Course Name- Nanophotonics, Plasmonics and Metamaterials

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Week-09

Lecture -26

Hello students, welcome to lecture 26 of the online course on Nanophotonics, Plasmonics and Metamaterials. Today's lecture will be on superlens, hyperbolic metamaterials and hyperlens. So here is the lecture outline. We will see the idea of perfect lens and to achieve that how far we have gone till now. We will see a version of superlens. We will then discuss the optics of anisotropic media that will allow us to understand how hyperbolic media works and this hyperbolic metamaterials will allow us to obtain hyperlens.

Lecture Outline

- The ideas of Perfect Lens
- Superlens
- Optics of Anisotropic Media
- Hyperbolic Media
- Hyperbolic Metamaterials: Hyperlens
- Hyperlens: Applications

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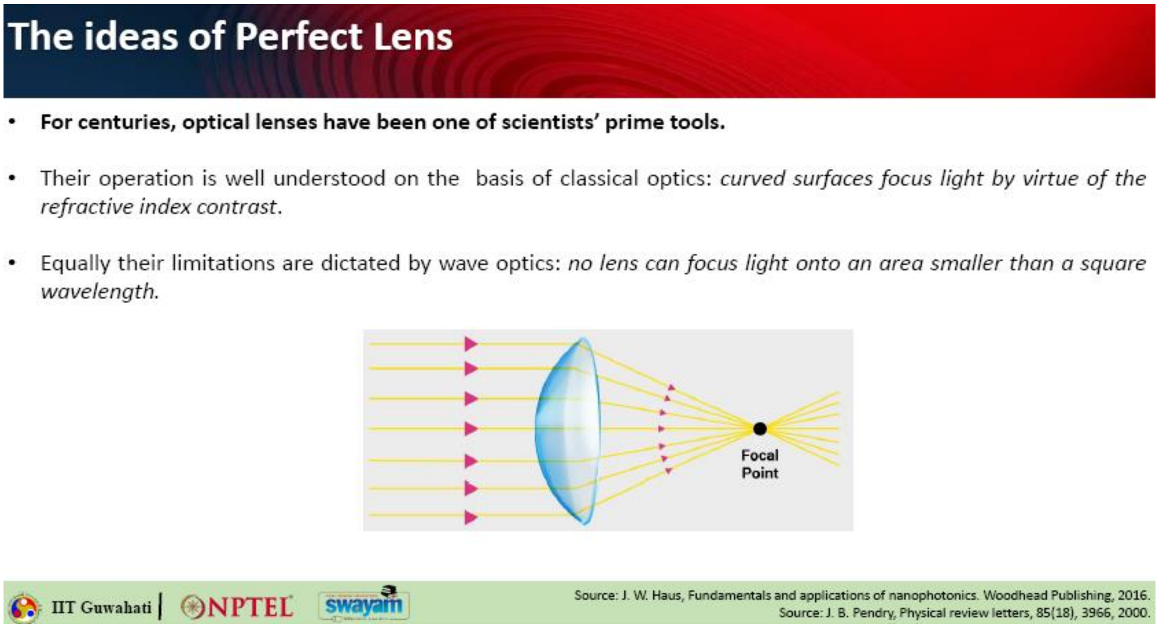
And then we will also see the applications of hyperlens. So for centuries optical lenses have been one of the prime tools of the scientists. It allows us to look at objects which are very very tiny.



So, you can focus light and that allows you to image a particular tiny object. That is how lens works. Now their operation is well understood on the basis of classical optics. So there is a curved surface and that helps us focus light by virtue of the refractive index contrast. Equally their limitations are also dictated by wave optics. So no lens can focus light onto an area which is smaller than a square wavelength.

So typically if there are two objects which are separated by a gap which is less than lambda by 2, we will not be able to see them differently. Now this is the problem with the conventional lenses. So what is the new to say that other than polishing the lens more perfectly or to invent slightly better dielectric which can have a better focusing ability, can we improve that resolution and go and see objects perfectly and that would be a perfect lens. And that quest was solved when Professor Pendry proposed the idea of negative index materials. So, if there is a material with negative refractive index n equals minus 1, you can use this as an alternative to your conventional lens.

The ideas of Perfect Lens

- **What is there new to say other than to polish the lens more perfectly and to invent slightly better dielectrics?**
- Material with negative refractive index ($\boldsymbol{n = -1}$)
  — **An unconventional alternative to a lens.**
- They focus light even in the form of a parallel-sided slab of material.
- The figure does obey Snell's laws of refraction at the surface as light inside the medium makes a negative angle with the surface normal.

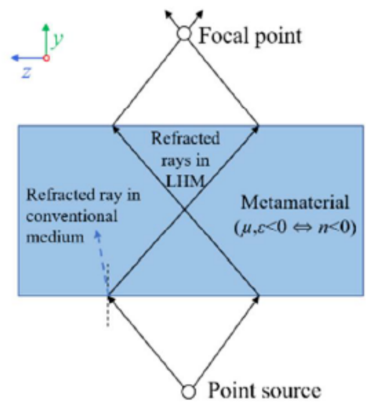


Figure: Ray diagram showing reversal of Snell's law in a metamaterial medium.

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Source: S. Datta *et al.*, Sensors, 21(14), 4782, 2021.
Source: J. B. Pendry, Physical review letters, 85(18), 3966, 2000.

So this kind of material, they can have parallel sides, okay, but then they are still able to focus light at a particular point as you can see here. So here is the point source, here is a slab of negative refractive index material, okay, and the light gets again over here and then when it exits, it again focuses at this particular point. So you are able to get a point source focused here with the help of a negative index material. If you look into the picture carefully, you can see that the Snell's law of refraction at the surfaces are actually followed just that at negative refractive index material, they actually make a negative angle with respect to the surface normal. So instead of bending this side, they are actually bending on the opposite side, okay.

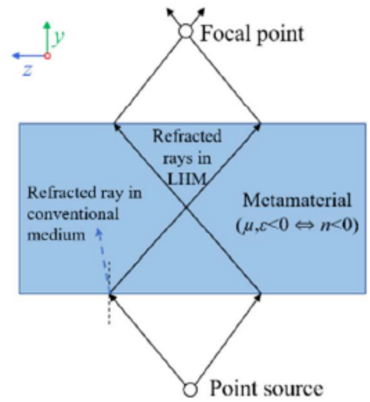
So they are bending towards negative theta angle, okay. And another important characteristic of the system is basically the double focusing effect. As you can see here, so it is actually doing double focusing, okay, and the underlying secret of this medium lies in the fact that both dielectric function permittivity and magnetic permeability both happens to be negative, okay, and you get n =- 1, okay. So how it works? So $n = \sqrt{\varepsilon\mu}$. When epsilon mu both are minus 1, okay, so you get, okay, to judge this particular square root, you can you have to take the negative sign for that, okay.

The ideas of Perfect Lens

- The other characteristic of the system is the double focusing effect revealed by a simple ray diagram.
- The underlying secret of this medium is that both the dielectric function ε and the magnetic permeability μ happen to be negative *i.e.* -1.
- At first sight this simply implies that the refractive index is that of vacuum,

$$n = \sqrt{\varepsilon\mu} \qquad \text{(L26.1)}$$

- But further consideration will reveal that when both ε and μ are negative, we must choose the negative square root.



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Source: S. Datta *et al.*, Sensors, 21(14), 4782, 2021.
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So that allows you to get n equals minus 1. But if you look into the impedance of the medium, impedance can be calculated as this $Z = \sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}}$. Now here if mu and epsilon are both negative, they do cancel out each other, okay, and in that case, you will get that, the impedance is basically same as that of the free space. So it is matching perfectly with the free space and that is why this kind of a interface will never show any reflection. Moreover, at the far boundary here again there is an impedance match and that is why light can, perfectly transmit into the vacuum.

The ideas of Perfect Lens

- The impedance of the medium:

$$Z = \sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}} \qquad \text{(L26.2)}$$

- Z retains its positive sign so that, when both $\varepsilon = -1$ and $\mu = -1$,the medium is a perfect match to free space and the interfaces show no reflection.
- At the far boundary there is again an impedance match and the light is perfectly transmitted into vacuum.

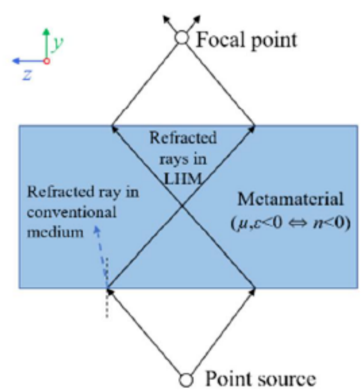

Figure: Ray diagram showing reversal of Snell's law in a metamaterial medium.

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Source: S. Datta *et al.*, Sensors, 21(14), 4782, 2021.
Source: J. B. Pendry, Physical review letters, 85(18), 3966, 2000.

So this is how you can actually realize a perfect lens. But then as you know that, for every wavelength to get material permittivity and permeability both to be negative and minus 1 is very difficult, okay. So people have tried some easier version of achieving this kind of a functionality and those kind of lenses are called superlens, okay. So they have focusing capability better than the conventional lens and that is why they are called superlens. So what actually decides the resolution of a conventional optical lens? That is with the lens which has got, permittivity positive and permeability positive, so they are typically constrained by the diffraction limit, okay.

Superlens: Introduction

- The resolution of traditional optical lenses **($\varepsilon > 0$, $\mu > 0$)** is generally **constrained by the diffraction limit**.
- This is because the evanescent wave that contains the subwavelength information decays exponentially with distance and is undetectable to image by conventional lenses (**Figure**).
- *So far, the information carried by evanescent waves can only be gathered by scanning near-field optical microscopy techniques.*
- Evanescent wave in NIM is magnified exponentially with distance and the subwavelength details are retained for imaging, which could break through the diffraction limit.

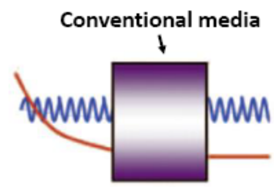

Figure. An EM wave transmitted in conventional media in propagation mode. The evanescent part decays exponentially with distance, which is only detectable in the near field.

