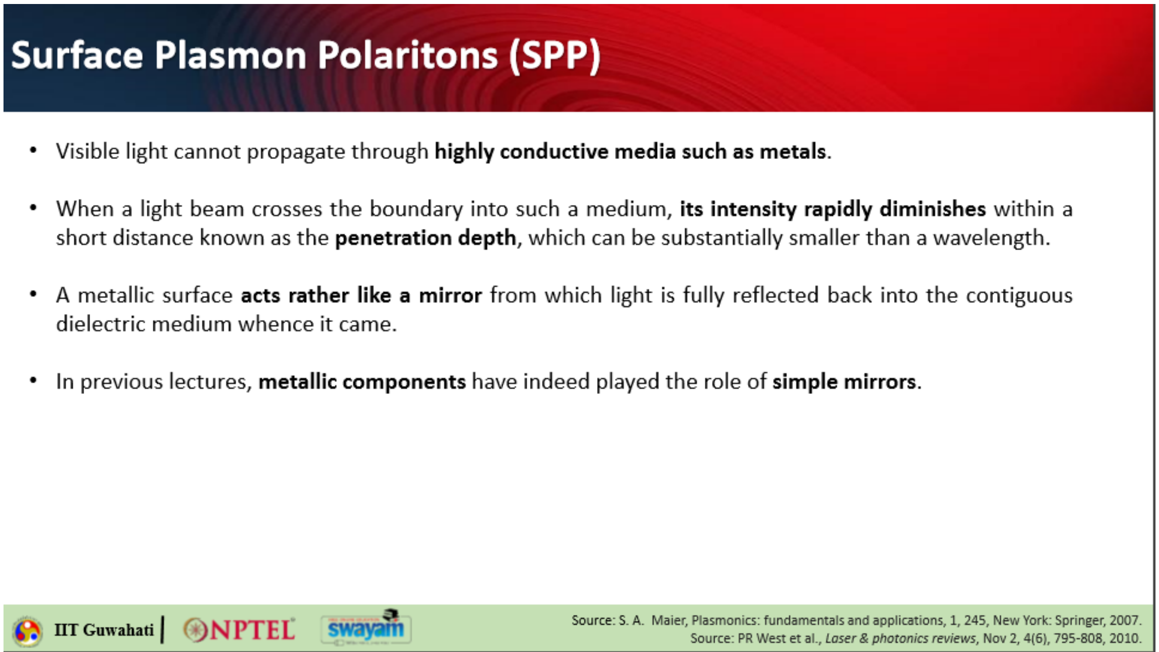
**Lecture -17**

Hello students, welcome to lecture 17 of the online course on Nanophotonics, Plasmonics and Metamaterial. Today we will be covering surface plasmon polaritons or in short SPP and we will look into their fundamentals. So, here is the lecture outline. So, we will discuss about surface plasmon polaritons, obtain the wave equation, we will look into SPPs at single interface, the dispersion relationship of this SPPs and we will also if time permits we will look into this generation or else we will take it into the next lecture ok.




So, surface plasmon polariton. So, what is that? So, visible light as we know cannot propagate through highly conductive media such as metals.

When a light beam crosses the boundary into such medium it its intensity rapidly diminishes within a very short distance which is called as the penetration depth ok and that is substantially smaller than a wavelength. So, a metallic surface rather acts as a mirror right from which light is fully reflected back into the medium or dielectric medium where the light came from. So, in the previous lectures the metallic components have played the role of simple mirrors right. If you remember the reflecting surfaces we were discussing.

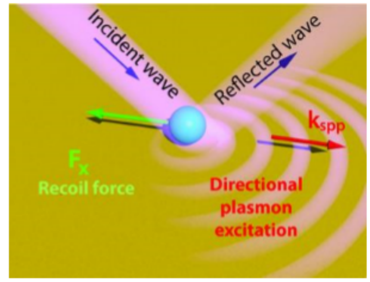



So, you can think of metallic components there and the purpose there was to simply act as mirrors. Now, if I say the metal also supports some kind of light waves ok. So, do not be surprised with that. So, typically what happens you know when you see incident wave there is a reflected wave, but there are conditions in which you are also able to excite some kind of surface waves on this metal surface ok. So, they actually travel along the boundaries of the metal ok and typically they are confined to sub wavelength dimensions.

So, such light waves they travel along a metal surface in the form of a guided surface wave. So, it may propagate on, but not in ok the metallic wires. So, that you have to understand it does not penetrate too much in to the metal rather it is mostly guided along the surface of the metal ok. So, they are basically useful in integrated photonic circuits as wave guides ok. So, you can actually transfer information across this metallic surfaces over a very very short distance.

## Surface Plasmon Polaritons (SPP)

- However, metals can **support light waves**, provided that they travel along the boundaries of the metal, in regions confined to sub-wavelength dimensions.
- Such light waves travel **along a metal surface** in the form of a guided surface wave. It **may propagate *on*, but not *in*, metal wires**, and it may be guided by subwavelength metallic structures configured as integrated-photonic circuits.
- Such structures may also serve as resonators within which light may be confined, or from which light can scatter strongly at specific resonance frequencies.






Source: S. A. Maier, Plasmonics: fundamentals and applications, 1, 245, New York: Springer, 2007.
Source: PR West et al., *Laser & photonics reviews*, Nov 2, 4(6), 795-808, 2010.

Such structures can also serve as resonators within which light may be confined or from which light can scatter strongly at a specific resonant wavelength or frequency. So, there are couple of applications with SPPs which we have covered briefly during our introduction lecture. So, here we will go into the fundamentals of this SPPs. We will also look into some applications in the next lecture, but today's lecture will be mainly on the fundamentals. So, surface plasmon polaritons or SPPs ok.

So, these are basically what electromagnetic waves. Where they travel? They travel along metal dielectric or metal air interface fine. And where they are seen which range they are seen? They are practically seen in the infrared and visible frequency range and they are called surface plasmon polaritons or SPPs. So, this is illustration of surface plasmon. So, this is the thin planar metallic film and this is the boundary of the metallic film.

On the top you have dielectric. So, what are happening here? You can see that the field or surface wave is allowed to propagate along the surface or the interface between this metal and dielectric. So, you can say more specifically that surface plasmon polaritons are electromagnetic excitations propagating at the interface between the dielectric and a conductor and they actually decay along the perpendicular direction. So, you can understand because they decay very sharply along the metal. So, you can only see only one such representation is shown ok.

## Surface Plasmon Polaritons (SPP)

- These electromagnetic waves that travel along a metal–dielectric or metal–air interface, practically in the infrared or visible-frequency are called **Surface plasmon polaritons (SPPs).**
- More specifically, **Surface plasmon polaritons** are **electromagnetic excitations** propagating at the interface between a dielectric and a conductor, evanescently confined in the perpendicular direction.
- These electromagnetic surface waves arise via **the coupling of the electromagnetic fields to oscillations of the conductor's electron plasma.**

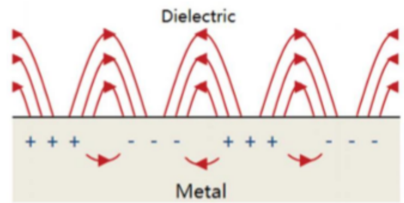



**Figure:** Illustrations of surface plasmon (in the planar metal film).

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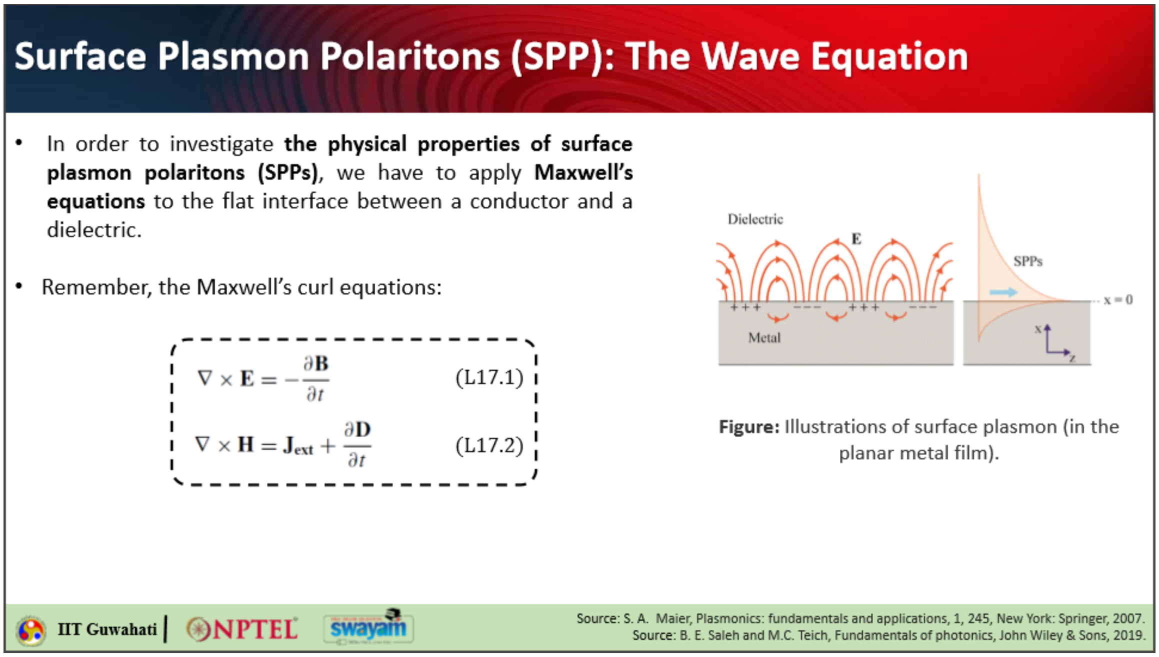
Source: S. A. Maier, Plasmonics: fundamentals and applications, 1, 245, New York: Springer, 2007.
Source: B. E. Saleh and M.C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.

