Course Name- Nanophotonics, Plasmonics and Metamaterials

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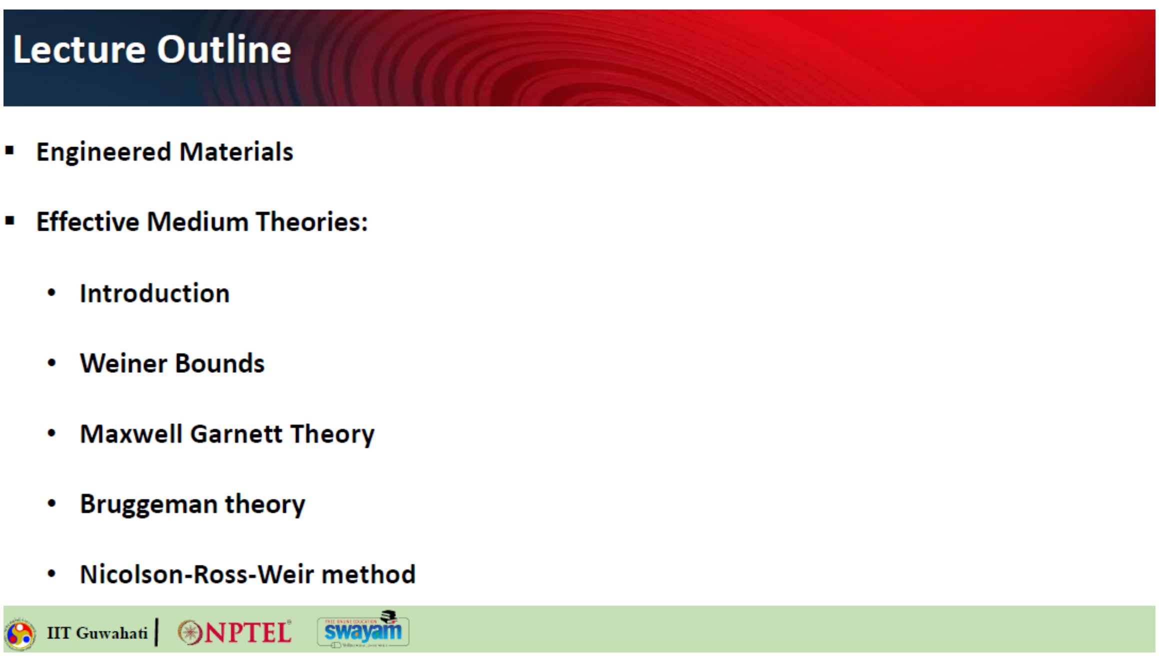
Week-08

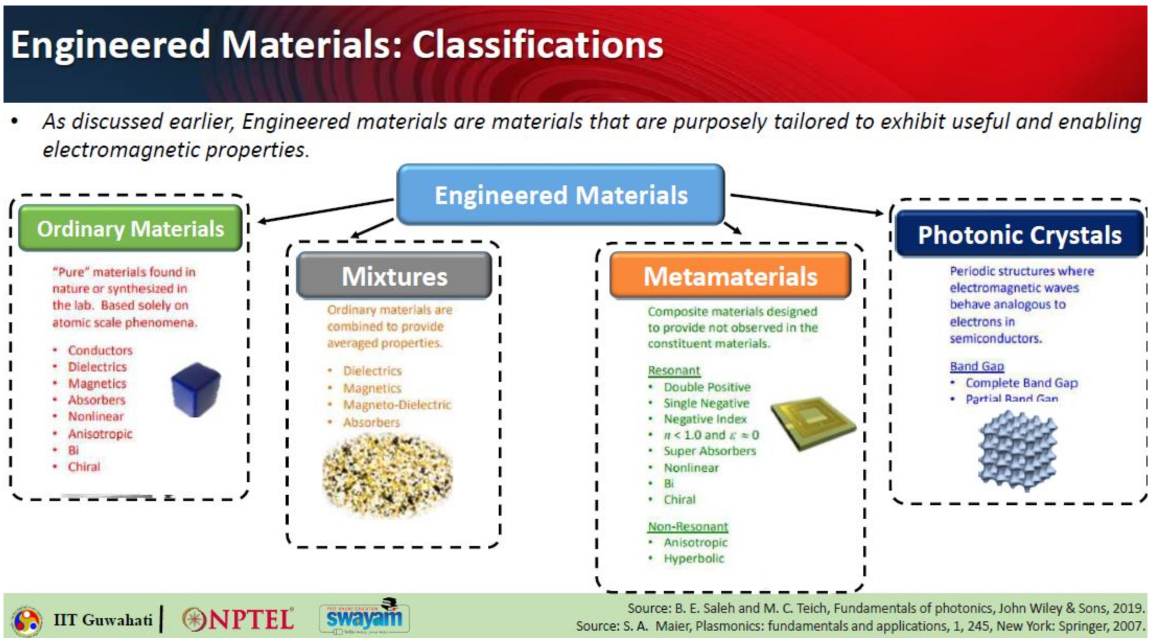
Lecture -23

Hello students, welcome to lecture 23 of the online course on Nanophotonics, Plasmonics and Metamaterials. Today's lecture will be on effective medium theories. If you remember the discussion on electro-tunable optical devices where we discussed nanoparticles in aqueous medium or any other kind of mixtures kind of cases where nanoparticles or any other dielectric nanoparticles which are scattered in a homogeneous medium. In those cases there is a requirement of estimating the effective permittivity of those medium. So this is where this kind of theories will be very very much applicable. So today we will quickly look into certain things like different classifications of engineered materials and here we will particularly focus on the mixtures and how do we characterize those mixtures in terms of their effective permittivity and we will look into these effective medium theories.



We will introduce these concept then we talk about Weiner bounds. We will discuss about Maxwell-Gernett theory, Bruggeman's theory and also Nicolson-Ross-Weir or NRW method. So I believe you know this particular slide now. So these are the different classifications of the engineered materials.

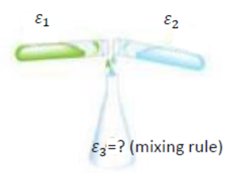
So in this lecture we will be mainly focusing on mixtures. So mixtures are basically they are made of ordinary materials which are combined together to get some averaged property. So that is very important here. So we are looking for some averaged property in this case. So what are the mixing rules for these mixtures or engineered materials? Now if you wish to mix multiple materials together to get some overall material property there has to be some kind of guidance through which we should achieve that like some kind of formula that relates the effective permittivity to the permittivity of the constituent materials and that will be seen here.



So, the effective dielectric constant of the mixture can be taken as epsilon effective and that depends on couple of things. So something like the shape of the materials then the size of the particles, electromagnetic properties of the particles, statistics of the particle distribution and the volume fraction of the constituent materials. So how much volume one particular material is covering that also plays a very important role. So effective medium theories as I mentioned why they are needed. So effective medium theories can provide a macroscopic model of inhomogeneous media based on analytical, numerical and sometimes experimental techniques.

So what are Mixing Rules for Engineered Materials?