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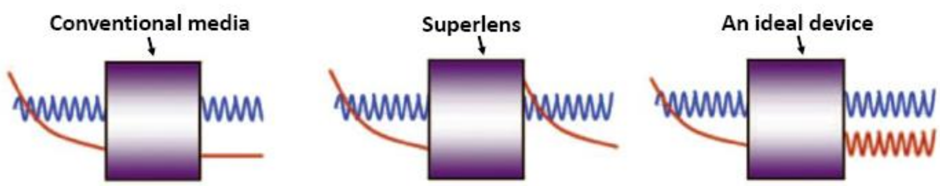
Source: J. W. Haus, Fundamentals and applications of nanophotonics. Woodhead Publishing, 2016.
Source: A. McGurn, Nanophotonics, Springer International Publishing, 2018.

And this happens because of the fact that, the evanescent wave which contains the sub-wavelength information that decays exponentially with distance and it is basically undetectable for the conventional lenses to image. So you do not, you are not able to capture any information which is coming from a sub-wavelength source or sub-wavelength object. So so far the information carried by this evanescent waves can only be gathered by ANSOM, that is near field optical microscopy techniques, okay. Now in negative refractive index media there is something magical that happens with this evanescent wave. Evanescent waves in negative index material can actually get magnified, okay, and that happens exponentially with distance and that helps you to get the sub-wavelength details retained for imaging by a conventional lens and that allows you to break the diffraction limit of light for traditional, for conventional imaging.

Superlens: Introduction

- However, because of the decay of the evanescent field outside of the NIM lens (superlens), the image of this superlens still requires the near-field detection.
- *A "superlens" amplifies the evanescent waves but does not change their decaying character.*
- An ideal device would convert evanescent waves to propagating waves for ease of detection and processing in the far-field.



Source: J. W. Haus, Fundamentals and applications of nanophotonics. Woodhead Publishing, 2016.
Source: A. McGurn, Nanophotonics, Springer International Publishing, 2018.

However, there is a small problem that inside the negative index material evanescent wave does get amplified but outside of that material it again decays, okay. So you cannot image this particular information from far field, you still have to do the detection at the near field, okay. So super lens does one good thing, it could amplify and extend the evanescent field but it cannot change the decaying characteristics. So if you put the conventional media, the normal lens that you see every day, they do not do any good to the evanescent waves. Super lens they magnify the evanescent waves but outside that that again decays but then you are actually looking for something that is capable of converting this evanescent waves in some form of propagating waves, okay.

Evanescent Waves in NIMs

Evanescent "wave" in ordinary material

$$E(z) = E_0 e^{j(k'+jk'')nz} = E_0 \cdot \underbrace{e^{-k''nz}}_{\text{evanescent component (decays)}} \cdot \underbrace{e^{jk'nz}}_{\text{oscillating component}}$$

Evanescent "wave" in negative-index material

$$E(z) = E_0 e^{j(k'+jk'')(-\tilde{n})z} = E_0 \cdot \underbrace{e^{k''\tilde{n}z}}_{\text{evanescent component (grows)}} \cdot \underbrace{e^{-jk'\tilde{n}z}}_{\text{oscillating component}}$$

Evanescent fields grow in amplitude inside NIMs

Source: J. W. Haus, Fundamentals and applications of nanophotonics. Woodhead Publishing, 2016.
Source: A. McGurn, Nanophotonics, Springer International Publishing, 2018.

So that is what you are looking for. If you are able to do this, then you will be able to image this object using a far field technique, okay, and this is where, hyper lens will come into picture, right. So as I mentioned, an ideal device would convert the evanescent waves to propagating waves for ease of detection and processing in the far field. So mathematically let us look into the situation how it looks like. Like if you take evanescent wave in ordinary material, you can write the electric field as E z that is given as

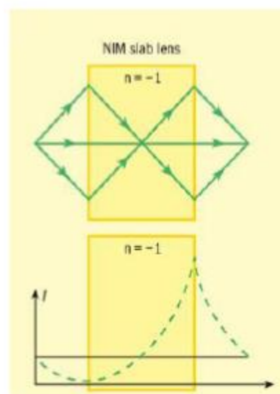
$$E(z) = E_0 e^{j(k'+jk'')nz} = E_0 \cdot e^{-k''nz} \cdot e^{jk'n}$$

.

So if you open this up, you will see that there are two components. One has got, e to the power j k prime n z and that is basically the oscillating component of the wave and then you have e to the power minus k double prime n z that is actually the decaying component or the evanescent component. If you write the same wave equation in a negative index material, you can replace this n by a minus n and then you will see that instead of this guy getting a minus k prime, you are actually getting a positive factor here. So this oscillating part remains same, but this decaying evanescent component is now a growing evanescent component. So that is the main difference between the evanescent waves when propagating in a negative index material.

NIM slab lens

- A schematic of a superlens where the green lines illustrate light rays incident and focused by a superlens.
- The upper plot illustrates how propagating waves are focused and the lower plot shows how evanescent waves are restored.



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Source: J. W. Haus, Fundamentals and applications of nanophotonics. Woodhead Publishing, 2016.
Source: A. McGurn, Nanophotonics, Springer International Publishing, 2018.

In short, evanescent field grows in amplitude inside negative index materials. So if you use a negative index material slab as lens, you can see that you are actually able to obtain a super lens where light can be focused at a particular point perfectly. And this particular plot shows the evanescent field which is getting amplitude. What is this one? This is showing the propagation like the propagating wave that is perfectly fine. So these are basically propagating waves.

So they are happily propagating inside this material without any change in its intensity or amplitude. However, if you see the evanescent field, they get amplified when they are traveling inside the negative index material, but as soon as they leave, they again decay, but then that gives you that extra distance to image them using near field techniques. Now let us look into a poor man's super lens. The name itself tells you that it is not a perfectly made super perfect lens, but it is a close approximation or you can say rough approximation of similar kind of feature. So nevertheless, applications that involve distances smaller than a wavelength such as near field microscopy, this kind of techniques may significantly benefit from a simplified version of the super lens.

So here is an object that you are trying to image and you can actually think of a poor man's super lens made of a material which has got just negative dielectric permittivity. So you are not looking for both epsilon and mu to be negative, you are just looking for negative permittivity material and the easy choice could be a silver super lens. So that is why it is called a poor man's super lens. Reason is that this negative dielectric permittivity alone is much easier to realize because there are naturally occurring materials like metals, gold, silver, they all have negative permittivity at optical frequencies. So if you put this kind of a thin slab material next to your object what happens? So the evenness in field will get amplified in it and that will help you to do the imaging.

So the purpose here is a thin slab of metal should have a permittivity equal and opposite to that of the surrounding media. The reason is you need to get rid of the reflection by doing impedance matching. So that can give you a poor man's super lens in the quasi-static limit. And what is the benefit of this kind of a super lens? They can form an image with sub wavelength resolution in the near field. So, remember here they are still doing the imaging in the near field just that this kind of a lens allow you to do sub wavelength imaging with ease.

Poor man's superlens

- Nevertheless, applications that involve distances much smaller than a wavelength, such as near-field microscopy, may significantly benefit from a simplified version of a superlens
- Also called **a "poor man's superlens"**, which has a negative dielectric permittivity only.

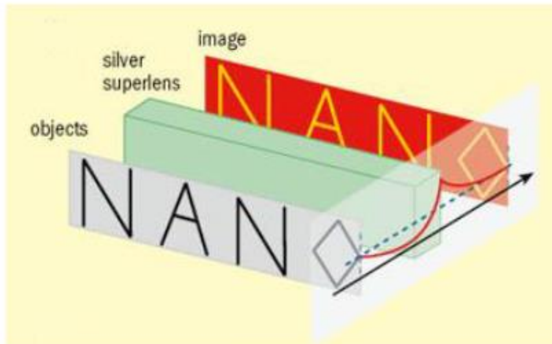

Figure — a "poor-man's" superlens *i.e. a negative dielectric permittivity.*

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Source: J. W. Haus, Fundamentals and applications of nanophotonics. Woodhead Publishing, 2016.
Source: A. McGurn, Nanophotonics, Springer International Publishing, 2018.

Now let us look into some super lens examples. So one example is this particular experiment where an arbitrary object, NANO, nano was imaged by a silver super lens. So this particular object was inscribed on a 50 nanometer thick chrome and then there was 365 nanometer light which was used for illumination. And there was a photoresist layer on the other side of the silver super lens to capture the image of this particular object. And these are the different types of images that has been taken.

Superlens: Introduction

- Negative dielectric permittivity alone is easier to realize since it is readily attainable in metals at optical frequencies.
- A thin slab of metal with permittivity equal and opposite to that of the surrounding media can be used as a poor man's superlens in the quasistatic limit.
- *Such a superlens can form an image with subwavelength resolution in the near field.*

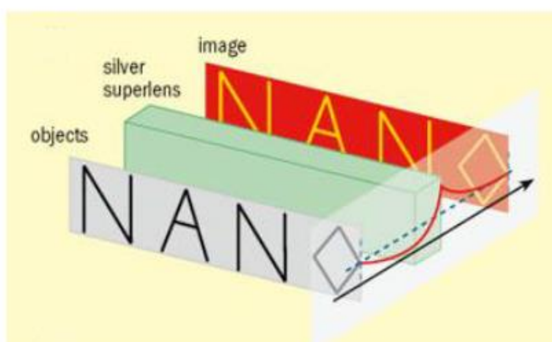

Figure — a "poor-man's" superlens *i.e. a negative dielectric permittivity.*

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Source: J. W. Haus, Fundamentals and applications of nanophotonics. Woodhead Publishing, 2016.
Source: A. McGurn, Nanophotonics, Springer International Publishing, 2018.

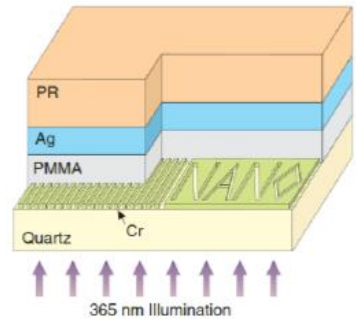
So this is the focused ion beam or you can say FIB image of the object. So here you can see that the line width of the NANO object is basically 40 nanometer. The second one shows the atomic force microscopy or AFM image that is developed with the photoresist and in the presence of silver super lens. And this is the same experiment without this silver super lens. Here the silver is basically replaced by this PMMA spacer.