So, electric field lines from plus to minus ok. And here also from plus to minus plus to minus it is denoted by an electric field line, but along dielectric you can see that the electric field is much more extended into the dielectric region ok in the perpendicular direction. So, this is the way they are propagating and the field is extended more towards the dielectric region. Now, this electromagnetic surface waves arise by the coupling of the electromagnetic fields ok, two oscillations of the conductors electron plasma. So, this is where you are basically you can actually use light to excite this kind of surface waves.

So, they are kind of interaction between the electromagnetic fields ok to the oscillation of the conductors in this case is metals electron plasma right. So, this is again similar kind of diagram, but only one thing is shown here extra that is the direction of a SPB propagation and this is what I was talking about that in the metal they extend less while towards the dielectric they can extend much more in the transverse direction. So, in order to investigate the physical properties of surface plasmon polaritons we have to apply Maxwell's equation to the flat interface between a conductor and a dielectric. So, these are again electromagnetic waves. So, they are bound to follow the Maxwell's equation.




So, if you remember the Maxwell's curl equations these are the two curl equations right. So, curl of E is $-\frac{\partial \mathbf{B}}{\partial t}$, J external ($\mathbf{J}_{\mathbf{ext}}$) that is the external surface current density, curl of H equals $\mathbf{J}_{\mathbf{ext}} + \frac{\partial \mathbf{D}}{\partial t}$ ok. Now, if you consider the case where there is no external charge and current density. So, you can actually get rid of this term and this term ok. So, no current charge so you can get rid of this one first ok.

So, you can actually combine these two and write $\nabla \times \nabla \times \mathbf{E}$ equals $-\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}$ ok. That is very simple you can take the curl you take the curl equation here ok. So, what you get you get $\nabla \times \nabla \times \mathbf{E}$ ok equals minus dot E of curl of B ok, B equals $\mu_0$H. So, you can actually take out $\mu$ ok and you can write like this ok. So, you are left with curl of H and curl of H you put from that formula by putting J external to be 0 ok very simple.

Now, the $\nabla \times \nabla \times \mathbf{E}$ can be written as you know gradient of divergence of E minus Laplacian of E ok. And this particular term ok we can expand further, but first let us see what happens to divergence of D. D is nothing but $\varepsilon\mathbf{E}$ ok. So, you can write this as $\mathbf{E} \cdot \nabla\varepsilon + \varepsilon\nabla \cdot \mathbf{E}$ ok. And if you remember that if there is no external stimulus something like you know free charges.

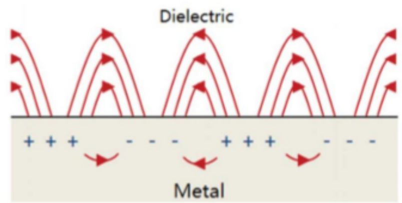
## Surface Plasmon Polaritons (SPP): The Wave Equation

- In the absence of external charge and current densities, the **Maxwell equations** (L17.1 & L17.2) can be combined to yield equation (L17.3) :

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} \qquad (L17.3)$$

- Using the identities $\nabla \times \nabla \times \mathbf{E} \equiv \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ as well as $\nabla \cdot (\varepsilon \mathbf{E}) \equiv \mathbf{E} \cdot \nabla\varepsilon + \varepsilon \nabla \cdot \mathbf{E}$, and remembering that due to the absence of external stimuli $\nabla \cdot \mathbf{D} = 0$, (L17.3) can be rewritten as:

$$\nabla \left( -\frac{1}{\varepsilon} \mathbf{E} \cdot \nabla \varepsilon \right) - \nabla^2 \mathbf{E} = -\mu_0 \varepsilon_0 \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad (L17.4)$$

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So, $\nabla \cdot \mathbf{D}$ is also 0 ok. So, that you can put here and you can simplify the equation further. So, in that case this term so, this is D. So, this $\nabla \cdot \mathbf{D}$ becomes 0 ok. So, what you require here you see from this so, $\nabla \times \nabla \times \mathbf{E}$ can be written as $\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ .

So, $\nabla^2 \mathbf{E}$ remains like this divergence of E can be obtained from this formula ok. So, from this equation so, this is there this part is 0. So, you can write this as $-\frac{1}{\varepsilon}$ will be going to the denominator. So, $\frac{1}{\varepsilon}$ and then you are left with this. So, increase there is a gradient of permittivity ok.

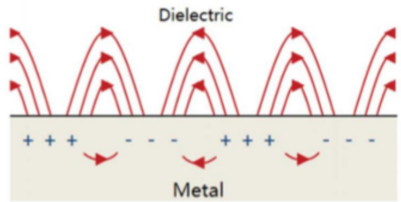
So, it is not an isotropic medium in that case it looks like this. And the right side remains like this ok it is pretty simple. So, D is simply replaced by $\varepsilon\mathbf{E}$ in this case ok. So, you also have a $\varepsilon_0$ coming into the picture because this $\varepsilon$ is basically the $\varepsilon(r)$ ok. So, for negligible variation of the dielectric profile that is if you consider that $\varepsilon$ is $\varepsilon(r)$ ok over distances of the order of one optical wavelength that means, you are not having much variation in the dielectric permittivity in that case you can safely take this term also to be 0.

## Surface Plasmon Polaritons (SPP): The Wave Equation

- For negligible variation of the dielectric profile $\varepsilon = \varepsilon(\boldsymbol{r})$ over distances on the order of one optical wavelength, ( L17.4 ) simplifies to **the central equation of electromagnetic wave theory**:

$$\nabla^2 \mathbf{E} - \frac{\varepsilon}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \text{(L17.5)}$$

- Practically, this equation has to be solved separately in regions of constant $\varepsilon$, and the obtained solutions have to been matched using appropriate boundary conditions.
- Equation (L17.5) needs to be casted in a form suitable for the description of confined propagating waves.

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So, you are just left with this term and this term ok. You remove the negative sign from both sides what you are left with $\nabla^2\mathbf{E} - \frac{\varepsilon}{c^2}$. So, that is basically $\mu_1$ over $c^2$ is nothing but $\mu_0\varepsilon_0$ ok. So, this is a simplified version of the wave equation ok. So, practically this equation has to be solved separately in regions of dielectric constant $\varepsilon$ ok if there are variations in different region.

And the obtained solutions have to be matched using appropriate boundary condition. Now here that is a requirement because the epsilon is not same throughout right because we are considering about the propagation along an interface. So, in one side of the interface the permittivity is different to the other side of the interface right. So, we have to solve them separately in the two regions and then you have to ensure that you know they are matched using proper boundary conditions.

So, that is the requirement. So, this particular equation needs to be casted in a form suitable for the description of the confined propagating waves. So, let us have a look how we can do this. So, we have seen the wave equation now let us try to solve this ok. So, we can assume a time harmonic electric field which is $\boldsymbol{E}(r,t)$ you can describe this as $\boldsymbol{E}(r)e^{-i\omega t}$ ok. And if you put that into that particular wave equation it looks like $\nabla^2\mathbf{E} + k_0^2\varepsilon\mathbf{E} = 0$.

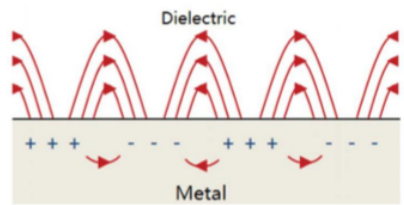
## Surface Plasmon Polaritons (SPP): The Wave Equation

- A harmonic time dependence $\boldsymbol{E}(r,t) = \boldsymbol{E}(r)e^{-i\omega t}$ of the electric field is assumed and inserted into (L17.5), which yields:

$$\nabla^2\mathbf{E} + k_0^2\varepsilon\mathbf{E} = 0 \qquad \text{(L17.6)}$$

  where $k_0 = \omega/c$ is the wave vector of the propagating wave in vacuum.