- We may wish to mix multiple materials together to realize some overall material property.
- The effective dielectric constant of the mixture ε_{eff} depends on many things:
  - Shape of the particles
  - Size of the particles
  - Electromagnetic properties of the particles
  - Statistics on the particle distribution
  - Volume fill fraction of the constituent materials



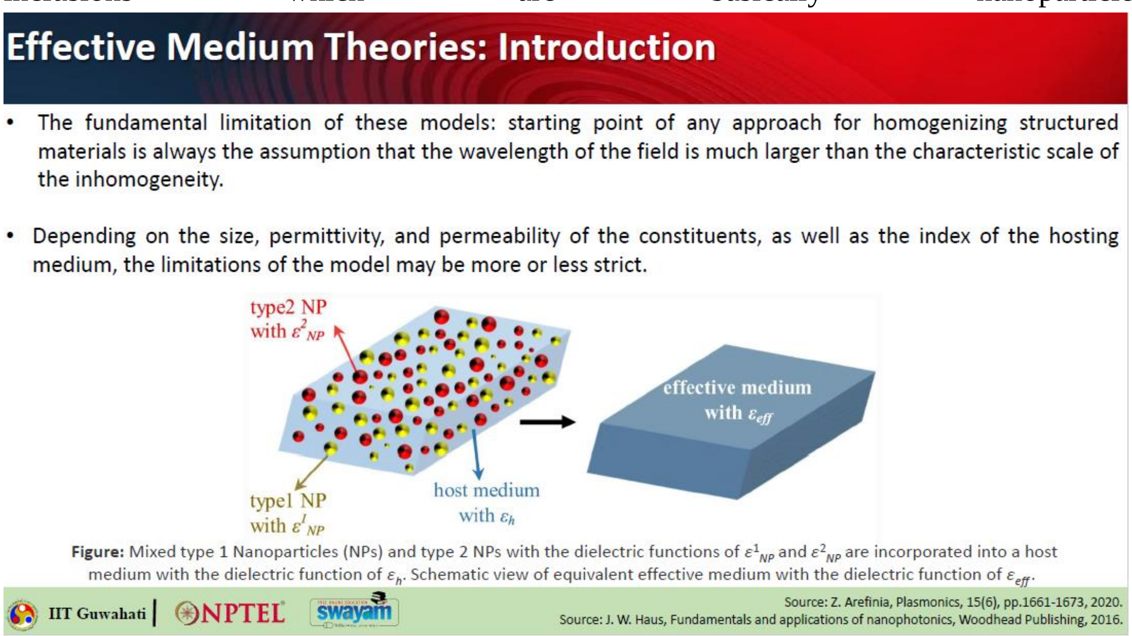
So these are basically a description of composite materials in terms of effective medium approximations okay. And this allows this is basically a very important tool as I told you is a valuable and versatile tool that allows you to investigate, predict and design the electromagnetic response of natural and structured materials. Now effective medium models they equip the macroscopic Maxwell's equation with very simple constitutive relations. So that is where you take care of the light matter interaction and in that case you do not need to go into the complexity of looking into all the minute details of light matter interaction at the constituent level rather you can actually deal with the macroscopic Maxwell's equation on a averaged manner for a particular mixture. So, when approaching an electromagnetic problem with effective medium theory what is very important is to know the limits in which this theory is valid.

Effective Medium Theories: Introduction

- Effective medium theories provide **macroscopic models** of inhomogeneous media based on analytical, numerical, and sometimes experimental techniques.
- A description of composite materials in terms of effective medium approximations is **a valuable and versatile** tool to **investigate, predict, and design** the electromagnetic response of natural and structured materials.
- Effective medium models **equip the macroscopic Maxwell's equations** with very simple constitutive relations, eliminating **the complexity of simulating light-matter interactions** at the constituents' level.
- When approaching an electromagnetic problem with an effective medium theory: **defining its limits of validity** is of extreme importance .
- Pushing any effective medium theory beyond these limits may lead to reasonable but only partially correct results or to wrong predictions.
- Effective medium models usually **depend on the electric and magnetic properties of the constituent materials**, **the volume fraction** of each constituent, and in some case **the geometry** of the structure at the constituent level.

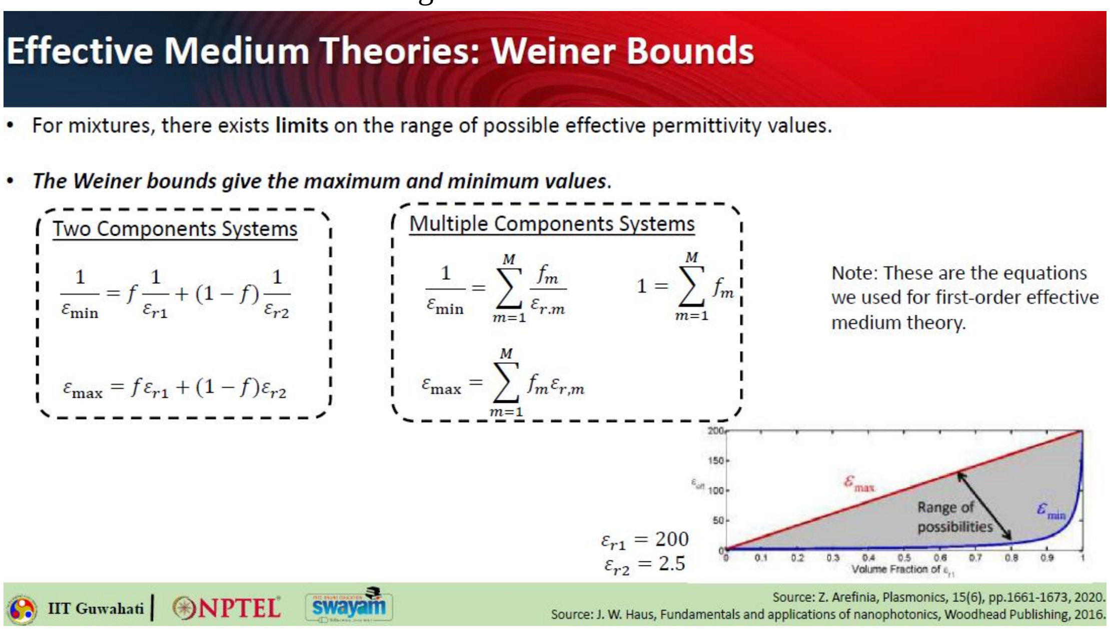
So that is basically coming from that the bounds are called as Weiner bounds which we will see in the next slide. So if you push the effective medium theory beyond these limits obviously it will fail. So you will not get correct predictions rather you may get wrong predictions okay. And effective medium theory usually depend on the electric and magnetic properties of the constituent materials, the volume fraction of the each constituent and in some case the geometry of the structure at the constituent level also gets into this particular theory. We look into the different theories and different cases where each of these theories can be used.

So the fundamental limitation of this model always you have to remember that every theory is valid for a certain range but then it also fails beyond certain limit and also there are some limitation or approximation or you can say simplification that has been considered while deriving these theories. So first thing is that when you are looking for any approach for homogenizing the structured materials it is always the underlying assumption that the wavelength of the light is much larger than the characteristic scale of the inhomogeneity. So in other words you can say all those inhomogeneity is basically much much smaller than the wavelength of light or they are simply sub wavelength in nature. Now depending on the size permittivity and permeability of the constituents as well as the index of the hosting medium or the background medium the limitations of the model can be more strict or less strict okay. So here is one example so here you see you have a host medium with permittivity epsilon H and then there is it has got some inclusions which are basically nanoparticles.



Now these are two different types of nanoparticles being mixed here. So one is type 1 nanoparticles which is given as epsilon 1 NP is the permittivity of that nanoparticle and the other type is this one that has got a permittivity of epsilon 2 NP okay and then these are all homogeneously distributed or you can say they are randomly distributed does not matter the distribution does not matter here okay. They are into this host medium and effective medium theory will be able to give you epsilon effective of this composite system which has got a background medium and two different type of constituent materials but you will be able to get a effective permittivity using the effective medium theory okay. So we will actually look into this aspect in today's lecture. So as I mentioned for mixtures there exist limits on the range of possible effective permittivity values.

So when you mix two or three different materials you cannot get very abrupt or out of the world permittivity values no that is not possible. So you can actually get some effective values which are limited by some upper bounds and lower bounds. So those bounds are known as Weiner bounds okay. So there are some formula that helps you to find out what is that minimum permittivity and maximum permittivity. So, if you consider a two component system it means you have one material and the other material so one material is filling a fraction of f in the in the volume.