Superlens: Examples

- ***Optical superlensing experiment***

  An arbitrary object "NANO" was imaged by silver superlens.

- The embedded objects are inscribed onto the 50-nm-thick chrome (Cr).
- The image of the object is recorded by the photoresist on the other side of the silver superlens.

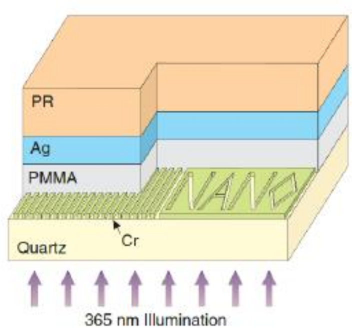


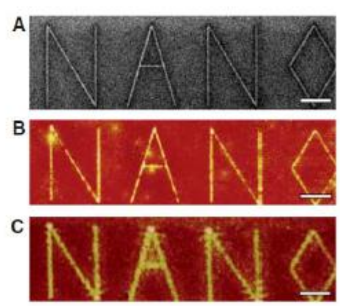
Source: N. Fang *et al.*, Science, 308(5721), 534-537, 2005.
Source: A. McGurn, Nanophotonics, Springer International Publishing, 2018.

So here you can see that the resolution has gone bad and the line width has substantially increased. So here you can see the cross section, the average cross section of the letter A shows an exposed line width of 89 nanometer. This is the case when you are having the silver super lens. But when you do not have it without the silver super lens the maximum width or you can say half maximum line width significantly increases and it goes to 321 nanometer. So, you actually lose the resolution. In that case you will not be able to see those if there are tiny objects which are very closely spaced you will not be able to image if you do not use the super lens.

Superlens: Examples

- (A) Focused Ion Beam (FIB) image of the object. The linewidth of the "NANO" object was 40 nm.
- (B) Atomic Force Microscopy (AFM) of the developed image on photoresist with a silver superlens.
- (C) AFM of the developed image on photoresist when the 35-nm-thick layer of silver was replaced by PMMA spacer as a control experiment.

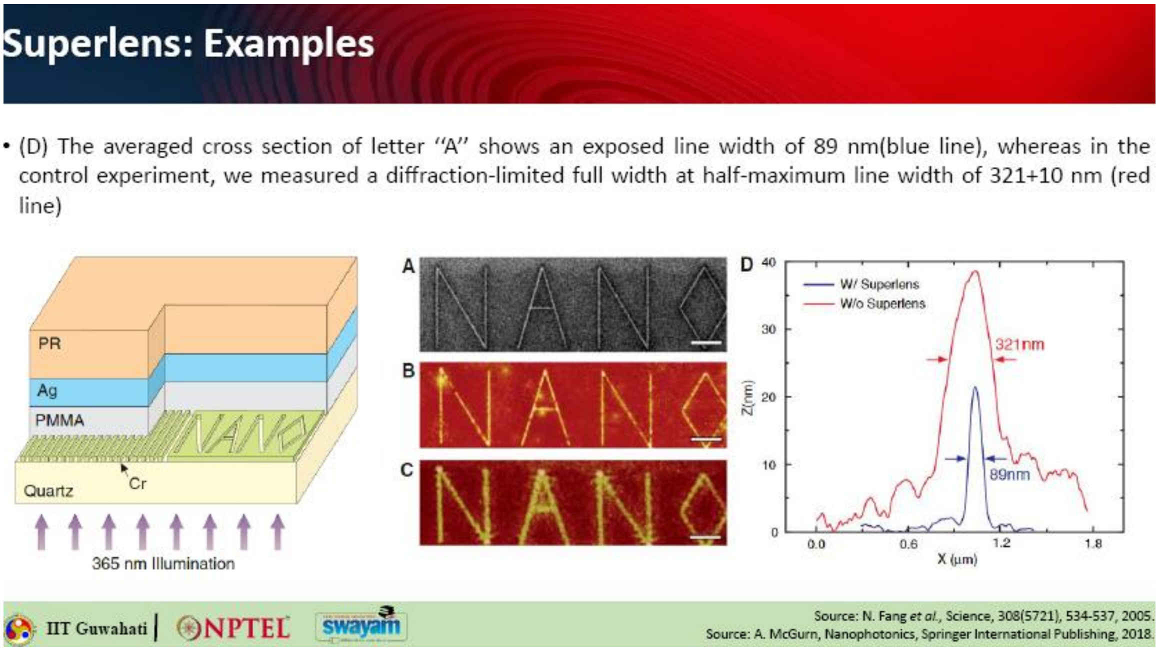




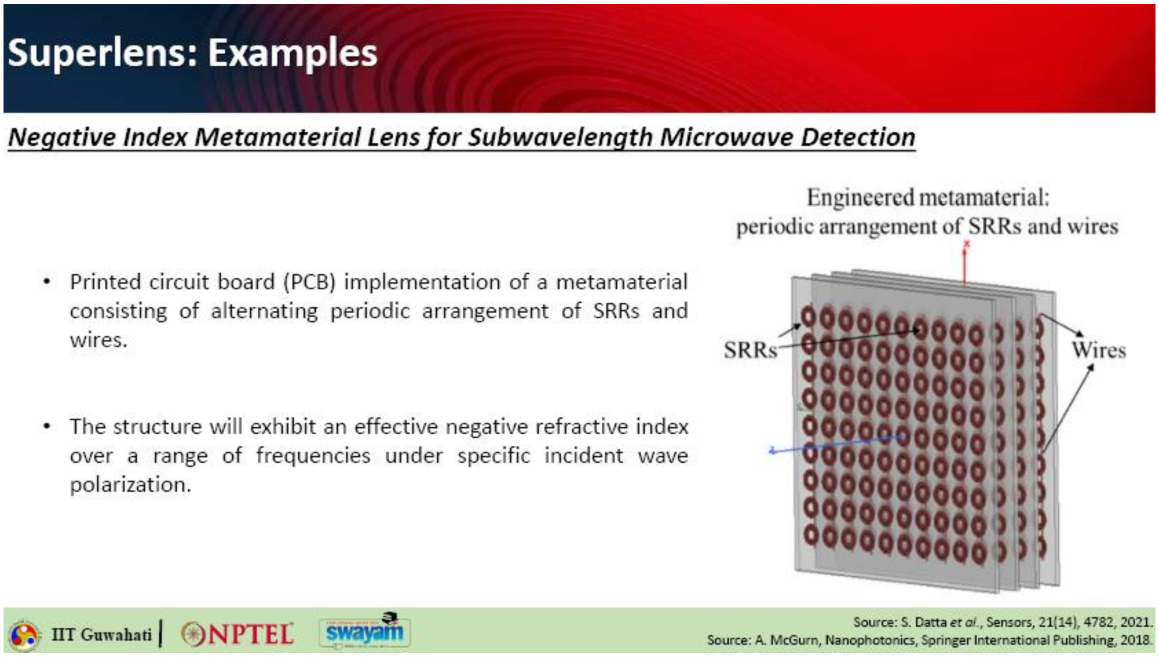
Scale bar in (A) to (C), 2mm

Source: N. Fang *et al.*, Science, 308(5721), 534-537, 2005.
Source: A. McGurn, Nanophotonics, Springer International Publishing, 2018.

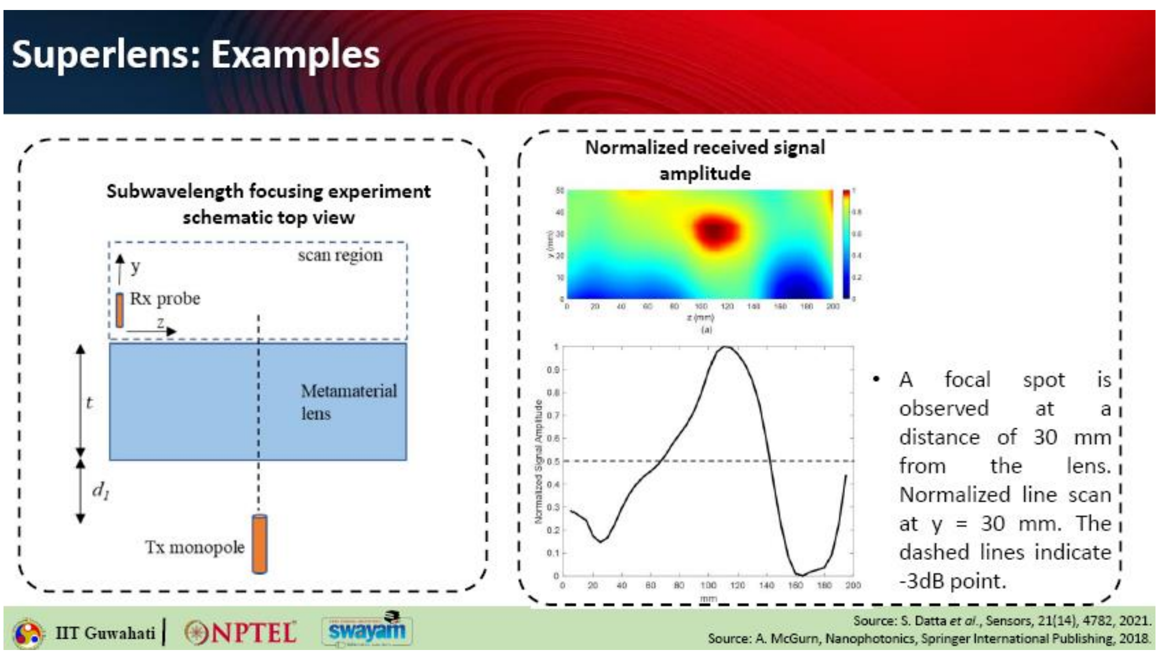
So, people have also used negative index metamaterial lens for sub wavelength microwave detection. So this is a very common structure that you see in microwave for realizing negative index metamaterial. So here it is a periodic arrangement of split ring resonators and wires. So, we have seen this kind of arrangement there are vertical wires and on top of that you have this ring kind of thing which are basically split ring resonators.