- Equation (L17.6) is known as **the *Helmholtz equation***.
- Next, the propagation geometry needs to be defined.

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So, what is $k_0$ that is $\omega/c$ ok. So, $k_0$ is the wave factor of the propagating wave in vacuum. So, this equation we have seen before this is nothing, but the Helmholtz equation right. So, now we have to see the geometry. So, the propagation geometry needs to be defined next.

So, let us assume a simplicity of a one dimensional problem ok. So, this is basically a one dimensional problem that along z something is changing, but you have along x and y there is no variation in the permittivity right. So, you can think of a ah planar wave kind of a geometry and the waves here propagate along x direction in the Cartesian coordinate system right. So, specifically if you consider this x propagation direction ok there is no variation along y. So, permittivity is only function of z.

## Surface Plasmon Polaritons (SPP): The Wave Equation

- Assume for simplicity a one-dimensional problem, *i.e.* $\varepsilon$ depends only on one spatial coordinate.
- Specifically, the waves propagate along the x-direction of a Cartesian coordinate system, and show no spatial variation in the perpendicular, in-plane y-direction (see **Figure**); therefore $\varepsilon = \varepsilon(z)$.
- Applied to electromagnetic surface problems, the plane $z = 0$ coincides with the interface sustaining the propagating waves, which can now be described as $\boldsymbol{E}(x,y,z) = \mathbf{E}(z)e^{i\beta x}$.
- The complex parameter $\beta = k_x$ is called the *propagation constant* of the traveling waves and corresponds to the component of the wave vector in the direction of propagation.

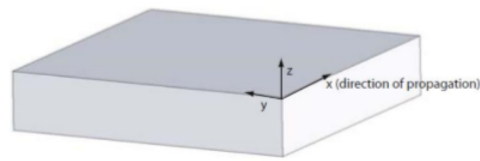


**Figure:** Definition of a planar waveguide geometry. The waves propagate along the x direction in a Cartesian coordinate system.



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So, applied to the electromagnetic surface problems the plane that is z equals 0 that is basically the x y plane ok that coincides with the interface sustaining the propagating wave because the propagating wave is actually in that particular surface ok. So, there you can describe the electric field as E (x, y, z) can be taken as E(z)e $^{i}\beta^{x}$. So, this clearly tells you that your wave is propagating along the x direction and the electric field is basically extended here in the z direction ok. So, the complex parameter which is $\beta$ ok that is nothing but $k_x$. So, if you think of a wave vector your x component of the wave vector is basically the propagation constant for this travelling waves and the cross that corresponds to the component of the wave vector as I mentioned ok.

So, if you take that and put this into the previous equation you will get this is the form of the wave equation. So, now, you are very specific that $\nabla^2$E where is it if you go back yeah this is the equation. So, $\nabla^2$E is now because E is now only in the z direction ok and propagation vector or propagation constant you can say is only along the x direction ok. So, field along z and this guy is only along $\beta$ is along x direction.

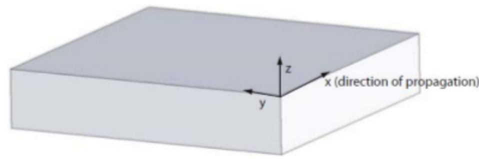
So, this is how you can write it ok. So, this becomes the wave equation for this particular case. So, you have $\frac{\partial^2 \mathbf{E}(z)}{\partial z^2} + (k_0^2\varepsilon - \beta^2)$ So, this is the propagation constant of the wave that is travelling in this interface ok times E $= 0$. Now, naturally such a equation also will exist for magnetic fields. So, you can have for magnetic fields ok just the directions would be different.

## Surface Plasmon Polaritons (SPP): The Wave Equation

- Inserting this expression into (L17.6) yields the desired form of the wave equation:

$$\frac{\partial^2 \mathbf{E}(z)}{\partial z^2} + \left(k_0^2 \varepsilon - \beta^2\right)\mathbf{E} = 0 \qquad \text{(L17.7)}$$

- Naturally, a similar equation exists for the magnetic field **H**.
- Equation (L17.7) is the starting point for the general analysis of guided electromagnetic modes in waveguides.
- In order to use the wave equation for determining the spatial field profile and dispersion of propagating waves, explicit expressions for the different field components of **E** and **H** need to be derived.

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So, you can take this equation as the starting point for the general analysis of the guided electromagnetic waves in any waveguide. Now, in order to use the wave equation for determining the spatial field profile and the other dispersion of the propagating waves explicit expressions need to be written for E and H ok that is very important. So, this can be done in a very straightforward way.

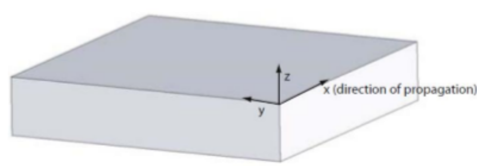
## Surface Plasmon Polaritons (SPP): The Wave Equation

- This can be achieved in a straightforward way using the curl equations (L17.1 & L17.2).
- For harmonic time dependence $\partial/\partial t = -i\omega$, we arrive at the following set of coupled equations:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\omega\mu_0 H_x \qquad \text{(L17.8)}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\omega\mu_0 H_y \qquad \text{(L17.9)}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\mu_0 H_z \qquad \text{(L17.10)}$$

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So, if you remember the curl equations that you have seen previously. Now, if you introduce the harmonic time dependence that is $\partial/\partial t$ as $-i\omega$, you can get this set of coupled equations ok. So, these are the equations that correlate your electric and magnetic field components ok. So, $H_y$ is basically connected to $E_z$ and $E_y$, $H_y$ is connected to $E_x$ and $E_z$, $H_z$ is connected to $E_y$ and $E_x$ ok. So, this is not a rocket science you can go and sit down and write down the vector equations and then put those components you will see you will get this after you apply those conditions ok. Similarly, you can also get the other set of equations that correlate electric fields to magnetic fields ok.

## Surface Plasmon Polaritons (SPP): The Wave Equation

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = -i\omega\varepsilon_0\varepsilon E_x \quad \text{(L17.11)}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_0\varepsilon E_y \quad \text{(L17.12)}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\omega\varepsilon_0\varepsilon E_z \quad \text{(L17.13)}$$

- For propagation along the x-direction $\partial/\partial x = i\beta$ and homogeneity in the y direction $\partial/\partial y = 0$, this system of equation simplifies to:

$$\frac{\partial E_y}{\partial z} = -i\omega\mu_0 H_x \quad \text{(L17.14)}$$

$$\frac{\partial E_x}{\partial z} - i\beta E_z = i\omega\mu_0 H_y \quad \text{(L17.15)}$$

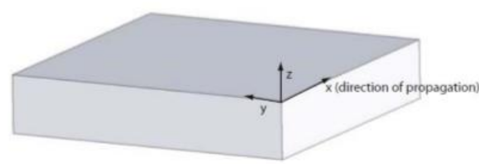


**Figure:** Definition of a planar waveguide geometry. The waves propagate along the x direction in a Cartesian coordinate system.

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So, these are interrelated equations fine. Now, we assume that the propagation happens along x direction. So, in that case $\partial/\partial x$ can be written as $i\beta$ ok and we have considered that the y direction is homogeneous. So, there is no variation along y. So, $\partial/\partial y = 0$ . So, if you put these conditions here things get even further simplified ok.

So, the first equation that you have seen ok this one ok this term becomes 0, this can be written as this particular form ok. So, you can simplify these equations and directly ah correlate $H_x$ to one electric field component and so on. So, these all these equation, the 6 equations again get further simplified because of this condition. Now, it can be easily shown that the system allows 2 sets of self consistent solution with different polarization properties of the propagating wave. So, there are basically 2 solutions which are possible.

## Surface Plasmon Polaritons (SPP): The Wave Equation

$$i\beta E_y = i\omega\mu_0 H_z \qquad \text{(L17.16)}$$

$$\frac{\partial H_y}{\partial z} = i\omega\varepsilon_0\varepsilon E_x \qquad \text{(L17.17)}$$

$$\frac{\partial H_x}{\partial z} - i\beta H_z = -i\omega\varepsilon_0\varepsilon E_y \qquad \text{(L17.18)}$$

$$i\beta H_y = -i\omega\varepsilon_0\varepsilon E_z \qquad \text{(L17.19)}$$

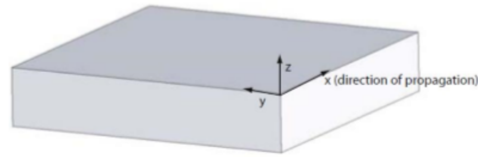


**Figure:** Definition of a planar waveguide geometry. The waves propagate along the x direction in a Cartesian coordinate system.