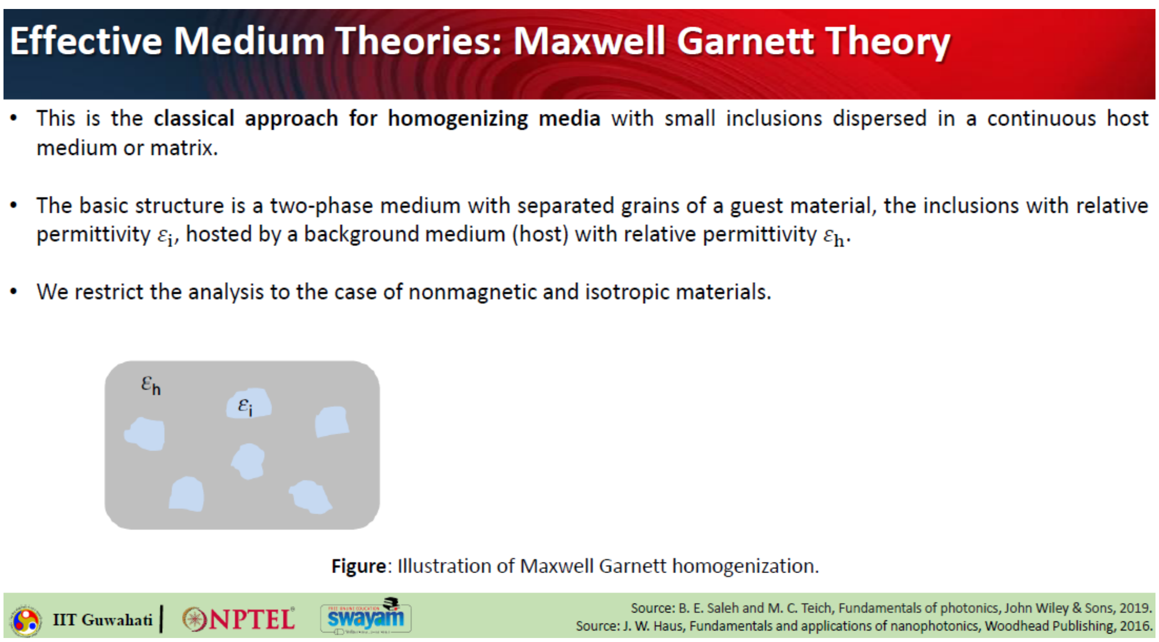


So the fraction for the other material is obviously 1 minus f and this is how you find out this formula tells you how do you find out the minimum limit. Similarly you can also find out what could be the maximum permittivity for this kind of a system. So if you think of a multi component system there also you can find out what is the minimum permittivity. So here instead of F you will have Fm so this is the fraction of the volume occupied by that material of m index. So now all the fractions if there is a two three or four different materials so it will be like f1 f2 f3 and f4 should add up to 1.

In this case it is only two variables so here f1 plus f2 equals 1 okay. So you will get this formula basically. So if you take one is f the other is basically 1 minus f and that you have seen here. So this is the formula that tells you about the minimum permittivity and maximum permittivity for the case of a multi component system. And now when you plot this if you take a two component system and you take ε_{r1} equals 200 and ε_{r2} equals 2.5.

So if you take this so it is basically like 2.5 is basically the background medium and then you have dielectric nanoparticle very high dielectric nanoparticle high permittivity dielectric nanoparticle inclusions okay. In that case you can see this is how you will your epsilon max will be and this is how the minimum will be okay. So the minimum value can go to you know the it is here. So you can actually when you when you keep on increasing the volume fraction okay that is when you are making f is close to 1 you can go up to 200.

And when you make f equals 0 that is that dielectric inclusions in the host permittivity is negligible in that case the effective permittivity is nothing but same as the host permittivity which is epsilon r2 that is 2.5 clear. So with that these are the bounds within which the effective permittivity will be. So these all these values are possible depending on what fraction you choose okay that is how you can always go and find this region is allowed okay. So that is how mixing can give you that many possibilities.



Now let us look into one of the most popular effective medium theories which is called Maxwell Garnett theory okay. So this is the classical approach for homogenizing media which has got small inclusions dispersed in a continuous medium or matrix. So one particular schematic will make it clear. So it is like this the basic structure is a two-phase medium with separated grains of the guest material. So here you see the guest material is this one which is called a permittivity of epsilon i.

The host medium has got a permittivity of epsilon h okay and you are supposed to find out the effective permittivity of this particular system okay using some theory and this is where Maxwell Garnet theory will come into the picture. So we restrict the analysis to the case of non-magnetic and isotropic materials. So there are some assumptions so initially I mentioned about the assumptions. So here the assumptions are that that materials we considered are non-magnetic and isotropic. So how do you start with that? First thing if those inclusions the small islands are sub wavelength in nature so you can easily adapt quasi-static approximation that you have already understood in previous lectures.

Effective Medium Theories: Maxwell Garnett Theory

- If the inclusions are small enough, then a quasi-static approximation can be adopted.
- For positive permittivity inclusions the following rule of thumb is considered to be conservative: ***the particle size should not exceed one-tenth of the effective wavelength***, which is the wavelength measured in the effective medium.
- However, in case of metallic or negative-permittivity inclusions, the limits of validity may be stricter, especially near the localized surface plasmon resonances.

Figure: Illustration of Maxwell Garnett homogenization.

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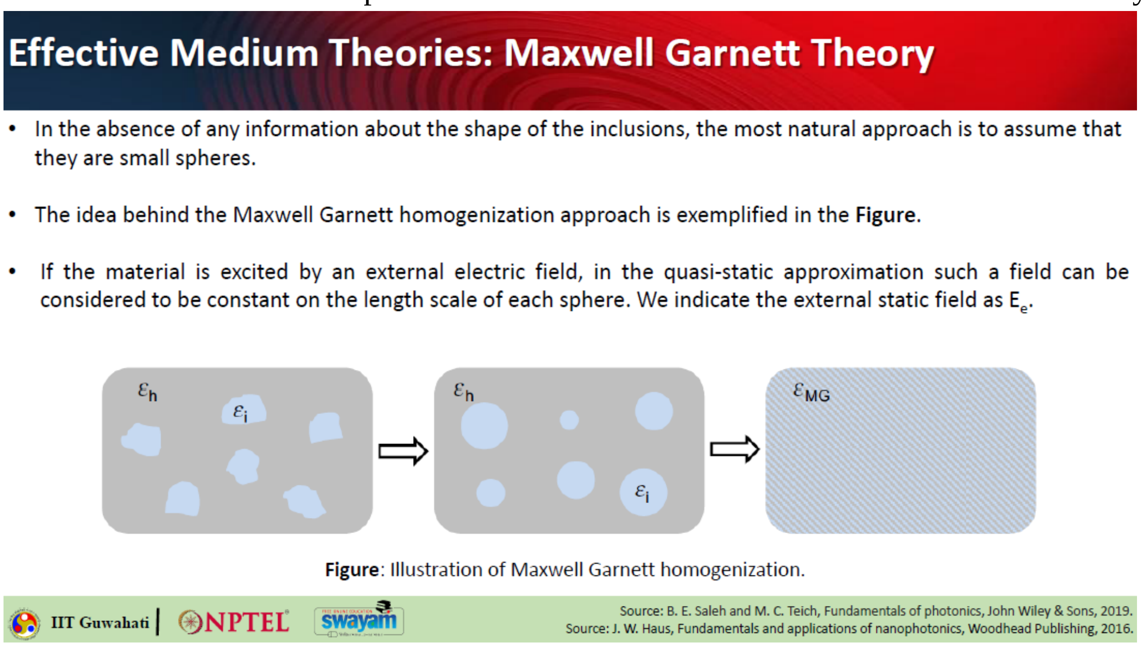
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: J. W. Haus, Fundamentals and applications of nanophotonics, Woodhead Publishing, 2016.

So in that case if you see that the inclusions are positive permittivity inclusions okay then the following rule of thumb is considered to be conservative. So you can actually go with this rule that the particle size should not exceed one tenth of the effective wavelength. So it should be less than lambda by 10 okay and lambda is what that is at this particular wavelength you are measuring the effective medium permittivity and if you get a metallic or negative permittivity inclusions the limit of the validity is much more stricter okay and this is the reason here is that because this negative permittivity materials as you have discussed before they can show localized surface plus 1 resonance. So in this case they need to be you need to be very very careful about this particular particle size limit okay. So let us see how do we go about that? So in case you do not have this information about the shape of the inclusion okay so the shape can be any arbitrary shape so the natural approach would be to assume each of those as very tiny spheres okay and that is how you can actually see how Maxwell-Garnett homogenization work so you can consider each of this inclusions act as spheres of different different size.