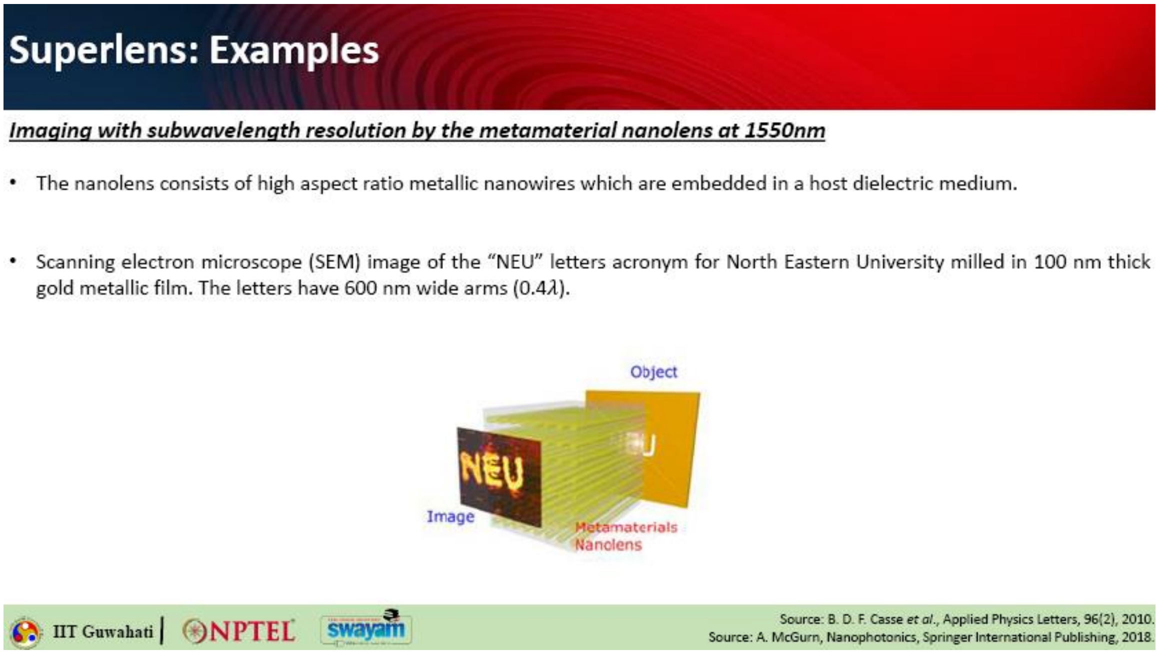
And you can print PCBs based on this and put them in a layer. And this particular structure exhibit effective negative refractive index over a range of frequencies upon certain specific incident wave polarization. Now what is the benefit of this? When you put this metamaterial lens and you have a transmitter monopole and this is the scan region.



So, when you do the scanning you could see that you are able to obtain a focal spot at a distance of 30 nanometer from the lens. And if you take the width of this focal spot you can get that it is around 30 nanometer. So, this is the full width half maxima because this is taken at the minus 3 dB point or where the amplitude is half.



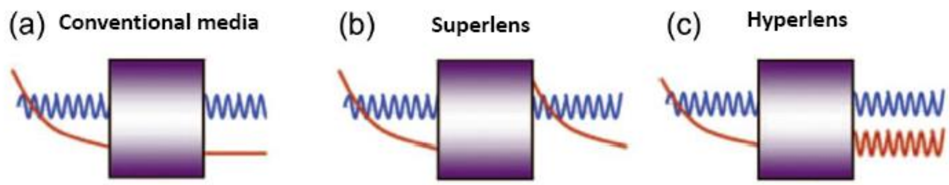
So that is the case that using this metamaterial lens you are able to focus or you are able to focus at this particular point. So there are experiments conducted at 1550 nanometer as well for imaging of sub wavelength resolution by metamaterial nano lens. So in that case you can think of nano lens which consists of high aspect ratio metallic wires which are embedded in a host dielectric medium. And then this is the object okay so and people have collected the SEM image of this NEU letter.



So, NEU stands for North Eastern University who conducted this experiment that was basically milled in a 100 nanometer thick gold metallic film as you see at the back okay. And the letters have 600 nanometer wide arms so this is the width of this arm that that you can see okay. And this is the lens and this lens actually allow you to capture the evanescent waves and amplify them so that you can clearly capture the image of this particular object. So that is the these are couple of examples of super lens okay that people have used in different different frequency regime okay. Now the whole idea here is to go for imaging at far field and that that is where hyper lens which are made from hyperbolic metamaterials would be very very interesting.

So in those cases what happens you are you will be able to convert the evanescent wave into a propagating wave and that is why you can do the imaging from far field. So once again if you look into this diagram the blue wave shows the normal propagating wave. So in all three medium the normal propagating wave simply propagates without any change in amplitude you can see that they are just propagating okay. But evanescent waves simply decays in a conventional media inside a super lens or negative index material you can see that evanescent waves get amplified but as soon as it leaves that material it decays again it remains like decaying evanescent wave. But in hyper lens evanescent wave enters and leaves as a propagating wave and that is something very cool because you can actually use this kind of material to extract sub wavelength information and image it from far field okay.

Towards 'Hyperlens' – from Hyperbolic Metamaterials



Hyperbolic Metamaterials are ***Anisotropic media***

Anisotropic media — *macroscopic optical properties depend on direction*

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

So hyperbolic metamaterials are basically anisotropic media. So when we say anisotropic media we are basically telling that the macroscopic optical properties of that media depend on directions. So let us quickly look into some of the optics of anisotropic media. The first parameter that comes to mind is permittivity tensor. So, if you remember that this epsilon is the permittivity but in the case of a linear anisotropic dielectric medium the displacement field Di looks like $D_i = \sum_j \varepsilon_{ij} E_j$.

Optics of Anisotropic Media: Permittivity Tensor

- For linear anisotropic dielectric medium: $D_i = \sum_j \varepsilon_{ij} E_j$

  The indices $i, j = 1, 2, 3$ refer to the x, y, and z components.

- The dielectric properties of the medium are therefore characterized by a 3 × 3 array of nine coefficients, $\{\varepsilon_{ij}\}$, that form the **electric permittivity tensor** ε.

  The **material equation**: $\boldsymbol{D} = \boldsymbol{\varepsilon E}$

- For most dielectric media, the electric permittivity tensor is symmetric, i.e., $\varepsilon_{ij} = \varepsilon_{ji}$. The relation between $\boldsymbol{D}$ and $\boldsymbol{E}$ is reciprocal, *i.e.*, their ratio remains the same if their directions are exchanged.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

So epsilon ij is basically the permittivity tensor. What are this indices i and j they are basically 1, 2, 3 they refer to the x, y and z components. So here you can see the dielectric property of the medium are basically characterized by a 3 by 3 array of 9 coefficients and that forms a electric permittivity tensor epsilon. And the material equation remains same that is D equals epsilon e but here epsilon is basically a tensor. Now for most of the dielectric media it has been found that electric permittivity tensor is physically symmetric.

Principal Axes and Principal Refractive Indices

- The elements of the permittivity tensor depend on how the coordinate system is chosen relative to the crystal structure.
- However, a coordinate system can always be found for which the off-diagonal elements of ε_{ij} vanish, so that:

$$D_1 = \varepsilon_1 E_1, \qquad D_2 = \varepsilon_2 E_2, \qquad D_3 = \varepsilon_3 E_3$$

where $\varepsilon_1 = \varepsilon_{11}$, $\varepsilon_2 = \varepsilon_{22}$ and $\varepsilon_3 = \varepsilon_{33}$.

- ***E*** and ***D*** are parallel along these particular directions so that if, for example, ***E*** points in the x direction, then ***D*** also does.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

So epsilon ij is same as epsilon ji. So in that case the relationship between D and E are also reciprocal that means their ratios remain same if their directions are changed. So in that case the elements of the permittivity tensor depend on how the coordinate system is chosen relative to a crystal structure. However it can be done very smartly because you can always find a coordinate system where the off diagonal elements of this epsilon ij will vanish so that you will be able to write $D_1 = \varepsilon_1 E_1, D_2 = \varepsilon_2 E_2, D_3 = \varepsilon_3 E_3$ where $\varepsilon_1 = \varepsilon_{11}, \varepsilon_2 = \varepsilon_{22}$ and $\varepsilon_3 = \varepsilon_{33}$. So these are basically the diagonal coefficients and all off diagonal elements are basically 0. Now E and D are parallel along this particular directions so that if for example E points in x direction D has to also point in x direction.

Principal Axes and Principal Refractive Indices

- This coordinate system defines the **principal axes** and principal planes of the crystal.
- The permittivities ε_1, ε_2, and ε_3 correspond to refractive indices

$$n_1 = \sqrt{\varepsilon_1/\varepsilon_0}, \qquad n_2 = \sqrt{\varepsilon_2/\varepsilon_0}, \qquad n_3 = \sqrt{\varepsilon_3/\varepsilon_0}$$

where ε_0 is the permittivity of free space; these are known as the **principal refractive indices.**

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

Now this coordinate system defines the principal axis and the principal planes of the crystal. So, the permittivity is ε_1, ε_2, and ε_3 will correspond to the refractive indices n_1, n_2 and n_3. You can write $n_1 = \sqrt{\varepsilon_1/\varepsilon_0}, n_2 = \sqrt{\varepsilon_2/\varepsilon_0}, n_3 = \sqrt{\varepsilon_3/\varepsilon_0}$. What is epsilon 0? There is a permittivity of free space and this n 1, n 2 and n 3 are known as the principal refractive indices. So that brings us to the different classification of crystals.