- It can easily be shown that this system allows two sets of self-consistent solutions with different polarization properties of the propagating waves.
- The **first set** are the transverse magnetic (TM or *p*) modes, where only the field components $E_x$ , $E_z$ and $H_y$ are nonzero, and the **second set** the transverse electric (TE or *s*) modes, with only $H_x$ , $H_z$ and $E_y$ being nonzero.



Source: S. A. Maier, Plasmonics: fundamentals and applications, 1, 245, New York: Springer, 2007.
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So, the first set are nothing but the transverse magnetic TM or you can also call p modes ok. And in this case you are only bothered about $E_x$, $E_z$ and $H_y$ ok. So, these are the components which are non-zero and in the second set you are basically talking about transverse electric or s modes. So, in this case it is like $H_x$, $H_z$ and $E_y$ which are non-zero ok. So, if you actually consider only the TM modes you can see that the system of governing equations.

So, 14 to 19 so we can think of from here to here these are the 6 equations interconnected equations. If you put these conditions that only $E_x$, $E_z$ and $H_y$ are non-zero then the 6 equations actually boil down to this one ok. That is very simple and from this you can also find out what is the equation of wave. So, this is what was not shown previously.

## Surface Plasmon Polaritons (SPP): The Wave Equation

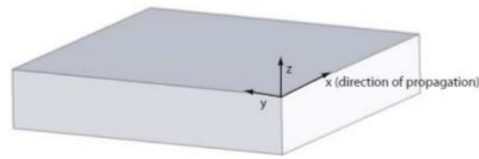
- For TM modes, the system of governing equations (L17.14 − 19) reduces to:

$$E_x = -i\frac{1}{\omega\varepsilon_0\varepsilon}\frac{\partial H_y}{\partial z} \qquad \text{(L17.20)}$$

$$E_z = -\frac{\beta}{\omega\varepsilon_0\varepsilon}H_y \qquad \text{(L17.21)}$$

- and the wave equation for TM modes is:

$$\frac{\partial^2 H_y}{\partial z^2} + \left(k_0^2\varepsilon - \beta^2\right) H_y = 0 \qquad \text{(L17.22)}$$

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So, this is in terms of magnetic field ok. So, electric field was written in terms of $E_z$ and magnetic field is written in terms of $H_y$ ok. So, this is how the wave equation for the TM modes look like $\frac{\partial^2 H_y}{\partial z^2} + (k_0^2\varepsilon - \beta^2)H_y = 0$ ok. Now, if you think of the TE modes they are having analogous set. So, again remember in TE these are the components which are non-zero ok $H_x$, $H_z$ and $E_y$ are non-zero.

## Surface Plasmon Polaritons (SPP): The Wave Equation

- For TE modes the analogous set is:

$$H_x = i\frac{1}{\omega\mu_0}\frac{\partial E_y}{\partial z} \qquad \text{(L17.23)}$$

$$H_z = \frac{\beta}{\omega\mu_0}E_y \qquad \text{(L17.24)}$$

with the TE wave equation

$$\frac{\partial^2 E_y}{\partial z^2} + \left(k_0^2\varepsilon - \beta^2\right) E_y = 0 \qquad \text{(L17.25)}$$

- With these equations, surface plasmon polaritons can be investigated mathematically.

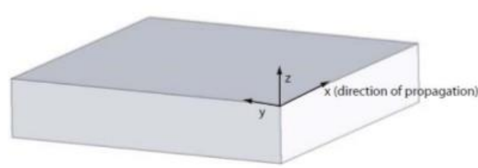


**Figure:** Definition of a planar waveguide geometry. The waves propagate along the x direction in a Cartesian coordinate system.

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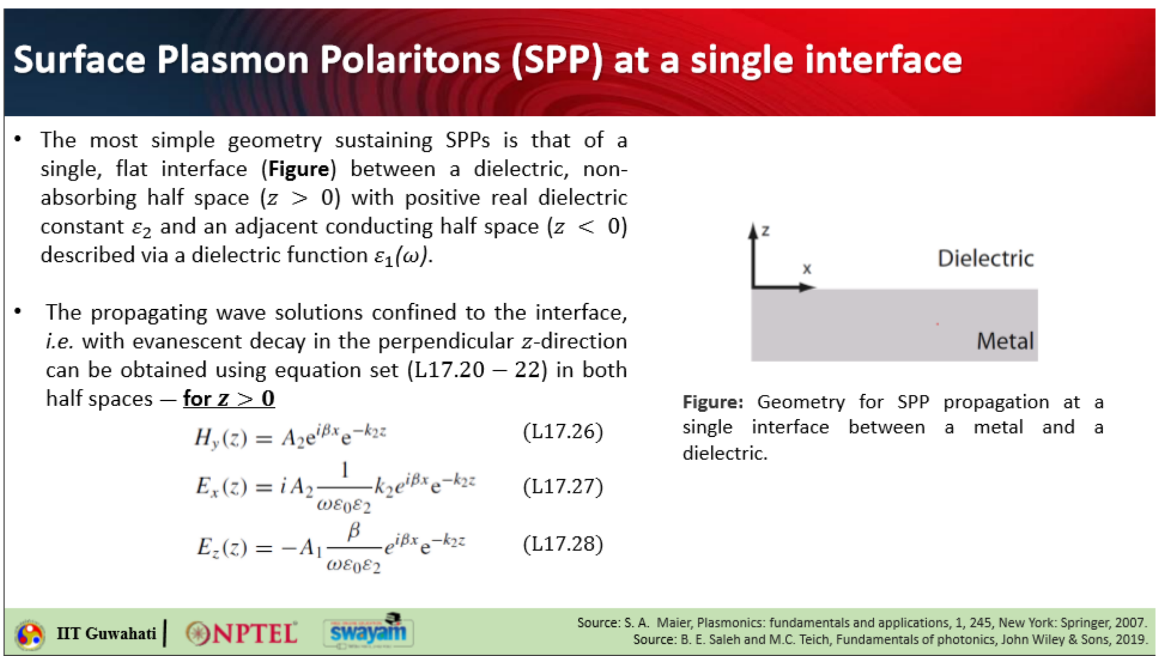
So, take those 6 equations and put only these 3 things as non-zero you will be left with these 2 equations and that will also give you the TE mode wave equation to be this one. So, this is what we have seen earlier right. Now, with these equations you are able to investigate the surface plus 1 polyetron modes mathematically because these are the 2 modes which are possible along this interface fine that can give you a solution. Now, let us take SPPs at a single interface. So, the most simple geometry that can support surface plus 1 polyetron mode is nothing, but a flat interface between a dielectric which is non-absorbing ok on one half that is we can take we can split the space with a boundary at z equals 0.

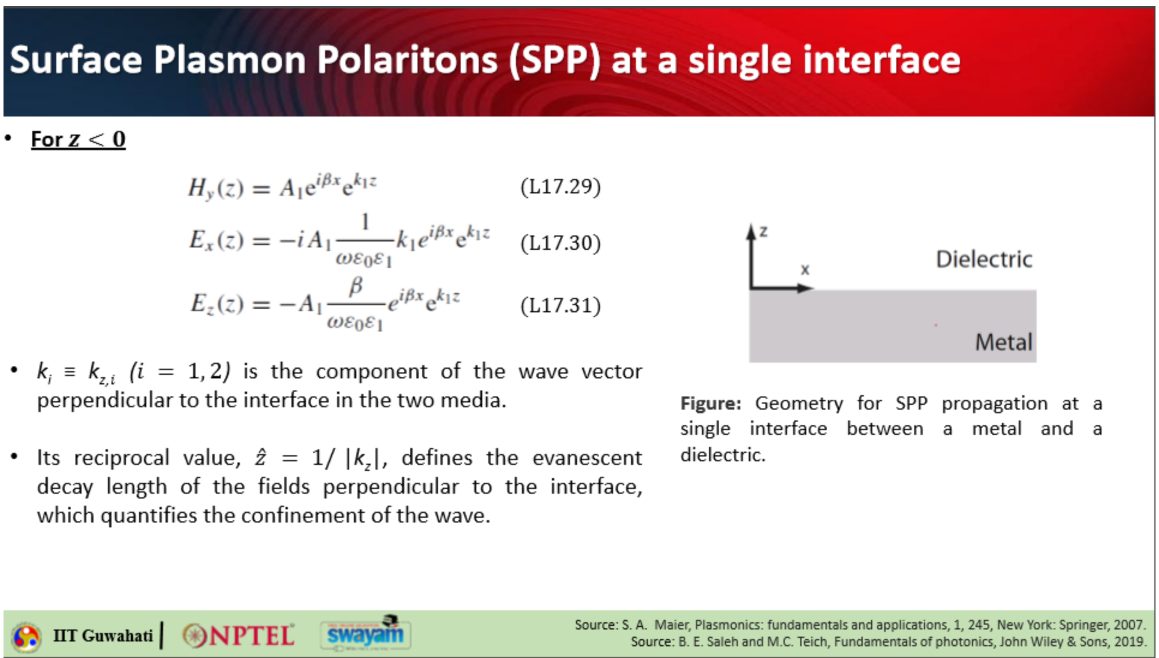
So, z greater than 0 you can take that half space to be dielectric non-conducting sorry non-absorbing as well and the bottom half you can think of you know z that is z less than 0 is nothing but a conducting half space fine. So, this one you can take a real dielectric constant that is $\varepsilon_2$ whereas metal usually have a complex dielectric constant and that has got also dispersion. So, metal does not have a flat dielectric profile. So, you can actually take it as a function of $\omega$ .