And if the material is excited by an external electric field so in quasi-static approximation we can understand that in this case the field on each of these tiny spheres that we have considered the field will be static and that is why it is called quasi-static approximation right and we are considering the external field to be Ee okay that is the notation we are using here Ee okay. So first we will so let us see into the derivation of this formula how Maxwell-Garnett theory helps us in getting the epsilon effective okay. So first of all we have to focus on the response of each isolated sphere to this excitation so there is an electric field and to this how these isolated spheres are responding okay. Now we have seen that when the sphere is very small it just acts as a point source with an electric dipole moment proportional to the applied field and that will be at the center of the particle. So, you can write that response of an isolated sphere where the host medium is epsilon h so that isolated sphere can be written as ph equals epsilon 0 epsilon h alpha Ee okay.



Now what is epsilon 0 that is the vacuum permittivity okay and alpha is basically the static electric polarizability of the spheres that you have considered. So, this you remember from the quasi-static formula that alpha is basically $\alpha = 3V \frac{\varepsilon_i - \varepsilon_h}{\varepsilon_i + 2\varepsilon_h}$ and what is V? V is the volume of that sphere okay. So now once we have this formula we can always write what is the electric field inside the sphere and that will be 3h over epsilon i plus 2 epsilon h times epsilon e and we consider this field to be uniform and parallel to the external electric field Ee okay. So with these two things we have also seen that the polarizability because the sphere is very tiny the polarizability of the sphere is considered to be isotropic since both permittivity and the shape are assumed to be isotropic. So, polarizability is also isotropic.

Effective Medium Theories: Maxwell Garnett Theory

- First we focus on the response of each isolated sphere to this excitation. Since the sphere is very small, it acts as a point source with an electric dipole moment proportional to the applied field.

- In other words, the response of an isolated sphere in the host medium is $\mathbf{p}_h = \varepsilon_0 \varepsilon_h \alpha \mathbf{E}_e$, where ε_0 is the vacuum permittivity, $\alpha = 3V \frac{\varepsilon_i - \varepsilon_h}{\varepsilon_i + 2\varepsilon_h}$ is the static electric polarizability of the sphere, and V is the sphere's volume.

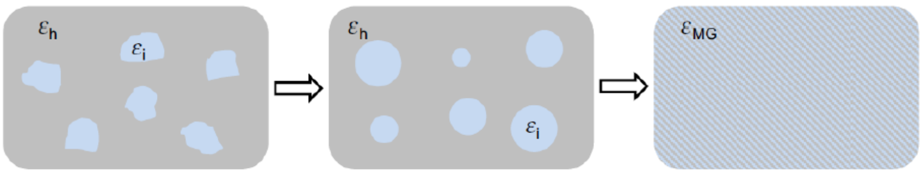


Figure: Illustration of Maxwell Garnett homogenization.

So what comes as the next? The next step would be to create an effective model of the distribution of the nanospheres. So that allows you to give you this kind of a homogenization picture. Now the spheres are now reduced to point dipoles. You can consider them as electric point dipoles and the field they are radiated okay each dipole is radiating will influence the all other dipoles in that particular medium. So, in such a case you need to get an information about how many such dipoles are there in unit volume.

Effective Medium Theories: Maxwell Garnett Theory

- The field inside the sphere, $\mathbf{E}_i = \frac{3\varepsilon_h}{\varepsilon_i + 2\varepsilon_h} \mathbf{E}_e$, is uniform and parallel to the external field $\mathbf{E}_e$.

- The polarizability of the sphere is isotropic since both the permittivity and shape of the inclusions are assumed to be isotropic.

- The next step is to create an effective model of the distribution of small spheres (***transition from the center to the right panel in Figure***).

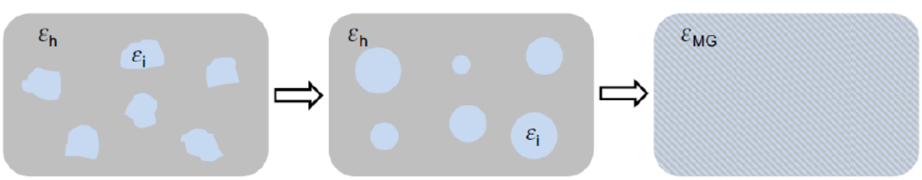


Figure: Illustration of Maxwell Garnett homogenization.

So let us assume that there are N such dipoles in that unit volume. So now you can define the effective permittivity based on the average or macroscopic constitution relationship that would link your displacement field the average displacement field to the average electric field. Normally what happens D equals epsilon E. So that is where the epsilon comes into the picture. So, in this case the D displacement field is also the average field the electric field is also the average field.

Effective Medium Theories: Maxwell Garnett Theory

- The spheres are reduced to electric point dipoles, and the field radiated by each dipole is now influenced by the presence of all the other dipoles.
- At this point the information required is the number of dipoles per unit volume, N.
- The definition of ***the effective permittivity*** is based on the average, or macroscopic, constitutive relation that links the average electric field $\langle \mathbf{E} \rangle$ to the average displacement field $\langle \mathbf{D} \rangle$.

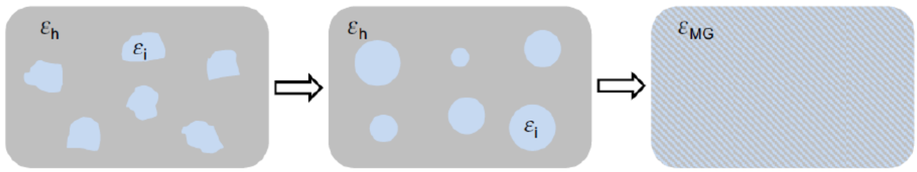


Figure: Illustration of Maxwell Garnett homogenization.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: J. W. Haus, Fundamentals and applications of nanophotonics, Woodhead Publishing, 2016.

So epsilon will be now the effective permittivity of that medium right. So the average operator integrates over sufficiently large volumes to provide an accurate description of the average fields of the original medium and so you can write this particular expression. So, you can write average D is nothing but $\varepsilon_0 \varepsilon_{MG} \langle \mathbf{E} \rangle$. So ε_{MG} nothing but the Maxwell Garnett effective permittivity of this particular system. And then multiplied by the average electric field.

Effective Medium Theories: Maxwell Garnett Theory

- The average operator integrates over sufficiently large volumes in order to provide an accurate description of average fields in the original medium. Hence, one can write:

$$\langle \mathbf{D} \rangle = \varepsilon_0 \varepsilon_{MG} \langle \mathbf{E} \rangle \qquad \text{(L23.1)}$$

where ε_{MG} is the effective (relative) Maxwell Garnett permittivity that models the original mixture, as shown in **Figure**.

- One can also see the average medium response as the average response of the host medium plus the average response of the dipoles, so that

$$\langle \mathbf{D} \rangle = \varepsilon_0 \varepsilon_h \langle \mathbf{E} \rangle + \langle \mathbf{P} \rangle \qquad \text{(L23.2)}$$

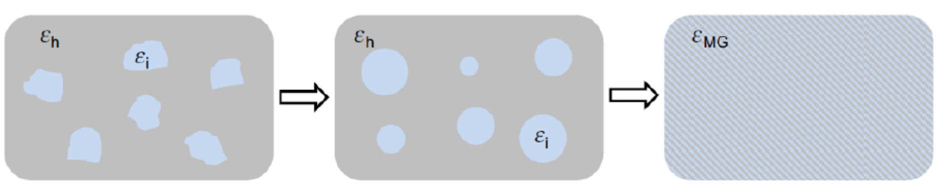

Figure: Illustration of Maxwell Garnett homogenization.