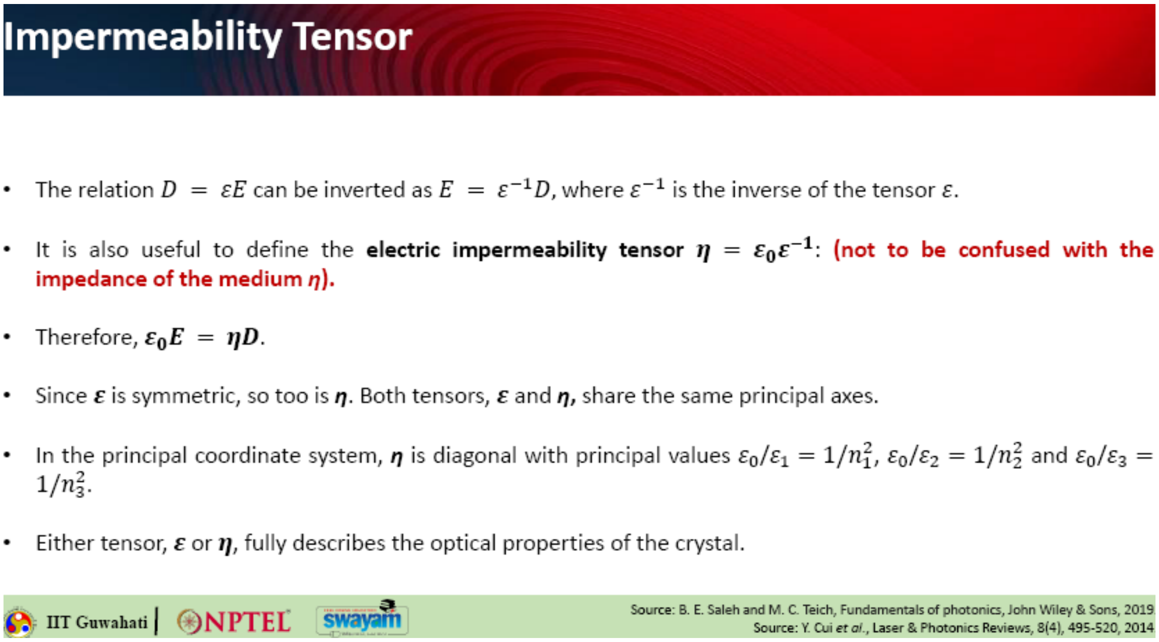
Biaxial, Uniaxial, and Isotropic Crystals

- **Biaxial Crystals** — the three principal refractive indices are different**.**
- **Uniaxial Crystals** — For crystals with certain symmetries, two of the refractive indices are equal ($n_1 = n_2$).

  In this case, the indices are usually denoted $n_1 = n_2 = n_o \Rightarrow$ **Ordinary indices**
  $n_3 = n_e \Rightarrow$ **Extraordinary indices**
- **Positive uniaxial** $\Rightarrow n_e > n_o$, and **Negative uniaxial** $\Rightarrow n_e < n_o$.
- The *z* axis of a uniaxial crystal is called the **optic axis.**
- In certain crystals with even greater symmetry (those with cubic unit cells, for example), all three indices are equal and the medium is **optically isotropic**.

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

So you can have biaxial crystals where the three principal refractive indices are basically different. You can have uniaxial crystal. So, these are the crystals with certain symmetries where two of the refractive indices are equal say $n_1=n_2$ and the third one is different. So, in case the indices are denoted like this n_1 is written as n is equal to n_2 can be written as n_o that is the ordinary indices and n_3 is basically n_e that is the extraordinary index.



So that will be index. So you can call the crystal to be a positive uniaxial crystal when the extraordinary refractive index is larger than the ordinary refractive index and you can call it negative if it is other way around that is n e is lesser than n o. And the z axis of the uniaxial crystal is called the optic axis. So, in certain crystals with even greater symmetry such as those with cubic unit cells all three indices will be equal $n_1 = n_2 = n_3$ and those are typically the optically isotropic crystals. Now the next important topic is impermeability tensor. Now when we read about this relation t equals epsilon e you can always think of an invert relationship that is basically e equals epsilon inverse d where epsilon inverse is basically the inverse of the tensor epsilon and it is useful to define this tensor as a electric impermeability tensor eta.

So eta is defined as epsilon naught epsilon inverse and do not confuse this eta with the impedance of the medium eta. These two are different concepts. So here you can write epsilon naught e equals eta d. So epsilon is symmetric that makes eta also symmetric.

So both epsilon eta they share the same principal axis. Now in this case the principal coordinate system in the principal coordinate system eta is diagonal with the principal values given as $\varepsilon_0/\varepsilon_1 = 1/n_1^2, \varepsilon_0/\varepsilon_2 = 1/n_2^2$ and $\varepsilon_0/\varepsilon_3 = 1/n_3^2$. So, either this tensor permittivity or impermeability they can fully describe the optical properties of the crystal. So you can either use any one of them. The next important parameter or the concept is called index ellipsoid.

Index Ellipsoid

- The **index ellipsoid** (also called the **optical indicatrixz**) is the quadric representation of the electric impermeability tensor $\eta = \boldsymbol{\varepsilon_0 \varepsilon^{-1}}$:

$$\sum_{ij} \eta_{ij}\, x_i x_j = 1, \qquad i, j = 1, 2, 3.$$

- If the principal axes were to be used as the coordinate system:

$$\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} = 1 \qquad \textbf{Index Ellipsoid} \qquad \text{(L26.3)}$$

- The optical properties of the crystal are completely described by the *index ellipsoid*.

- *For a uniaxial crystal, the index ellipsoid reduces to an ellipsoid of revolution; for an isotropic medium it becomes a sphere.*

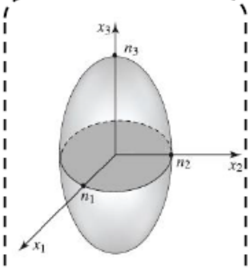


Figure: An index ellipsoid.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

So index ellipsoid is also called as optical indicators. So this is a quadratic representation of the electric impermeability tensor which is eta and as you have seen it is written as epsilon naught epsilon inverse. So you can write this equation the summation over ij eta ij xi xj should be equal to 1. What is ij? They are 1 to 3. They show the different directions. So if the principal axes were to be used as the coordinate system you come up with this equation that is $\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} = 1$ and that is the equation of a ellipsoid and we call this an index ellipsoid.

And this is how it will look like. So n_1, n_2 and n_3 are these points and x_1, x_2 and x_3 are the coordinates or you can say the principal axes. So the optical properties of the crystal are completely described by its index ellipsoid. For a uniaxial crystal the index ellipsoid reduces to an ellipsoid of revolution. So it looks like a ellipsoid like this. So, it can be a prolate or oblate shape depending on whether it is a positive uniaxial or negative uniaxial system.

Dispersion Relation: The k Surface

- Consider x-y-z be a coordinate system that coincides with the principal axes of a crystal.
- ***Normal Modes:*** The normal modes of light propagation through such crystal are the states of polarization that are not changed when the wave is transmitted through the system.
  - Normal modes for propagation in z directions are linearly polarized waves along x and y
- To determine the normal modes for a plane wave traveling in the direction $\hat{\mathbf{u}}$ and the material equation $\mathbf{D} = \varepsilon\mathbf{E}$ given, Maxwell's equations reduce to:

$$k \times H = -\omega D \qquad \text{(L26.4)}$$

$$k \times E = \omega \mu_o H \qquad \text{(L26.5)}$$

Substituting (L26.5)into (L26.4) leads to:

$$k \times (k \times E) = -\omega^2 \mu_o D \qquad \text{(L26.6)}$$

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

Prolate means like a kind of rice grain and oblate looks like a gems. And for isotropic medium where n_1, n_2 and n_3 are equal you can understand this ellipsoid will reduce to a sphere. So in all direction the property is the same and that is what isotropic means. So index ellipsoid for a isotropic material is a sphere. The next important concept is the dispersion relation the k surface.

Dispersion Relation: The k Surface

$$k \times (k \times E) = -\omega^2 \mu_o D \qquad \text{(L26.6)}$$

Using the relation $\boldsymbol{D} = \boldsymbol{\varepsilon E}$, we obtain:

$$k \times (k \times E) + \omega^2 \mu_o \varepsilon E = 0 \qquad \text{(L26.7)}$$

- This vector equation, which **E** must satisfy, translates to three linear homogeneous equations for the components E_1, E_2, and E_3 along the principal axes, written in the matrix form:

$$\begin{bmatrix} n_1^2 k_o^2 - k_2^2 - k_3^2 & k_1 k_2 & k_1 k_3 \\ k_2 k_1 & n_2^2 k_o^2 - k_1^2 - k_3^2 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & n_3^2 k_o^2 - k_1^2 - k_2^2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(L26.8)}$$

where (k_1, k_2, k_3) are the components of $\boldsymbol{k}$, $k_o = \omega / c_o$, and (n_1, n_2, n_3) are the principal refractive indices.

When the determinant of the matrix is set to zero, a nontrivial solution to these equations yields:

$$\sum_{j=1,2,3} \frac{k_j^2}{k^2 - n_j^2 k_o^2} = 1 \qquad \textbf{Dispersion Relation: } \boldsymbol{k} \textbf{ surface}$$