So, you can take this as $\varepsilon_1(\omega)$ ok. Now the providing wave solutions they confined to this particular interface ok that is we have assumed that the evanescent fields are basically decay along the z direction in both case. So, if there is a case let us first investigate these two halves separately. First let us look into z equals 0 that is into the dielectric region. So, these are the equations we have understood that for this particular case $H_x$,$H_y$, $E_x$ and $E_z$

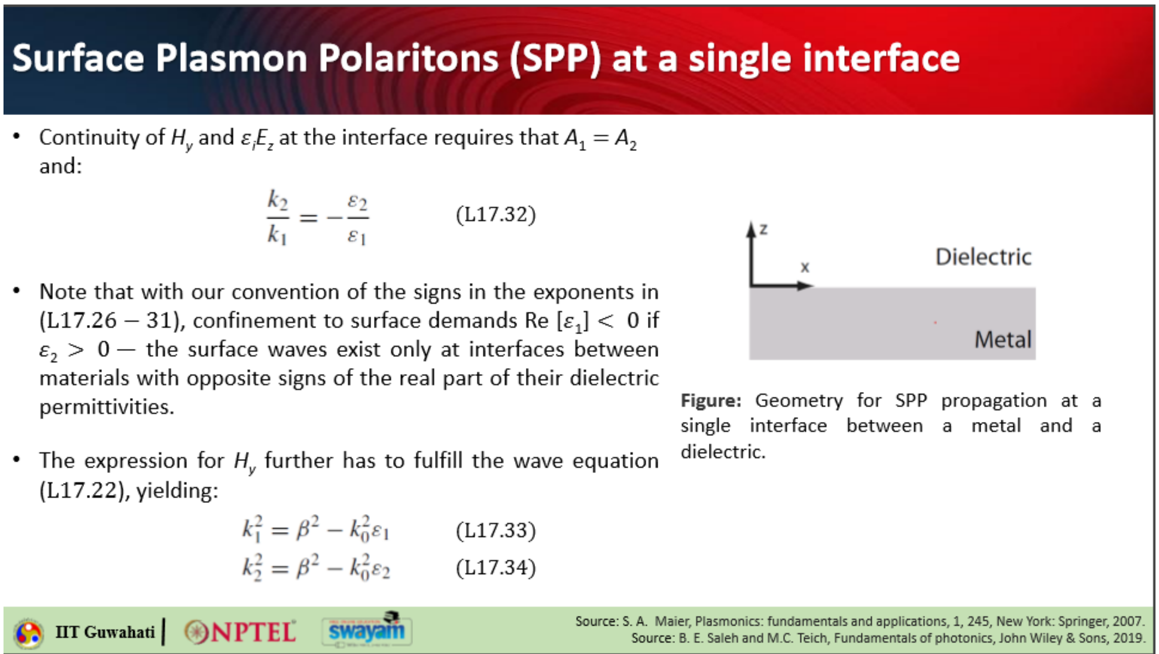
are non-zero and we can write the equations to be like this ok.

So, that is for z equals 0. So, if you consider z equals, z is less than 0 in that case all this z's will be actually replaced by minus z. So, here if you put z equals minus z ok. So, you get the equation in this particular form ok. So, I would recommend you guys to write down the equations on a piece of paper to look at them at the same time. So, that will possibly help you to understand this better ok. So, these are basically which are these equations ok what are these values. So, if you go back you have your answer here ok. So, $H_x$ ,$H_z$ and $E_y$ ok. So, that is for TE modes and if you look into the TM modes yeah look at the TM modes $E_x$, Ez and $H_y$. So, these are the three components that you require and these are the three components that has been described here.



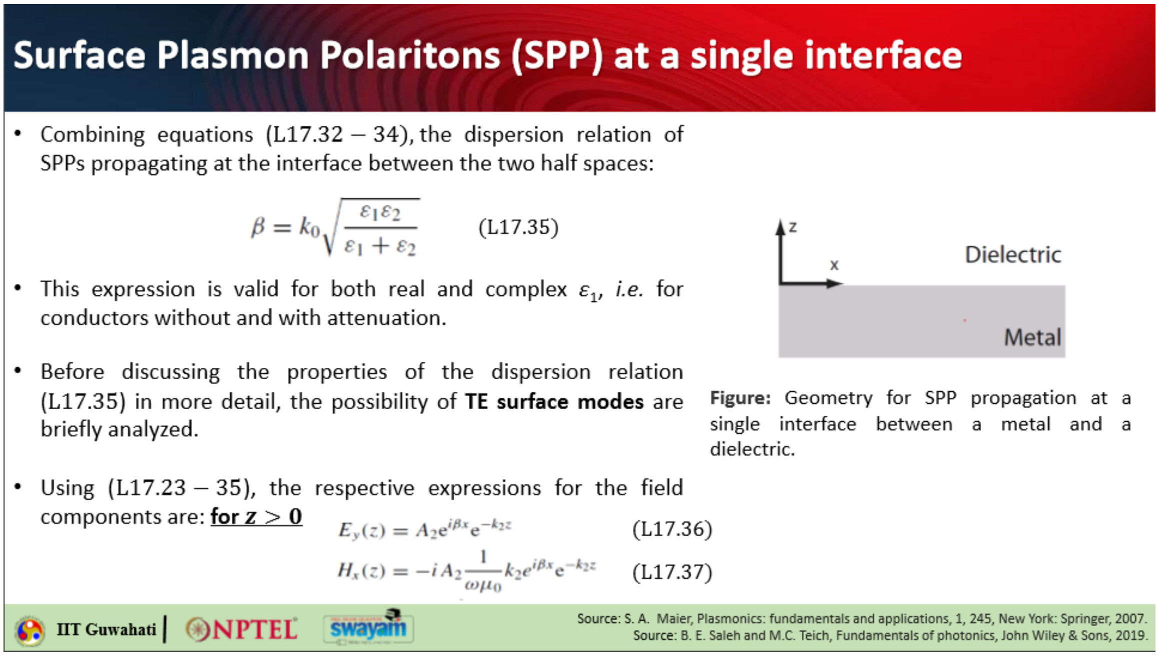

So, we are basically discussing about the TM modes in this case. So, with that what we figured out that $k_i$ can be written as $k_{z,i}$. So, i is nothing but 1 and 2 that is that these two particular region. So, this is 1 this is 2 ok and you can also find out that it has got a reciprocal vector that can be described as $\hat{z}$ which is $1/|k_z|$ ok. So, what is again what is $k_z$ that is the component of the wave vector perpendicular to this interface between the two media and you can also think of a reciprocal to that particular vector and that defines the evanescent decay length to the field which is perpendicular to the interface and this also quantifies the amount of confinement along each side ok. So, if you actually discuss about the continuity of magnetic field and electric field at the interface ok you will see that this two amplitudes a 1 and a 2 they should be same.

So, what are these two amplitudes? So, these are the amplitudes describe here ok and they should be same ok. I am not going into the exact details of each of this. So, that poses the requirement that $\frac{k_2}{k_1}$ should be equal to $-\frac{\varepsilon_2}{\varepsilon_1}$ that means the two permittivities should have opposing values ok. So, with that convention you can write that the confinement to the surface what you have seen here that demands that the real part of permittivity of $\varepsilon_1$ should be negative if you want the other one to be positive ok.




So, that is possible when you choose one is metal and another is dielectric ok. So, that is what we understood that a metal dielectric interface will actually be able to support surface plus 1 waves. So, in that case the expression for $H_y$ gets further simplified and that has to further fulfill the wave equation and that gives you these two values of $k_z$ ok. So, $k_1$ square that is the component of the wave vector along this interface this one in the metal and this will be $k_2$ that is the component of the wave vector along z direction into the dielectric.