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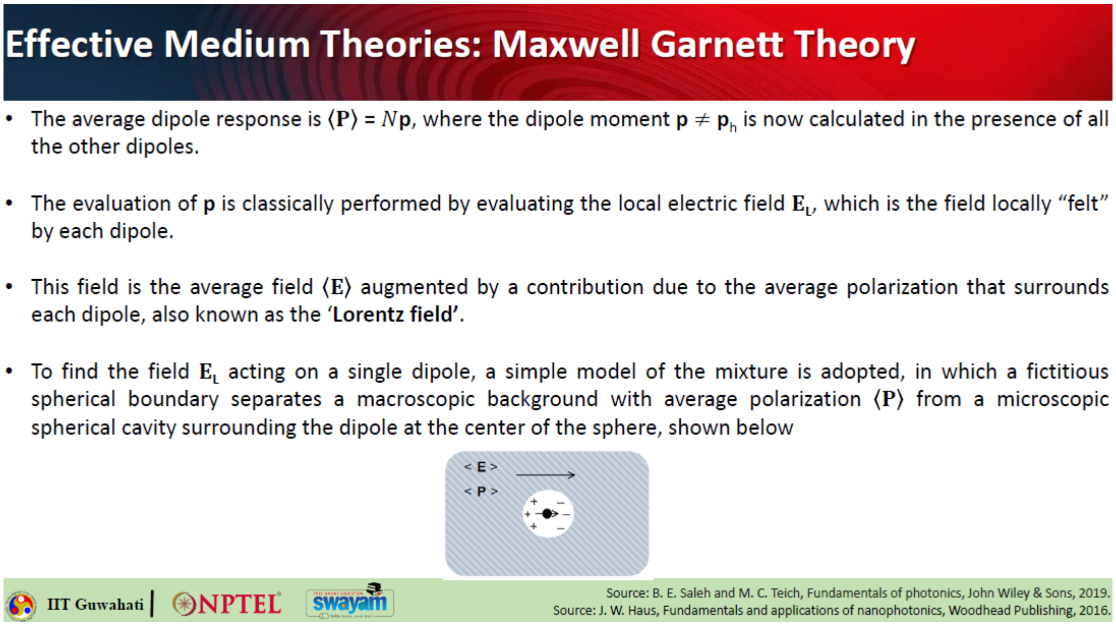
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: J. W. Haus, Fundamentals and applications of nanophotonics, Woodhead Publishing, 2016

And you can also see that this we have understood that this dielectric displacement field has got actually two parts. One is the average response of the host medium. So this is from the host medium alone and plus there is some average response of the point dipoles which are basically the inclusions right. So if you look into the point dipoles this point dipoles the average polarization is basically N times small p. And here remember that this small p is not same as that isolated spheres polarization or dipole moment okay.



Here this small p is basically the dipole moment that is calculated in the presence of all other dipoles in that system okay. So, the evaluation of this small p is classically performed by evaluating the local electric field which is denoted by E_L. And this is the local electric field which is felt or experienced by each point dipole in the presence of all other dipoles okay. So the field this field this local electric field is basically the average field augmented by a contribution due to the average polarization that surrounds each dipole and this is also known as the Lorentz field okay. Now how do you find out this E_L? To find out E_L acting on a single dipole a simple model of the mixture can be adapted where a fictitious spherical boundary like this can separate ok this dipole from the background media.

Effective Medium Theories: Maxwell Garnett Theory

- The local field can be straightforwardly written as:

$$\mathbf{E}_\mathrm{L} = \langle \mathbf{E} \rangle + \frac{\langle \mathbf{P} \rangle}{3\varepsilon_0 \varepsilon_\mathrm{h}} \qquad \text{(L23.3)}$$

- And the dipole moment as:

$$\mathbf{p} = \varepsilon_0 \varepsilon_\mathrm{h} \alpha \langle \mathbf{E} \rangle + \alpha \frac{\langle \mathbf{P} \rangle}{3} \qquad \text{(L23.4)}$$

- We can now retrieve the Maxwell Garnett permittivity in terms of the polarizability α and the number density N,

$$\varepsilon_\mathrm{MG} = \varepsilon_\mathrm{h} \left(1 + \frac{N\alpha}{1 - \frac{N\alpha}{3}} \right) \qquad \text{(L23.5)}$$

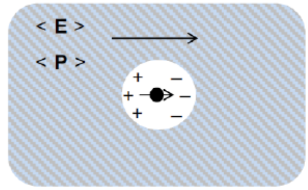


Figure: A fictitious spherical cavity separates a homogeneous background from a microscopic phase in which a dipole is included

- For very diluted media $1 - \frac{N\alpha}{3} \approx 1$, and the effective permittivity is simply $\varepsilon_\mathrm{MG} \approx \varepsilon_\mathrm{h}(1 + N\alpha)$. The same expression can easily be obtained when the local field is $\mathbf{E_L} = \langle \mathbf{E} \rangle$. This approximation is fully justified in diluted mixtures where the interaction between dipoles is weak.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: J. W. Haus, Fundamentals and applications of nanophotonics, Woodhead Publishing, 2016.

Now this background media has got a average polarization of P ok and there is a average electric field of E but this dipole is in a kind of spherical boundary which is separated from this macroscopic background. So, with that you can write down local electric field E_L will be nothing but the average electric field plus the polarization divided by 3 epsilon 0 epsilon h. And from that you can also write down what is the dipole moment small p that is given by this expression. Now in this expression when you put your you can from this you can retrieve the Maxwell coordinate permittivity in terms of polarizability alpha and number density N. So, you can write $\varepsilon_\mathrm{MG} = \varepsilon_\mathrm{h}\left(1 + \frac{N\alpha}{1 - \frac{N\alpha}{3}}\right)$

So I am not showing the overall calculation here but this can be obtained from this formula and in the case of very diluted media you can consider so N is very diluted media means N is very small capital N is very small. So, you can actually consider these as close to 1. So, in that case the effective permittivity will be simply $\varepsilon_\mathrm{h}(1 + N\alpha)$. So this is how you can find out the effective permittivity and the same expression can be easily obtained when you put that your local electric field is only this one. So the contributions from the neighboring dipoles are also negligible.

So this kind of approximation is fully justified in the case of a diluted mixture where the interactions between the dipoles is weak. That means when the spheres that you have seen are far away from each other that they do not interact with each other. In that case also in that case this particular approximation will be valid and using this formula you should be able to find out the effective permittivity very easily. Now you have seen this particular case that N α by 3 so from this you can find out this one and then you can also write $\frac{N\alpha}{3} = \frac{\varepsilon_\mathrm{MG} - \varepsilon_\mathrm{h}}{\varepsilon_\mathrm{MG} + 2\varepsilon_\mathrm{h}}$. And this equation is also known as the Clausiuse-Mossotti formula or it is also called as Maxwell's formula or Lorentz-Lorenz formula.

Effective Medium Theories: Maxwell Garnett Theory

- The following form of the relation in Eqn (L23.5)

$$\frac{N\alpha}{3} = \frac{\varepsilon_{\mathrm{MG}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{MG}} + 2\varepsilon_{\mathrm{h}}} \qquad \text{(L23.6)}$$

Known as The Clausiuse Mossotti formula, Maxwell's formula, or the Lorentz-Lorenz formula.

- Substitution of the expression of the polarizability α (from quasi-static theory) in the Clausiuse Mossotti relation gives:

Rayleigh formula
$$\frac{\varepsilon_{\mathrm{MG}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{MG}} + 2\varepsilon_{\mathrm{h}}} = f\,\frac{\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{i}} + 2\varepsilon_{\mathrm{h}}} \qquad \text{(L23.7)}$$

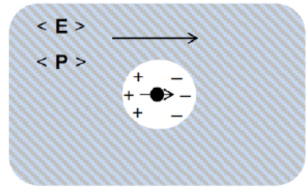

Figure: A fictitious spherical cavity separates a homogeneous background from a microscopic phase in which a dipole is included

- Relates the effective permittivity to the constituents' permittivities and to the parameter $f = NV$, which is the volume fraction of the inclusions (in this case spheres) in the medium.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: J. W. Haus, Fundamentals and applications of nanophotonics, Woodhead Publishing, 2016.