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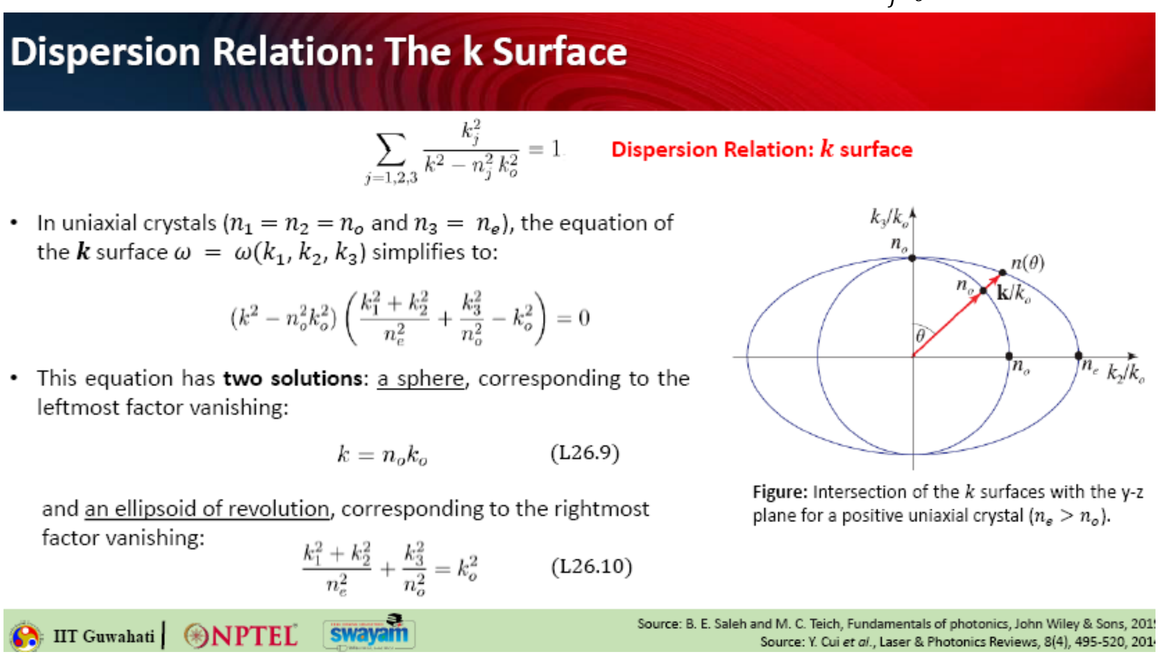
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

So consider x, y, z be a coordinate system that coincides with the principal axes of a crystal. Now you are considering light propagating through it. What will be the normal modes? Now the normal modes of a light propagation through this kind of a crystal are basically those states of polarization that are not changed when the wave is transmitted through the system. So if you consider normal modes for propagation in z direction that will be linearly polarized waves along x and y.

So there are two normal modes in this case. So to determine the normal modes for a plane wave that is traveling in the direction of u cap it can be any arbitrary direction. The material equation D equals epsilon e that allows you to write down the Maxwell's equation like this. So we know this from the Maxwell's equation and when you put this value into this equation the value of H you can have everything in terms of E and then D you can replace with epsilon E and you finally come to this particular equation. So this vector equation which the electric field has to satisfy it basically translates to three linear homogeneous equations for the components E1, E2 and E3.

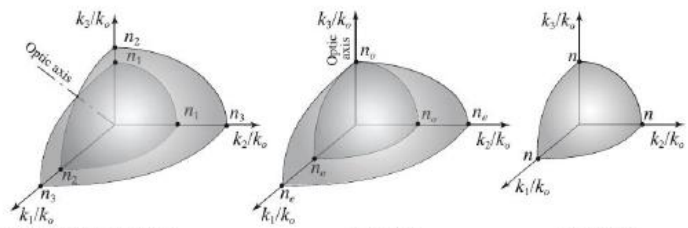
So these are basically the components along the three principal axis. And you can write down the vector equation in terms of a matrix or in the form of a matrix and it looks like this. So here k_1, k_2, k_3 are basically the components of the wave vector k. k naught is the vacuum wave number which is omega over c naught and n_1, n_2, n_3 are the three principal refractive indices. So when the determinant of this matrix goes to 0 a non-trivial solution for this equations yield the dispersion relation. So, this is a compact form of the equation it reads $\sum_{j=1,2,3} \frac{k_j^2}{k^2 - n_j^2 k_o^2} = 1$.



So that is the equation of the dispersion relation or the k surface. So, in the case of uniaxial crystals you have $n_1 = n_2$ which are basically the ordinary one n_0 and n_3 is ne and in that case the equation of the k surface so the equation looks like omega equals omega. Now this is basically function of k so k_1, k_2, k_3 and you can write this equation in this form. So here you will see that this particular equation has got two solutions. One of the sphere that corresponds to the left most factor vanishing so you can write k equals n naught k naught and other one is this one that corresponds to a ellipsoid of revolution that vanishes this factor and you can write that equation as this plus this equals k naught square.

The k Surface

- The **k** surface is a centrosymmetric surface comprising two sheets, each corresponding to a solution (a normal mode).
- For biaxial crystals ($n_1 < n_2 < n_3$), the two sheets meet at four points, defining two optic axes.
- In the uniaxial case ($n_1 = n_2 = n_o$, $n_3 = n_e$), the two sheets become a sphere and an ellipsoid of revolution that meet at only two points, thereby defining a single optic axis (the *z* axis).
- In the isotropic case ($n_1 = n_2 = n_3 = n$), the two sheets degenerate into a single sphere.



One Octant of the k surface for (*a*) Biaxial (*b*) Uniaxial (*c*) Isotropic

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

So this is how you can put it on the k surface. So here is basically the plot of the k surfaces with the yz plane so you are only showing k_2 and k_3 here and we are considering a positive uniaxial crystal and that is why you are able to see ne larger than no. So here you see the sphere that corresponds to this one and this is the ellipsoid which is this one. Now when you say k surface what are k surface? It is a centrosymmetric surface which comprises two sheets so each corresponding to one solution that is a normal mode. Now in the case of biaxial crystal and if you consider n1 smaller than n_2 smaller than n_3 then this sheets they meet at four points defining two optic axis. So a biaxial crystal will have two optic axis and you can see that if you consider one octant of the k surface so there you will see that there are already you can see the one optic axis in the other octant you will be able to see the other one.

In the case of uniaxial crystal where $n_1 = n_2 = n_o$ that is the ordinary refractive index n_3 is the extraordinary one the two sheets become a sphere and a ellipsoid of revolution and they meet at only two points that means that will have only one single optic axis and that is also along z. So you can actually see they are meeting here and also at the bottom so this is the top you will also have same thing happening at the bottom so you can actually see that this is along the z axis and in the case of isotropic where n_1 ,n_2 , n_3 all are equal to n the two sheets degenerate into a single sphere so they overlap and you will just get this one right so this builds up the foundation to introduce hyperbolic media here. Now let us look into the concept of isotropic media which has got permittivity and permeability tensors with positive and negative principal values. Till now we have considered all the positive values. Now the designation of the medium as double positive single negative double negative will all then be dependent on the direction of propagation of the wave as well as on its propagation polarization.

Introduction to Hyperbolic Media

- Anisotropic media may have permittivity and permeability tensors with positive or negative principal values.
- The designation of the medium as double-positive (DPS), single negative (SNG), or double negative (DNG) is then dependent on the direction of propagation of the wave as well as on its polarization.
- To demonstrate one facet of this rich behavior, we consider media endowed with <u>isotropic magnetic properties and positive permeability μ</u>, but with <u>anisotropic dielectric properties</u>.
- Wave propagation in anisotropic media is described for media with positive principal values of the electric permittivity tensor $\boldsymbol{\varepsilon}$, namely ε_1, ε_2 and ε_3.
- If these parameters take on <u>mixed signs</u> instead, wave propagation exhibits a number of unusual properties.

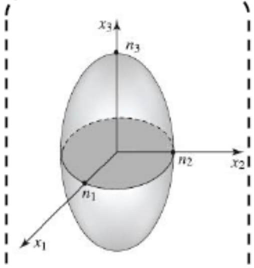

Figure: An index ellipsoid.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014

So to demonstrate one kind of this rich behavior so let us consider a media with isotropic magnetic properties and let us consider positive permeability but we have anisotropic dielectric properties. So one more time we are considering mu to be isotropic and positive so we are not we were just keeping it fixed and same throughout but then epsilon is anisotropic and in this in such a media wave propagation will be described by its the positive values of the electric permittivity tensor epsilon that is epsilon 1 epsilon 2 and epsilon 3 and in the case this these parameters when they take up mixed science the wave propagation will start showing some unusual properties which we have not seen in the previous cases. So to understand this better let us first consider the special case of a uniaxial crystal with epsilon 1 and epsilon 2 to be positive and compare the wave propagation when the media has got a positive epsilon 3 and a negative epsilon 3. So let us first consider the positive epsilon 3 case and here we have seen that the dispersion surfaces which are also called the k surfaces or you can say omega k for the ordinary wave are basically like a sphere. So, you have got k equals n naught k naught where n naught is basically square root of epsilon 1 over epsilon 0 but the extraordinary wave it saw a quadric surface and that is basically the ellipsoid.

Towards Hyperbolic Media

- For simplicity, we consider **the special case of a uniaxial medium** with $\varepsilon_1 = \varepsilon_2 > 0$, and compare wave propagation in media with **positive** and **negative** ε_3.

- The dispersion surfaces, also called the $\boldsymbol{k}$ surfaces or surfaces of constant $\omega(\boldsymbol{k})$, **for the ordinary wave** are sphere of radius:

$$k = n_o k_o$$

where $n_o = \sqrt{\varepsilon_1/\varepsilon_0}$

- **For the extraordinary wave**, a quadric surface:

$$\frac{k_1^2 + k_2^2}{\varepsilon_3} + \frac{k_3^2}{\varepsilon_1} = \frac{k_o^2}{\varepsilon_0} \qquad \text{(L26.11)}$$

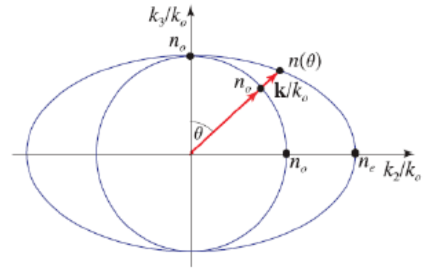


Figure: Intersection of the k surfaces with the y-z plane for a positive uniaxial crystal ($n_e > n_o$).

Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

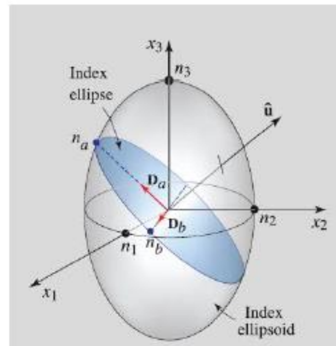
So you can write this as this particular ellipsoid equation. Now in the uniaxial crystals the index ellipsoid is basically an ellipsoid of revolution. So this is how you can represent that. Now if you consider a wave that is traveling along a direction arbitrary direction of u cap and it forms an angle theta with the optic axis. So what will happen in that case? So, it will have an index ellipse with half lengths n naught and n theta. Now what is this new concept called index ellipse? So this basically when you have this direction given to you that is the direction in which the wave is propagating you can draw a plane through the origin of the index ellipsoid that is normal to u cap and the intersection of this plane with the ellipsoid will give you this index ellipse and the half lengths of the major and the minor axis of the index ellipse are basically the refractive indices n a and n b of the two normal modes.