So, you can actually obtain these two values. Now if you combine these two these equations this ratio and these values from here you are able to obtain the dispersion relation of the surface plus 1 polaritons which are propagating along the interface. So, you can obtain what is $\beta$ that can be written as $k_0\sqrt{\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}}$. So, that is a very very important relationship that defines the relation between the propagation constant and the frequency. Now you must be wondering where is the frequency term here where is the propagation constant term here. So, $\beta$ if you remember this is basically the propagation constant along the x direction right.



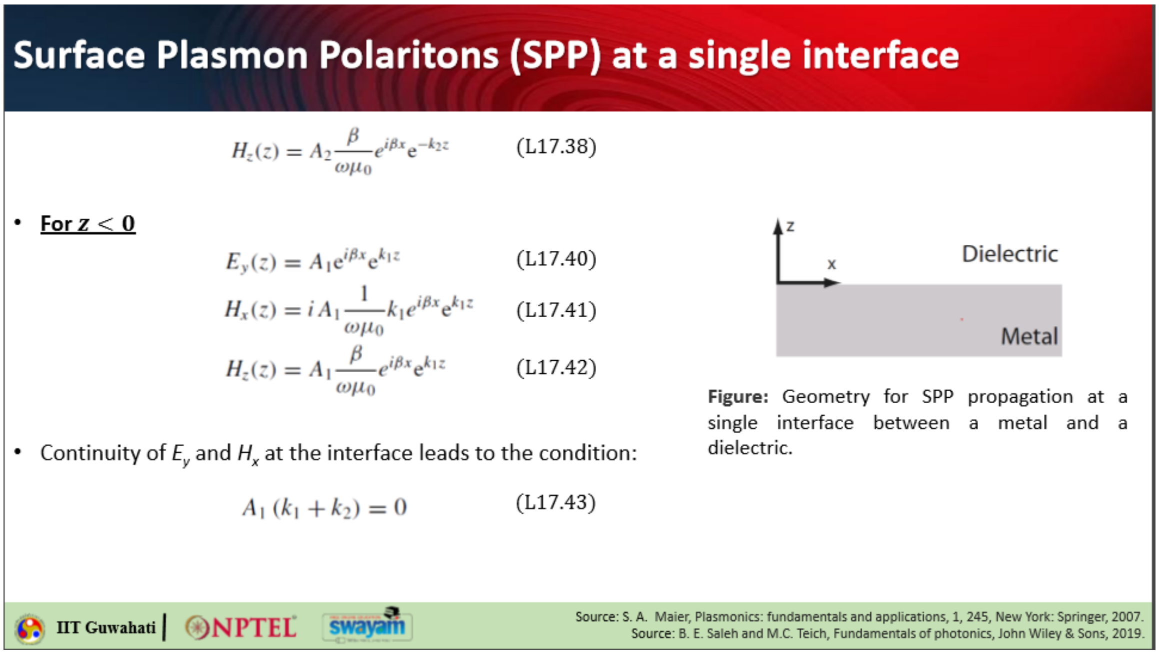

So, this is basically where the k part is coming from ok and then $\varepsilon_1$ as I mentioned $\varepsilon_1$ is implicitly $\varepsilon_1$ function of $\omega$. So, this is where this clips in. So, you actually get $\omega$-k relationship that is this dispersion relation for SPPs propagating along this particular interface which is in x direction. Now this expression is valid for both real and complex $\omega_1$ that is for conductors with or without attenuation. So, you may think of a metal which is lossless in that case also it will work ok, but that the requirement is that you need to have this one satisfied ok.

So, $\frac{k_2}{k_1}$ should be equal to $-\frac{\varepsilon_2}{\varepsilon_1}$ this has to be satisfied ok. Now before we go into discussing the dispersion relation and what exactly it conveys let us also have a look into the possibility of TE surface modes ok. So, this was what we have seen is for the TM surface modes. Now similar kind of approach so we will start with the TE set of equations that is basically equation number 23 to 35 ok.

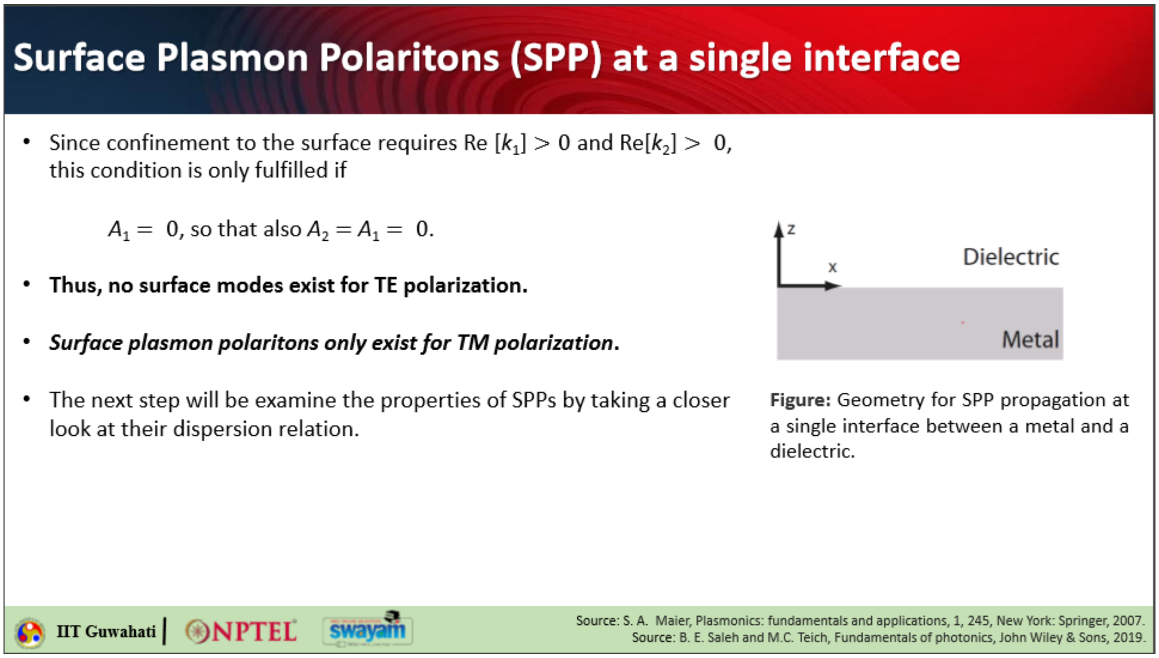
So, in each case we have to see that what happens for the TE modes. So, we will be only discussing with those components which are nonzero for TE case and that is basically $E_y$, Hx and $H_z$ right. And we write this for z equals 0 also we write it for z sorry z greater than 0 that is in the dielectric we also write the same sort of equations for the space z less than 0 ok that is what happens in the metal ok. So, direct correlation because here this figure this one and this one are on the same page.

So, I will be able to compare it. So, quickly you see how they are different. So, $H_z$ function of z equals $A_2$. So, $A_2$ is the amplitude in the dielectric in metal you have the amplitude of $A_1$ ok. So, this is a parameter that is different then you have $\beta\omega\mu_0$ that remains same propagation constant is also along this one. So, that remains same $e^{-\beta x}$, $e^{-\beta x}$ that is fine. What is different the decay the decay constant in dielectric is $e^{-k_2 z}$ that is fine ok.

Now, in this case that is greater than 0 when z is already negative you can replace this minus z by z because z itself is negative. So, you simply have z and the constant decay constant in metal is $k_1$ ok in dielectric it was $k_2$. So, this is how you can actually see the equations are being different ok that is all. So, all these equations here are $A_1$ $A_1$ $A_1$ for dielectric you will have $A_2$ $A_2$ $A_2$ ok very simple similar to the previous case.