Now in this one if you put alpha from the quasi-static theory that is alpha equals 3 V $\frac{\varepsilon_i - \varepsilon_h}{\varepsilon_i + 2\varepsilon_h}$ you will be able to get this kind of a formula which is also known as Rayleigh formula. So, in this what is epsilon MG that is basically the effective permittivity of this medium and what is f small f f is basically NV that is a volume fraction of the inclusions and in this case it is sphere. So from that you can find out what is the effective permittivity that is epsilon MG in terms of epsilon i epsilon h and f. So these are the three things you got to know you should know the permittivity of the host medium permittivity of the inclusions and the filling fraction of this material in this entire volume. So you add up all these volumes ok and divide by the total volume that is your filling fraction fine.

So this simple formula represents a classical approach to homogenization of the composite media and it is widely used in many many applications. Another here we have to make sure that the nanoparticles are far away from each other so that they are not interacting ok. So this is the case. Now it is also important to notice that the only necessary parameter for retrieving the Maxwell's garnett permittivity are basically three things ok. As I mentioned the volume fraction and the two permittivities and this formula does not require the spheres to be of the same size and they should be located at a specific location.

Effective Medium Theories: Maxwell Garnett Theory

- The so-called Maxwell Garnett formula derives from (L23.7) and it is written as follows:

$$\varepsilon_{\mathrm{MG}} = \varepsilon_{\mathrm{h}}\left[1 + 3f\frac{\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{i}} + 2\varepsilon_{\mathrm{h}} - f(\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}})}\right] \quad \text{(L23.8)}$$

- This simple formula represents the classical approach to homogenizing composite media, and it is widely used in many applications.

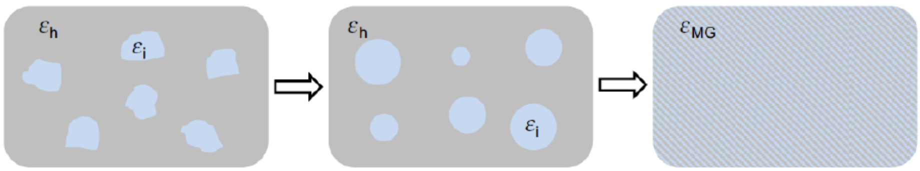

Figure: Illustration of Maxwell Garnett homogenization.

So all these requirements are not there. So you do not want your spheres particularly to form an array or something nothing like that. So the only requirement here is that the wavelength in the medium must be much larger than the size of the inclusion so that the quasi-static approximation remains valid. So Maxwell garnet theory also predicts the following. So, if you put f equals 0 you get effective permittivity to be same as the most one and if you put f equals 1 you will get the effective permittivity to be same as the inclusions one ok.

Effective Medium Theories: Maxwell Garnett Theory

- It is interesting to notice that the only necessary parameters for retrieving the Maxwell Garnett permittivity are the permittivities of inclusions and host medium and the volume fraction of the inclusions.
- The formula ***does not require that the spheres are of the same size and located at specific positions*** (e.g., periodic arrays). The only requirement is that the wavelength in the medium should be much larger than the size of the inclusions.
- The Maxwell Garnett theory predicts that $\varepsilon_{\mathrm{MG}} = \varepsilon_{\mathrm{h}}$ for $f \to 0$, and $\varepsilon_{\mathrm{MG}} = \varepsilon_{\mathrm{i}}$ for $f \to 1$.

Figure: Illustration of Maxwell Garnett homogenization.

That is pretty simple. So that that tells you about the Maxwell's garnett theory. The next important and popular theory is Bruggeman's theory. So we have seen that Maxwell garnett formula represents a valid homogenization model for mixtures with a well-defined host medium and inclusions right. And they result more accurately for small values of inclusion volume factor f. So smaller the f is you will get more and more accurate result.

Now there could be aggregate mixtures which has got random distribution of two or more constitutive materials and in those case the effective medium theories should be based on some statistical formulation right. And these are the cases where you have continuous boundaries and any of this material can be of any permittivity and they may have a different fill factor right. So these are the cases where you actually use Bruggeman's theory. So here also what you will do you will try to find a medium. So, you will try to find what is epsilon br that is the effective medium.

Effective Medium Theories: Bruggeman theory

- The **Maxwell Garnett formulas** represent a valid homogenization model for mixtures with a well-defined host medium and inclusions, and they result more accurate for relatively small values of the inclusion volume factor *f*.
- For aggregate mixtures with random distributions of two or more constituents effective medium theories based on a statistical formulation are more suitable.
- The classic theory for this class of inhomogeneous mixtures is the **Bruggeman theory**.

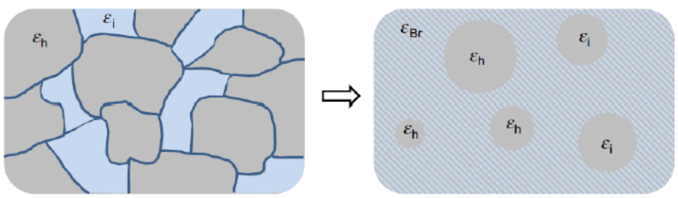

Figure: Illustration of Bruggeman Theory Model.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: J. W. Haus, Fundamentals and applications of nanophotonics, Woodhead Publishing, 2016.

So you have the host medium and the inclusion medium they are basically capturing different regions. So you will see that how do we handle this particular class of inhomogeneous mixture through Bruggeman's theory. So let us consider two phase microstructure of the type that is shown in this figure where the constituent material of permittivity epsilon i has filled a volume factor f and in that case the other permittivity material epsilon h will have a volume fill factor of 1 minus f ok. So this mixture will be now modeled as a continuum continuous medium hosting a distribution of small and large spherical inclusions of the two different dielectric permittivities. So, one is h epsilon h another one is epsilon i ok and the background one or the overall one is basically epsilon Br not the background one the overall or the effective one.

Effective Medium Theories: Bruggeman theory

- We now consider a two-phase microstructure of the type illustrated in **Figure**, where the constituent with permittivity ε_i has volume fill factor f, and the constituent with permittivity ε_h has volume fill factor $1 - f$.
- This mixture is now modeled as a continuous medium hosting a distribution of small spherical inclusions of two different dielectric permittivities.
- **The probabilities of finding spheres with permittivity ε_i and ε_h are f and $1 - f$**, respectively, which correspond to the volume fill factors of the two phases in the original mixture.

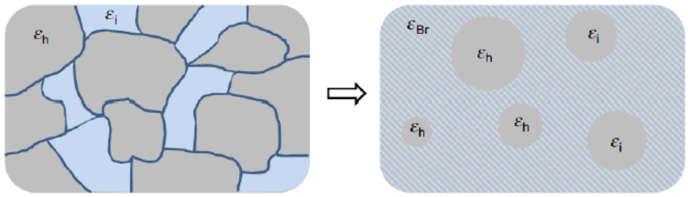

Figure: Illustration of Bruggeman Theory Model.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: J. W. Haus, Fundamentals and applications of nanophotonics, Woodhead Publishing, 2016.

So the probabilities of finding spheres of permittivity epsilon i will be f and for finding spheres with permittivity epsilon h the probabilities 1 minus f ok. So this basically corresponds to their volume fill factors in the original mixture. Now we can assume that the host medium for this Bruggeman mixture has unknown effective permittivity. So what is that the effective permittivity we write as epsilon Br ok. So how this will be related so you can write down the basic form of the Bruggeman theory as $f\frac{\varepsilon_i-\varepsilon_{Br}}{\varepsilon_i+2\varepsilon_{Br}} + (1-f)\frac{\varepsilon_h-\varepsilon_{Br}}{\varepsilon_h+2\varepsilon_{Br}} = 0$.