Uniaxial Crystal: Positive ε_3

- In uniaxial crystals ($n_1 = n_2 = n_o, n_3 = n_e$), the index ellipsoid of is an ellipsoid of revolution.
- For a wave (whose direction of travel is û) forms an angle θ with the optic axis, the *index ellipse* has half-lengths n_o and $n(\theta)$
- $n(\theta)$ is determined from the index ellipsoid equation by making the substitutions $x_1 = n(\theta)\cos\theta$, $x_2 = 0$, and $x_3 = -n(\theta)\sin\theta$.
- The result is: $\dfrac{1}{n^2(\theta)} = \dfrac{\cos^2\theta}{n_o^2} + \dfrac{\sin^2\theta}{n_e^2}$ (L26.12)

  so that the normal modes have refractive indices $n_b = n_o$ and $n_a = n(\theta)$.

$$\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} = 1$$



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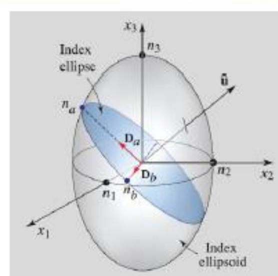
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

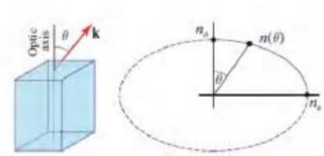
Now here we have seen that this is n o and this will be n theta. So n theta needs to be determined from the index ellipsoid equation. So there you can actually make the so this is the index ellipsoid equation. So there if you make certain substitution like $x_1 = n(\theta)\cos\theta, x_2 = 0$, and $x_3 = -n(\theta)\sin\theta$ you can get this is the equation that relates your n theta with n o and n e. So you can actually obtain that the normal modes will have refractive indices so n b is basically n o and n a is basically your n theta. So why this is required this is basically a generic case if the wave propagation was along the axis optic axis then it is very simple but if it is not then this is how you should be able to find.

Uniaxial Crystal: Positive ε_3

The normal modes have refractive indices $n_b = n_o$ and $n_a = n(\theta)$.

- The first mode, called the **ordinary wave**, has a refractive index n_o regardless of θ.
- In accordance with the ellipse, the second mode, called the **extraordinary wave,** has a refractive index $n(\theta)$ that varies from n_o when $\theta = 0°$, to n_e when $\theta = 90°$.





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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

You have to find out the normal modes and what are the refractive indices of the normal mode. So you have to take help of the index ellipse. So normal modes have refractive indices $n_b = n_o$ and $n_a = n(\theta)$ and the first mode which is the ordinary wave as you can see that it has got a refractive index that is independent of theta. But in accordance with the ellipse the second mode which is called the extraordinary wave it has got a refractive index n theta and that varies from n_o to n_e when your value of theta is changing from 0 to 90 degree.

Uniaxial Crystal: Positive ε_3

- When ε_3 is positive, the extraordinary $\boldsymbol{k}$ surface is an ellipsoid of revolution (**Figure**).
- The ordinary and extraordinary waves have refractive indices n_o and $n(\theta)$, respectively, where $n(\theta)$ varies between n_o and $n_e = \sqrt{\varepsilon_3/\varepsilon_0}$, as the angle θ varies between 0 and 90°.

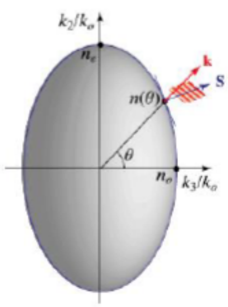

Figure: Contour of the $\boldsymbol{k}$ surface in the k_2–k_3 plane for a uniaxial anisotropic medium with principal values of the dielectric tensor $\varepsilon_1 = \varepsilon_2 > 0$ with $\varepsilon_3 > 0$.

The contour is shown only for the extraordinary wave (for the ordinary wave, it is a sphere). The $\boldsymbol{k}$ contour is an ellipse and the medium is DPS for all directions of propagation.

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Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: Y. Cui *et al.*, Laser & Photonics Reviews, 8(4), 495-520, 2014.

So that can be seen here. So, this is another representation so if you have the axis this is the k vector that is making a theta angle with it and this this one is your basically n_o and this is n_e and this is the angle theta. So, when theta is 0 degree your n theta is basically n_o and when theta is 90 your n theta is basically n_e. So you can actually land up anywhere depending on the direction at which the wave is propagation through the crystal. So, we understood that when ε_3 is positive the extraordinary k surface is basically an ellipsoid or you can say it is an ellipsoid of revolution it is a 3D thing. Now in this particular figure what is shown is the contour of the k surface in k_2 k_3 plane and we have considered a uniaxial anisotropic medium where we consider ε_1 and ε_2 to be equal and positive and ε_3 is also taken to be positive.

Uniaxial Crystal: Negative ε_3

- When ε_3 is negative, the extraordinary $\boldsymbol{k}$ surface is instead a **hyperboloid** of revolution (in two sheets) and the material is known as a **hyperbolic medium**.

- The refractive index $n(\theta)$ for an extraordinary wave at an angle θ in the $k_2 - k_3$ plane is then given by:

$$\frac{1}{n^2(\theta)} = \frac{\cos^2\theta}{n_o^2} - \frac{\sin^2\theta}{n_e^2} \quad \text{(L26.13)}$$

where $n_e = \sqrt{|\varepsilon_3|/\varepsilon_0}$

So we have marked the theta angle here for reference. So this contour is shown only for the extraordinary wave. You already know that for ordinary wave is basically sphere. So here the k contour is ellipse and this medium is DPS double positive for all the direction of propagation because in all cases mu we have already taken as positive and isotropic and in all cases ε_1 ,ε_2 and ε_3 they are all positive. So this is a DPS medium right. So, this we have already seen is that n o is basically the ordinary index and n e that is given a square root of ε_3 over epsilon naught ok that can vary from so n theta can actually vary between n_o and n_e when the angle of theta is changing from 0 to 90 degree and ε_3 is positive here so n_e is always positive.

Hyperbolic Media

$$\frac{1}{n^2(\theta)} = \frac{\cos^2\theta}{n_o^2} - \frac{\sin^2\theta}{n_e^2} \quad \text{(L26.13)} \qquad \text{where } n_e = \sqrt{|\varepsilon_3|/\varepsilon_0}$$

- As θ increases from 0 to $\theta_{\max} = \tan^{-1}(n_e/n_o)$, the refractive index $n(\theta)$ increases from n_o to ∞, signifying that the wavelength in the medium becomes progressively smaller and the wave slows to a halt at $\theta = \theta_{max}$.

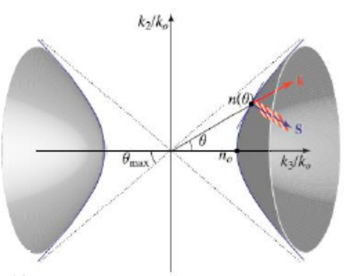

Figure: Contour of the $\boldsymbol{k}$ surface in the k_2–k_3 plane for a uniaxial anisotropic medium with principal values of the dielectric tensor $\varepsilon_1 = \varepsilon_2 > 0$ with $\varepsilon_3 < 0$.

The $\boldsymbol{k}$ contour is a hyperbola and the waves can propagate only in directions that lie within a cone of half-angle θ_{max}.

Outside this cone, the medium acts like a SNG medium, which does not support propagating waves.

So that makes it a DPS medium. Now consider the case when ε_3 is negative. So in that case the extraordinary k surface which was previously a ellipsoid that will take the form of a hyperbola or you can say it is a hyperbola of revolution ok and this such medium is known as hyperbolic medium. So, in this case the refractive index n theta or for the extraordinary wave at an angle theta in the k_2 k_3 plane will be given as this 1 over so you can do the same process that we have seen before and you will get this particular expression. Here n_e is basically square root of mod ε_3 over ε_0 why not because ε_3 here is negative. So if you do that you also know the theta can increase from 0 to theta max. So what is theta max? Theta max is given as tan inverse of n_e over n_o ok and that can make your refractive index n theta starting from n_0 to infinity.

It means you can go to a very very large refractive index using this kind of a medium. So this signifies that the wavelength inside this medium can be very very small ok because n is almost infinity very large and when n is very large you can also slow down the wave to almost a halt it is like extremely slow speed of wave propagation when your theta is approaching theta max. So this is the same contour of the k surface that you have seen before just for the case of here it is epsilon 3 is negative. So, the contour of the k surface in k_2 k_3 plane for uniaxial anisotropic medium where epsilon 3 is negative ε_1 and ε_2 are same and positive. So, the k contour is basically hyperbola and the wave can propagate in the direction that lie between the cone this particular cone ok of the half angle theta max.

Hyperlens based on Hyperbolic Metamaterials

- An **important property** of a hyperbolic medium is that for a plane wave with wave vector components (k_1, k_2, k_3), no matter how large k_1 and k_2, there is a real value of k_3 that satisfies:

$$\frac{k_1^2 + k_2^2}{\varepsilon_3} + \frac{k_3^2}{\varepsilon_1} = \frac{k_o^2}{\varepsilon_0}$$

when ε_3 is negative, indicating that the wave can propagate through the medium.

- This signifies that spatial frequencies greater than an inverse wavelength in any plane do not correspond to evanescent waves, as they do for ordinary media, but rather can be transmitted over long distances.
- Although propagation is accompanied by Fresnel diffraction, the hyperbolic medium has a transfer function with no spatial frequency cutoff.
- Moreover, Fresnel diffraction may be significantly reduced in a hyperbolic medium.