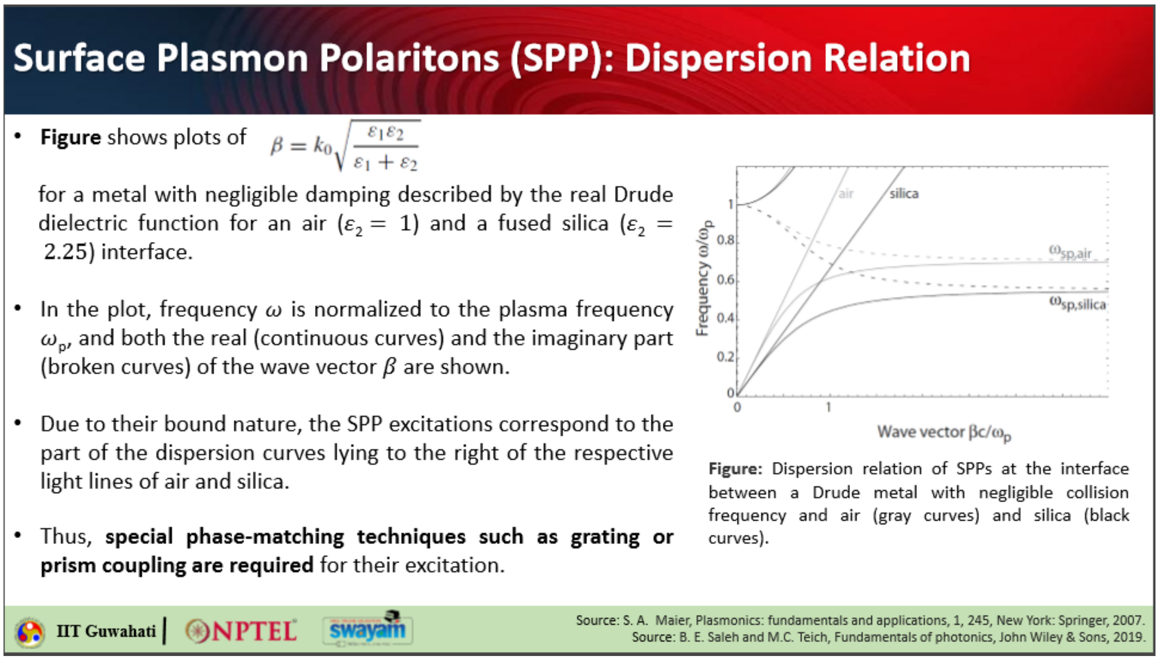

Now, if you look into the continuity of $E_y$ and $H_x$ ok across the interface. So, these are the 2 components you have to see they are continuous then this is the condition that comes out to be. So, you will get $A_1(k_1 + k_2)$ equals 0. So, if you take that particular condition we have to see what can give us that particular this particular condition ok what actually gives us that. So, since the confinement to the surface now you have to make sure that there is confinement on the surface. So, you will require that the real part of this $k_1$ is positive and also real part of $k_2$ is also positive then only they will be confined along the surface ok.




So, this condition can only be fulfilled if $A_1$ is 0 and if $A_1 = 0$, $A_2$ also becomes 0 that means, there will be no amplitude possible when you are considering the TE polarization. It means no surface modes can exist for TE polarization. So, in simple word we can say that SPPs are only TM polarized waves clear ok. So, once we understood that SPPs are purely TM polarized waves and we have also seen a glance of their dispersion relation.

Let us look into the dispersion relation and see what it actually conveys. Now here is a plot ok dispersion relation of the SPPs. So, at the interface between a Drude metal with negligible collision frequency. So, you can think of as almost lossless ok, but it is having a negative permittivity, but it is lossless it is a real one ok. So, Drude metal permittivity you have seen previously in the previous lectures. If you take gamma that is a collision frequency to be 0 your Drude metal permittivity will be simply $\varepsilon_1$ function of $\omega$ will be nothing but 1 minus $\omega_p$ square by $\omega$ square as simple as that ok. So, if you take that and you consider the two cases in one case it is this Drude metal and air interface, in other case it is this Drude metal and silica interface. So, the air interface you are drawing it using gray curve. So, this light color lines are basically for that case and if you see the black curves they are basically for the metal silica interface. Now this we have seen this is the dispersion relation ok. So, now what are the parameter values for $\varepsilon_2$? If it is air, $\varepsilon_2$ is 1 and fused silica if you think of fused silica $\varepsilon_2$ is nothing but 2.25 fine.

Now in this particular plot you see one thing that the frequency is basically mentioned as a normalized frequency and normalized to what? Normalized to the plasma frequency of the metal. So, this kind of graph you can use for any metal if you actually know the plasma frequency you can actually find out what is the frequency actual frequency we are talking about. So, this is a normalized one. Again, the wave vector is also a normalized one you can see it is basically $\beta_c$ over $\omega_p$ ok. Now, here what is happening that the solid curves that you see they are basically the continuous curves they are the real part of the wave vector beta and if you take the dotted part or the dashed part or the broken curves they basically correspond to the imaginary part of the $\beta$ .




Now due to the bound nature the SPP excitation correspond to the part of the dispersion curve that lie to the right of the respective light lines. So, these are basically the light lines. So, here light lines means here the dispersion relation is simply $\omega = ck$ ok. And this one what is the dispersion relation for this one? $\omega$ = c by $nk$ ok.

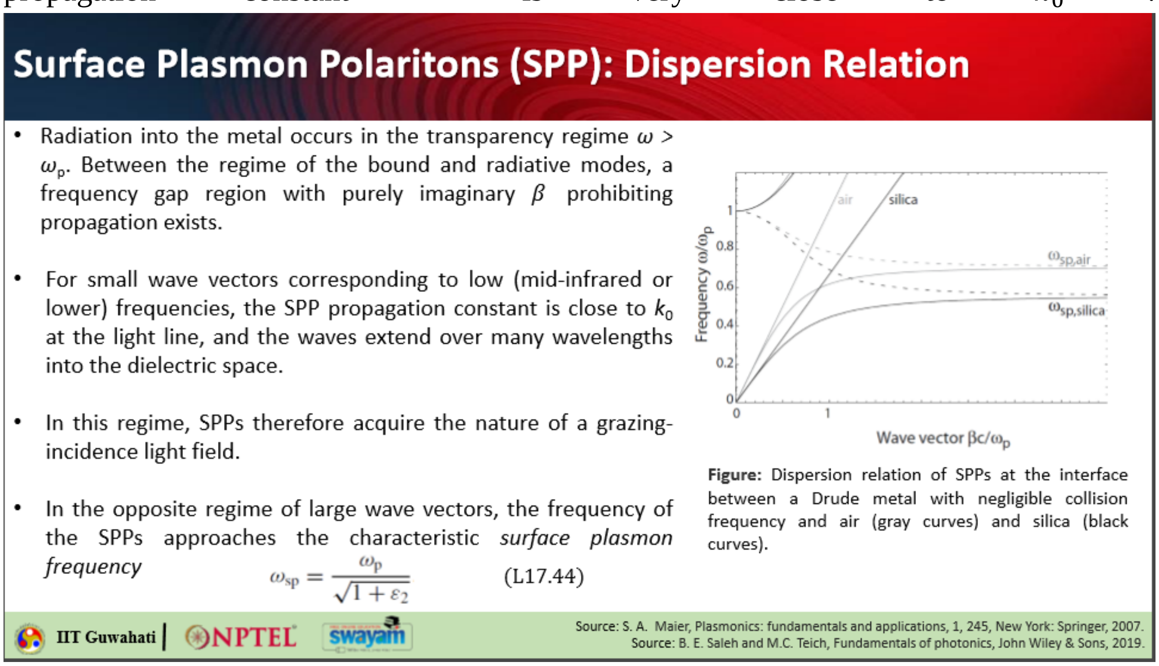
So, these are called the light lines ok. So, you will see that in both case the SPP relation are basically lying towards the right of this Now if you look into the case that they do not exactly match. So, for air you see there is no case where there is a direct matching of momentum. So, you can take this as a momentum vector or k vector. So, for a given energy you will see that there is no match between the momentum of the SPP is generated in the air interface with the photons that are in the air ok. So, you have to do some kind of phase matching or momentum matching using some techniques that we will describe maybe in the next lecture ok.

And we will see how we can able to excite surface plus bonds on air interface or silica interface. So, for silica this black line you have to look into this particular line as your ah light line. For SP in air interface this curve you have to look for this particular grey line ok fine. Now there are certain regime in this particular graph which also needs special attention.

First of all radiation into the metal occurs in the transparency region. So, what is the transparency region when the frequency is more than the plasma frequency that is when $\omega / \omega_p$ is greater than 1. So, this particular is region. So, if you draw an imaginary line horizontal line at this $\omega / \omega_p$ equals 1. So, anything above this is basically the transparency region ok.

So, in that case the radiation into the metal will occur in the transparency region. Now between this region and the bound mode. So, these are basically two bound modes the SPP modes ok. So, there is basically a region like this you see here no solid lines are possible. And what is only possible here is nothing but purely imaginary $\beta$ .