Effective Medium Theories: Bruggeman theory

- We now assume that the host medium of the Bruggeman mixture has the unknown effective permittivity ε_{Br}, as indicated on the right side of **Figure**, and invokes the transparency or "invisibility" condition for the distribution of the spherical inclusions.
- In our distribution of spheres ε_p can be either ε_i with probability f or ε_h with probability $1 - f$. The resulting "averaged transparency" condition reads as follows:

Basic form of the Bruggeman theory

$$f\frac{\varepsilon_i-\varepsilon_{Br}}{\varepsilon_i+2\varepsilon_{Br}} + (1-f)\frac{\varepsilon_h-\varepsilon_{Br}}{\varepsilon_h+2\varepsilon_{Br}} = 0 \quad \text{(L23.9)}$$

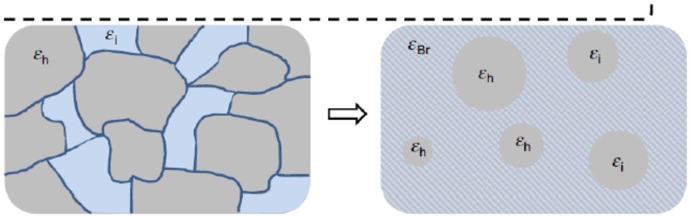

Figure: Illustration of Bruggeman Theory Model.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
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So if you see can you find out what are these basically these are basically the contributions coming from spheres of permittivity epsilon i in a effective background medium of this one plus the volume fraction of the other other type of spheres which has got permittivity of epsilon h in a background of epsilon Br. So this is how you add up these two contributions ok and that is basically the basic form of the Bruggeman's theory. Now if you have more inclusions not only two say if you have a multi phase aggregate in that case it will be simply $\sum_{m=1}^{M} f_m \frac{\varepsilon_m - \varepsilon_{Br}}{\varepsilon_m + 2\varepsilon_{Br}} = 0$. So, what is f_m, f_m is the filling factor or volume fill fraction of the mth point of the mixture and there are total m number of capital M number of phases right. So, the limitation of the Maxwell-Garnett theory that the particles with very small depolarization factors.

Effective Medium Theories: Bruggeman theory

- The formula for spherical inclusions can be easily extended to multiphase aggregates by adding more terms in the relation Eqn (L23.9) yielding:

$$\sum_{m=1}^{M} f_m \frac{\varepsilon_m - \varepsilon_{Br}}{\varepsilon_m + 2\varepsilon_{Br}} = 0 \qquad (L23.10)$$

where f_m is the fill factor of the mth constituent of the mixture, and M is the number of phases.

- Limitation of the Maxwell Garnett theory — particles with very small depolarization factors. A low depolarization factor implies an elongated shape of the particle with the result of stronger particle-particle interactions. In this situation, which is similar to the scenario of a mixture with large inclusions' fill factor, the Bruggeman prediction should be adopted.

- Another scenario in which the Bruggeman theory provides a more realistic electromagnetic description is for mixtures with large differences in the permittivities of the constituents e.g., metal-dielectric mixtures, where a percolation phenomenon above a threshold of the metallic phase occurs.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: J. W. Haus, Fundamentals and applications of nanophotonics, Woodhead Publishing, 2016.

Now what do you mean by depolarization factor? A low depolarization factor means it actually polarizes ok. So you are you may think of elongated shapes like ellipsoids and all these things and this kind of shapes they may result in strong particle particle interaction and as I mentioned previously that in Maxwell-Garnett you do not actually like those kind of contributions or interactions to come into play. So in this situation which is similar to the situation of a mixture of large inclusions fill factor the Bruggeman's prediction can be adapted ok. So if the grain boundary is very specific and the density is less you go for Maxwell-Garnett but if you see that the large inclusions are there and thus in this kind of situation you can go for Bruggeman's theory. Another situation in which Bruggeman theory looks more realistic will be for the mixtures with large difference of in the permittivities of the constituents.

Something like if you have metal dielectric mixture ok where percolation phenomena above a threshold of the metallic phase takes place. So in those cases you know you should go for Bruggeman theory. Now if you do not know what this particular phenomena is this is basically a threshold that is the critical metal filling factor above which there is a formation of kind of long connectivity between the metal grains and the optical response of the mixture will change abruptly. So there is if you are going for a metal dielectric kind of mixture there is a threshold beyond which you should not have metal fill fraction ok. So thus last kind of the third type of effective medium theory that we will discuss today is the Nicholson-Ross-Weir method.

Effective Medium Theories: Nicolson-Ross-Weir method

- This is a homogenization method based on the inversion of Fresnel formulas relative to the transmission and reflection coefficients through slabs of homogeneous media.
- The technique was conceived to estimate the complex permittivity and permeability of an unknown material from the measured transmission and reflection spectra of a finite thickness sample.
- It was originally proposed in the time domain for pulsed measurement systems and then adapted to higher resolution, frequency domain systems.
- The transmission and reflection spectra may be retrieved with experiments or with numerical simulations.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
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So, this is a homogenization method based on the inversion of Fresnel formula. Now if you remember the Fresnel formula, the Fresnel formula allows you to calculate the reflection and transmission coefficient for the interface of two different materials of permittivity of different permittivity ok. So in this case you are using the reverse of it. So you are based on the inversion of Fresnel formula relative to the transmission and reflection coefficient through the slabs of homogeneous medium. Now this technique was conceived to estimate the complex permittivity and permeability of unknown material from the measured transmission and reflection spectrum of finite thickness sample.

Effective Medium Theories: Nicolson-Ross-Weir method

- A slab of thickness d of the unknown natural or artificial mixture is modeled as a slab of a homogeneous medium with effective (relative) permittivity ε_{eff} and permeability μ_{eff}.
- It is supposed that the thickness of the homogeneous slab is equal to d. The (complex) reflection and transmission coefficients R and T under normal incidence plane wave excitation are somehow known, via an experiment, a theoretical prediction, or a numerical simulation.

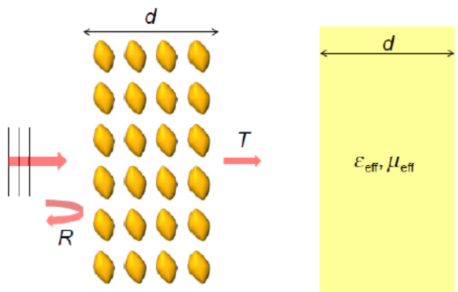

Figure: On the left is a slab of a composite medium with thickness d illuminated at normal incidence. R and T are the complex reflection and transmission coefficients. On the right is the homogenized slab with effective (relative) permittivity ε_{eff} and permeability μ_{eff}.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: J. W. Haus, Fundamentals and applications of nanophotonics, Woodhead Publishing, 2016.

So, to do reverse engineering to find out what kind of material could actually provide this kind of permittivity and permeability. So it was originally proposed in the time domain for pulsed measurement systems and then it was adapted for higher resolution systems like frequency domain systems ok. So the transmission and reflection spectra can be taken from the measurements of experiments or you can actually get them by doing simulations. So in this case what happens a slab of thickness d of unknown natural or artificial mixture something like this that can be modeled as a slab of homogeneous medium with effective relative permittivity epsilon effective and mu effective that is the effective permeability. So, in this case you are measuring what is the reflectance, what is the transmittance and then you try to find out what is the effective permittivity and permeability that gives that kind of a transmission and reflection coefficient ok.

Effective Medium Theories: Nicolson-Ross-Weir method

- The Fresnel reflection (R) and transmission (T) coefficients may be written in the following form:

$$R = \frac{\Gamma(1 - e^{-2ikn_{\text{eff}}d})}{1 - \Gamma^2 e^{-2ikn_{\text{eff}}d}} \qquad \text{(L23.11)}$$

$$T = \frac{(1 - \Gamma^2)e^{-ikn_{\text{eff}}d}}{1 - \Gamma^2 e^{-2ikn_{\text{eff}}d}} \qquad \text{(L23.12)}$$

which are dependent on the effective refractive index of the slab $n_{\text{eff}} = \sqrt{\mu_{\text{eff}}\varepsilon_{\text{eff}}}$, free-space wavenumber $k = \omega/c$, and the reflection coefficient Γ across the first interface between the input medium and the semi-infinite homogeneous slab with relative parameters (ε_{eff}, μ_{eff}).