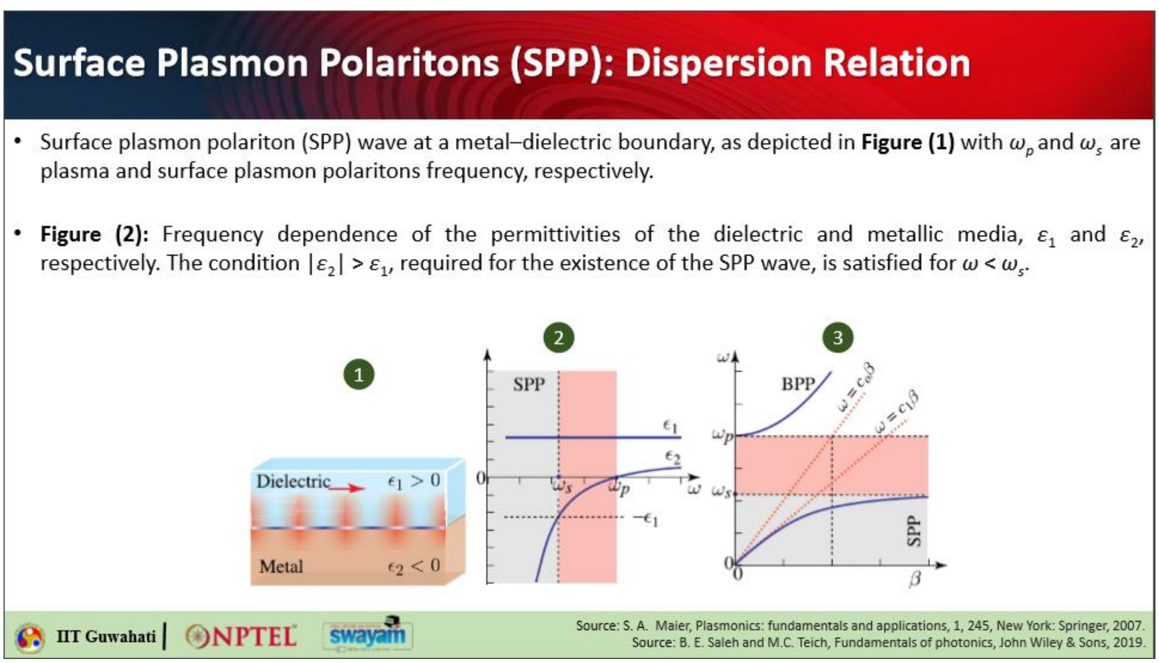
So, imaginary $\beta$ does not allow you propagation right. So, there is basically a frequency gap region that this particular region you are not able to excite any mode. Now if you look into the region of small wave vectors here. So, when the wave vector is small ok that is you can consider about mid infrared or lower frequencies ok. You can see that the SPP propagation constant is very close to $k_0$ .




So, $k_0$ will be the propagation constant in the light lines. So, you see here they are very close the curve and the straight lines they are almost overlapping ok and the waves they extend over many wavelengths into the dielectric space ok. But in the regime ok and in this particular regime SPPs therefore, acquire the nature of grazing incidence light field ok. So, here it is more or less you know at very small value of wave vector SPPs have very similar k naught as the light lines. And if you look into the opposite regime here where the wave vector is very large in that case ok the frequencies of the surface plasmon they actually reaches a saturation kind of thing and that is basically the frequency of surface plasmon resonance ok. So, that is a characteristic surface plasmon frequency at a particular interface and your SPPs will actually try to go and reach that particular frequency ok.

Now what is the good thing about this is that for a given frequency you are able to get SPPs with very large wave vector ok or wave number. So, that actually allows you to have very short wavelength and this is where the amazing thing about SPPs are. For similar frequency you are able to have wavelengths which are very very tiny. So, you can actually confine electromagnetic radiation into those small region or you can manipulate electromagnetic radiation in that very small scale that is not possible by photons.

So, that is why so these are light lines ok. So, here $\omega$ equals $ck$ that is it. So, there is a relationship between the frequency $\omega$ and the lambda ok it is a linear relationship. But when you come here they are having a non-linear relationship and that allows you to basically do this wonder that for a given frequency you are able to bring your wave vector as large as possible or you can say wave number as large as possible that allows you to have wavelength very very small or as small as possible. It cannot be infinitely small because there is a saturation here which corresponds to the surface plus 1 frequency and this is the wonderful thing about SPPs ok. So, you can actually see that what is the characteristic frequency $\omega_{\mathrm{sp}}$ it is the $\frac{\omega_{\mathrm{p}}}{\sqrt{1+\varepsilon_2}}$ . In this case epsilon d is the dielectric permittivity, here $\varepsilon_2$ is that permittivity right.

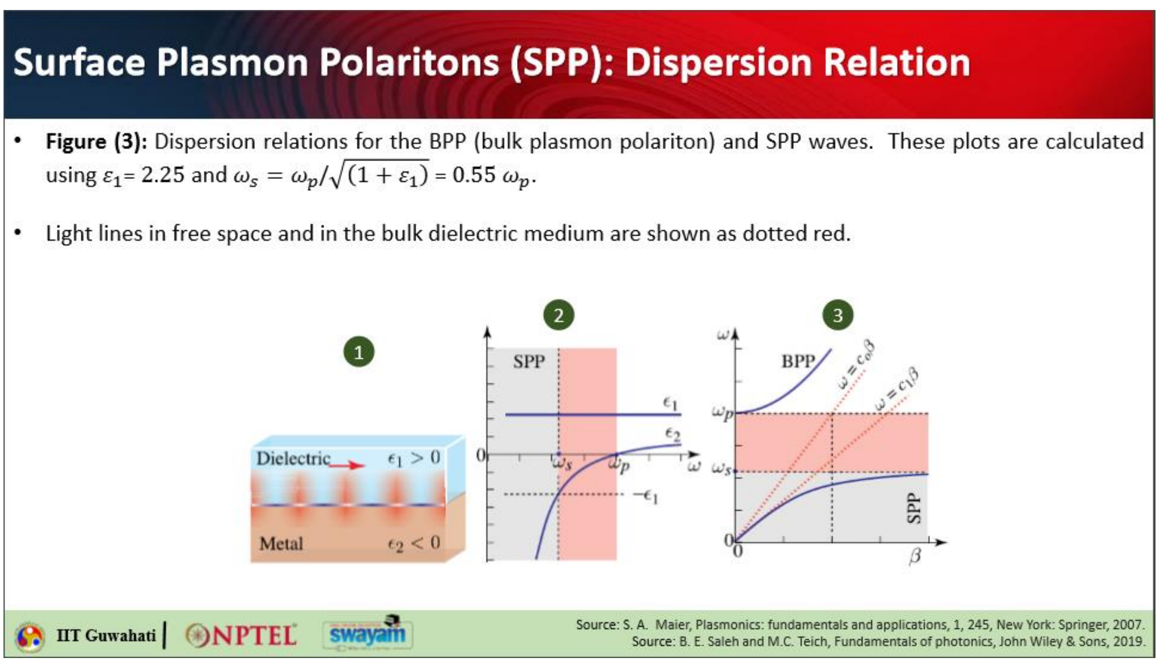


So, these are the cases let us quickly look into this. So, surface plus 1 polyetron wave at metal dielectric boundary this is what we have seen that this actually propagates along the boundary high low high low high low and so on. But the field extends more into the dielectric region less into the metal this is how it propagating nicely. Now if you look into figure 2. figure 2 actually plots the good things about ok.

So, here there is a reversal in the denomination. So, the parameter for dielectric is $\varepsilon_1$ here. So, let us consider $\varepsilon_1$ ok. So, it is a positive value. So, you have drawn it here $\varepsilon_2$ is negative because that is the metal.

So, let us plot this one. So, if you remember the Drude model. So, $\varepsilon_2$ in this case will be 1 minus omega p square over omega omega square. So, at $\omega$ equals $\omega_p$ the value is 0 ok. And at $\omega$ equals $\omega_s$, you will see that this value is same as minus $\varepsilon_1$ and this is the resonance ok. So, the frequency dependence of the permittivities of the dielectric and the metallic media are shown in this case. And the condition that modulus of $\varepsilon_2$ should be greater than $\varepsilon_1$ that is required and that is satisfied for $\omega$ less than $\omega_s$ .

So, if you have frequency below this ok this is the case this is the region where your SPPs will exist. Figure 3 is basically the dispersion relation we have just seen. So, this one as I told you this allows transparency. So, this is basically the bulk surface plasmon or you can say bulk plasmon polariton BPPs ok. And these are the these two dotted ones are the light line.



Why the slope is different $\omega$ equals c by n times k or beta ok. So, that is why the slope for silica is different than air the air one will have the highest slope. And this is how it actually gets saturated towards omega s that is this particular characteristic plasmon frequency. So, in this case $\omega_s$ is $\omega_p/\sqrt{(1+\varepsilon_1)}$ . here because permittivity($\varepsilon_1$) is 2.25. So, you can do this calculation it is 0.55 times $\omega_p$ . So, you can mark this point. So, this is the final value towards which this this curve will approach ok. I believe this particular concept is clear. So, with that we will stop here today and in the next lecture we will look into the mechanism of exciting surface plasmon polariton atoms and some of their applications. So, thank you if you have got any queries you can drop an email to me at this particular email address mentioning MOOC on the subject line. Thank you.