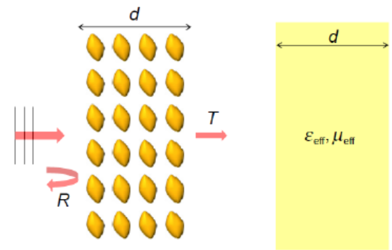

Figure: Illustration o Nicolson-Ross-Weir method.

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: J. W. Haus, Fundamentals and applications of nanophotonics, Woodhead Publishing, 2016.

So it is supposed that the thickness of the homogeneous slab is equal to d. So what you do the complex reflection and transmission coefficients. So note that in Fresnel theory we used to use small r and small t for coefficients. So here in this particular book they have used capital R capital T for the coefficients but you can also change them to small r small t to follow the continuity ok. So what they do here you have small r and small t or capital R capital T here ok I am just talking about the reflection transmission coefficient ok.

So it is depending on gamma as well as some n effective ok. So what is this n effective it is basically the effective refractive index of the slab. So it comes from square root of mu effective and epsilon effective ok. And what is k? k is basically the wave number that is omega by c and the reflection coefficient capital gamma is basically across the first interface between the input medium and this semi infinite homogeneous slab that you have made ok. So you can see the gamma is basically taking of this particular form. So $\Gamma = \frac{\eta_{\text{eff}} - \eta_0}{\eta_{\text{eff}} + \eta_0}$ in the case of normal incidence and you can also find out what is eta effective ok.

So from that this is basically the impedance and eta naught is the intrinsic impedance of the input output medium. So from that you can find calculate what is eta effective. So I think this there is a typo here this should be n effective not eta and this is eta effective ok. So eta effective is correlated to reflection and transmission coefficient using this formula ok. And you can also find out what is the quality Q ok and this particular Q is given as $Q = e^{-ikn_{\text{end}}\, d}$.

Effective Medium Theories: Nicolson-Ross-Weir method

- At normal incidence $\Gamma = \frac{\eta_{eff}-\eta_0}{\eta_{eff}+\eta_0}$, $\eta_{eff} = \sqrt{\mu_{eff}\varepsilon_{eff}}$ is the effective intrinsic impedance of the homogeneous slab, and η_0 is the intrinsic impedance of the input/output medium.
- The inversion of Eqns (L23.11) and (L23.12) leads to the following expression of the effective impedance:

$$\eta_{eff} = \pm\eta_0\sqrt{\frac{(1+R)^2-T^2}{(1-R)^2-T^2}} \qquad (L23.13)$$

- and the following expression for the quantity $Q = e^{-ikn_{eff}d}$

$$Q = \frac{T}{1-R\frac{\eta_{eff}-\eta_0}{\eta_{eff}+\eta_0}} \qquad (L23.14)$$

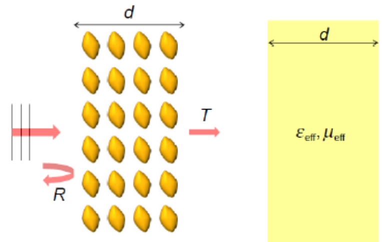

Figure: Illustration o Nicolson-Ross-Weir method.

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So $Q = \frac{T}{1-R\frac{\eta_{efT}-\eta_0}{\eta_{ef}+\eta_0}}$. So these are some formulas that actually tell you how to obtain n effective ok. So I will not go into the details of this formula the whole idea is to tell you that that it is also possible to find out the effective permittivity of a kind of system which has got metamaterial kind of design that you have inclusions or you have particles which are in arranged in this kind of a format ok. So with this you can obtain epsilon effective and mu effective. So what are the observations in this case? The first thing is that the choice of sign for the effective impedance and the refractive index does not alter the values of the effective parameters which are obtained via this equation ok.

Effective Medium Theories: Nicolson-Ross-Weir method

- From Eqns (L23.13) and (L23.14), one can write the effective refractive index as follows:

$$n_{eff} = \frac{i}{kd}\log(Q) = \frac{1}{kd}\{iLog|Q| - [Arg(Q) + 2m\pi]\} \quad (L23.15)$$

- Here $\log(Q)$ is the complex, multiple-valued logarithm of (Q), $Log|Q|$ is the ordinary real logarithm of $|Q|$, $Arg(Q)$ is the argument in the principal branch ($m = 0$), and $m = \pm1, \pm2 \ldots\ldots$ indicates the branch of $\log(Q)$.
- The effective parameters can finally be written as follows:

$$\varepsilon_{eff} = \frac{kn_{eff}}{\omega\eta_{eff}}, \quad \mu_{eff} = \frac{kn_{eff}\eta_{eff}}{\omega} \qquad (L23.16)$$

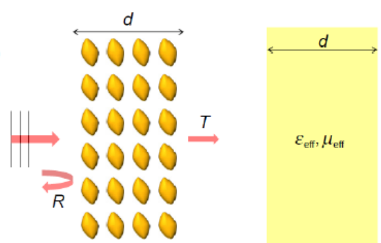

Figure: Illustration o Nicolson-Ross-Weir method.

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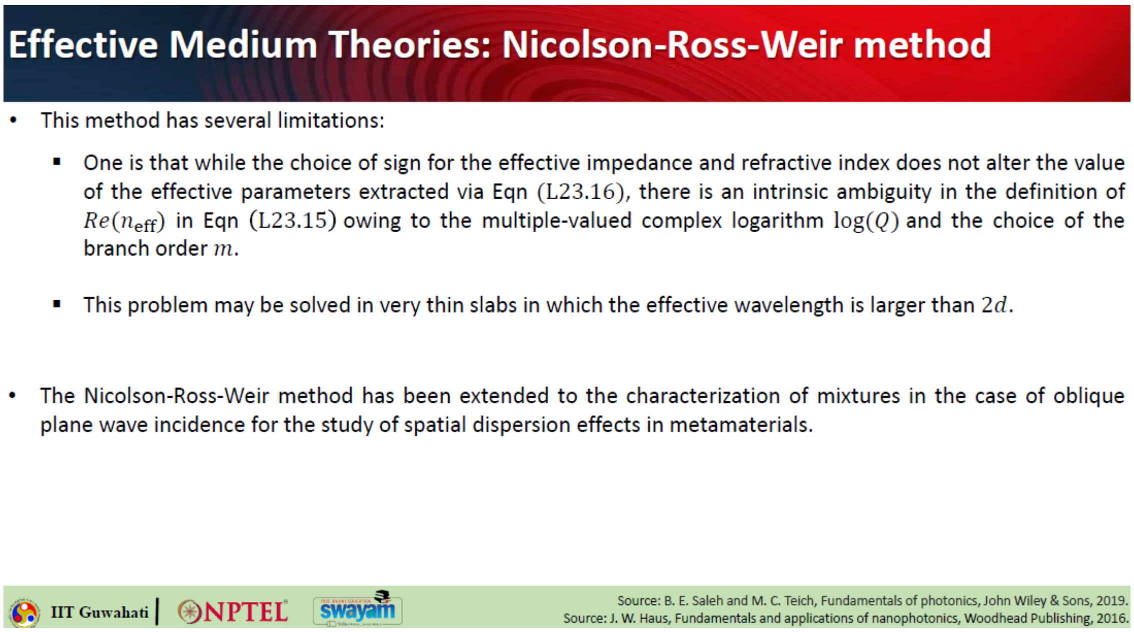
And there is some intrinsic ambiguity in the definition of this n effective ok because it is coming from this multi valued complex logarithm and the choice of your branch order

m. Anyways the problem may be solved in very thin slabs in which the effective wavelength is much larger than 2D. So this kind of method has been extended to characterization of mixtures in the case of oblique plane wave incidence for the study of spatial dispersion effects in metamaterials as I was mentioning. So, this kind of techniques can be used for studying the effect of oblique plane wave incidence for spatial distribution effects in metamaterials.



So, with that we will stop here and in the next lecture we will discuss single and double negative metamaterials design and thank you. If you have got any queries you can drop any email to my email address. Thank you.