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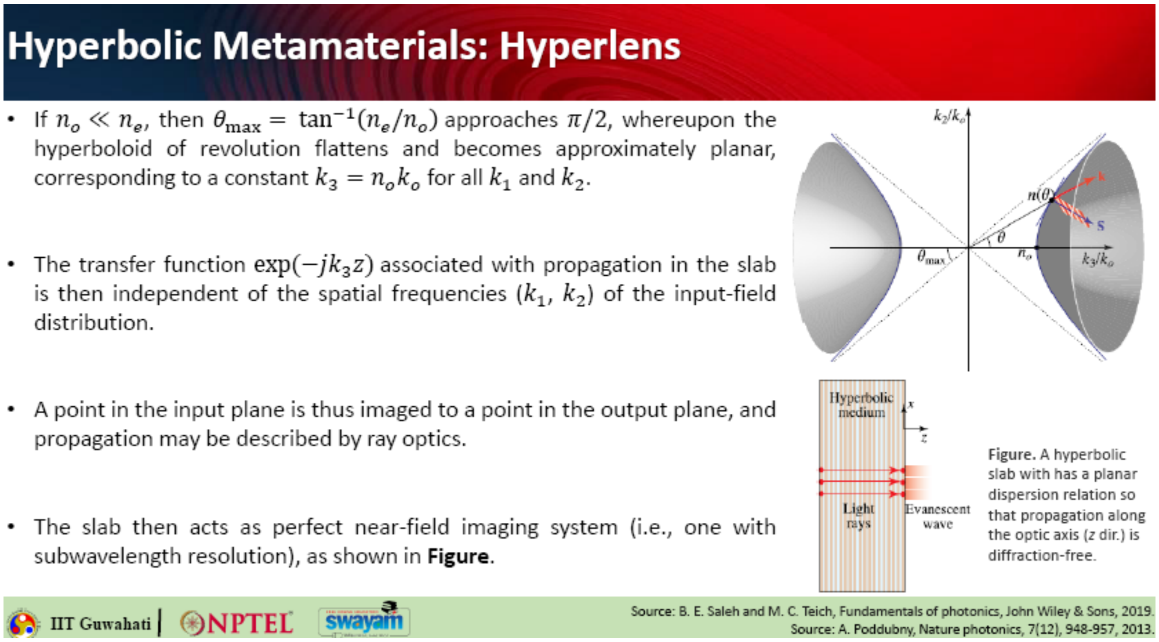
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019.
Source: A. Poddubny, Nature photonics, 7(12), 948-957, 2013.

So what is theta max? So this is theta max that you can see here ok. So if you actually go beyond theta max the medium will act like a single negative medium and it will not support propagating waves. So you have to be within this half angle theta max ok. So you be within this cone. Now how do you make hyper lengths based on this hyperbolic metamaterial? Now an important property that you have seen of hyperbolic medium is that the plane wave with wave vector components k_1 k_2 and k_3 it does not matter how large your k_1 and k_2 are ok there is always a real value for k_3 that will satisfy this equation. Now that tells you that when epsilon 3 is negative ok the wave can actually propagate through that medium ok.



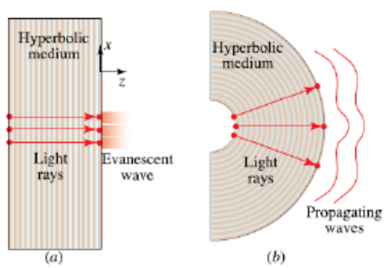
And this signifies that the spatial frequencies greater than an inverse wavelength in any plane do not correspond to evanescent wave in such medium ok. In ordinary medium they do but here it is not here they are basically they can be transmitted over long distances and this is the beauty of this hyperbolic metamaterial. So although propagation is accompanied by Fresnel diffraction the hyperbolic medium has a transfer function with no spatial frequency cut off. So it can actually take care of all spatial frequencies.

So moreover Fresnel diffraction may be significantly reduced in a hyperbolic medium. We will not go into that details of Fresnel diffraction here but we will just tell you the concept how this hyperbolic metamaterials can be used as a hyper lens. So, we have seen that n_o is in the case n_o is much much lesser than n_o ok. What you can have you can have theta max which is given as tan inverse. So, n_e over n_o and n_o is very very small in that case theta max approaches 90 degree.

So this surface 90 degree means like this. So this surface is almost becoming a plane ok. So, the hyperbola of revolution flattens and therefore approximately becomes planar and when it becomes planar you can see that it will correspond to a constant k_3 ok. So, when it becomes planar so it will have a constant k_3 for all the values of k_1 and k_2 right and the transfer function e to the power minus j k 3 z which is associated with the propagation in the slab will then be independent of the spatial frequencies k_1 and k_2 of the input field distribution and how it helps a point in the input plane is thus imaged to a point in the output plane and the propagation may be described by ray optics. This particular slab in this situation can act as perfect near field imaging system ok like this. So, you can actually have sub wave length resolution but because of this hyperbolic medium you can use ray optics information and get this tiny information converted into waves ok.

Hyperbolic Metamaterials: Hyperlens

- The slab may be curved in such a way that the image is geometrically magnified, as shown in **Figure**.
- If the spatial frequency components of the magnified image become smaller than the inverse wavelength, they generate propagating waves that may be captured with the help of a conventional lens, thereby forming a far-field image.
- A cylindrical slab such as this is known as a **hyperlens**.



(a) A hyperbolic slab with $\varepsilon_3 < 0$ and $0 < \varepsilon_1 = \varepsilon_2 \ll |\varepsilon_3|$ has a planar dispersion relation (k_3 = constant) so that propagation along the optic axis (z direction) is diffraction-free.

(b) A hyperbolic slab curved to form an inhomogeneous cylindrical structure, with the local optic axis pointing in the radial direction, acts as a magnifier of subwavelength details. If the details of the magnified image are larger than the wavelength, it produces propagating waves in the outer medium.

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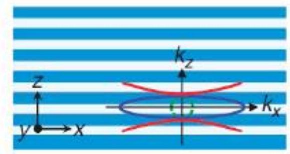
Source: B. E. Saleh and M. C. Teich, Fundamentals of photonics, John Wiley & Sons, 2019
Source: A. Poddubny, Nature photonics, 7(12), 948-957, 2013

So this is basically showing a hyperbolic slab with a planar dispersion relation. How do you get planar dispersion relation? When theta max approaches pi by 2 you will get like a plane ok and it is becoming k_3 becomes constant for all values of k_1 and k_2 so you are getting a planar dispersion relation. So the propagation along your optic axis direction that is the z direction which is this one will become diffraction free. So this kind of slab can then be curved in a way that image can be geometrically magnified as you can see in this particular figure ok. So here ok you are still having evanescent waves ok but then when you curve it like this ok you can actually have this three components magnified and you are getting a much magnified version of image here and this can actually give rise to this can be imaged as propagating waves from the far field. So, the concept here is if the spatial frequency components of the magnitude image becomes smaller than the inverse wavelength they generate propagating waves.

Hyperlens: Applications

- Hyperbolic metamaterials in a curved geometry result in hyperlenses (with corresponding dispersions represented in cylindrical coordinates).
- For a subwavelength object placed at the inner boundary, wave propagation along the radial direction gradually compresses its tangential wave-vectors, resulting in a magnified image at the outer boundary.
- The magnified image, once larger than the diffraction limit, will be resolved in the far field.



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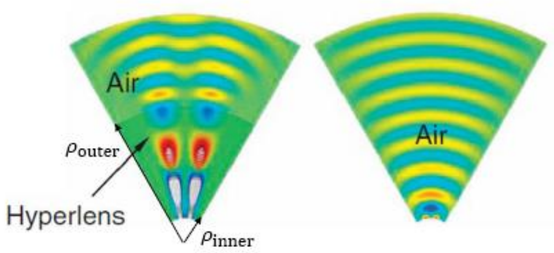
Source: D. Lu *et al.*, Nature communications, 3(1), 1205, 2012.
Source: A. Poddubny, Nature photonics, 7(12), 948-957, 2013.

Now this is the funda behind this hyper lens and once you generate propagating waves they may be captured with the help of conventional lens. So you do not need a near field microscopy technique to do this you can simply do the far field image. So let us look into this diagram once again because this is the heart and soul of hyper lens. So here you have a hyperbolic slab which has got epsilon 3 negative what is happening to epsilon 1 and epsilon 2 both are positive ok and equal but they are much smaller than the value of epsilon 3 which is the modulus of epsilon 3. This is required to get a planar dispersion function ok so that you get k_3 equals constant ok and that will make your propagation along the optic axis that is in z direction diffraction free.

So you can simply propagate like this without deviating here and there. Now when you make a hyperbolic slab curved into something like this so it form a inhomogeneous cylindrical structure with the local optic axis pointing in the radial direction ok. So this acts as a magnifier of the sub wavelength details. So if you have some objects a few objects which are in sub wavelength scale you can actually magnify them and bring it to a larger scale. And if the details of the magnified image are larger than the wavelength they can produce propagating waves and that can go to the outer medium.

So this is the concept. So why you are using this curved thing? Carving allow you to separate them understood and getting this planar dispersion relation makes it diffraction free. So you can specifically say that this point is now magnified onto this one without any kind of overlap understood. So that allows you to do the imaging and this cylindrical slab is known as the hyperlens. So here are some examples if you can see.

Hyperlens: Applications



Two sub-diffraction-limited line sources separated by a distance of ~80 nm can be clearly resolved by using hyperbolic metamaterials (left), but not by air alone (right).

- In the simulation, the radii of the hyperlens at the inner and outer boundaries are ρ_{inner} =240nm and ρ_{outer} =1,200 nm, respectively.
- The permittivity for metal and dielectric in the hyperbolic hyperlens is 2.3− 0.3i and 2.7, respectively.
- The filling ratio of metal in both hyperlenses is 50%.

Source: D. Lu *et al.*, Nature communications, 3(1), 1205, 2012.
Source: A. Poddubny, Nature photonics, 7(12), 948-957, 2013.

So here are some sub wavelength objects that were imaged using hyperlens. So hyperbolic metamaterials in curved geometry they result in hyperlenses ok. And with corresponding dispersion represented in cylindrical coordinates here you can see we are using cylindrical coordinate system rho theta z ok. This is planar so you are simply having Cartesian coordinate system but in when you convert into cylindrical shape you bend it you use this coordinate system. Similarly, you can write k rho and k theta here you are just talking about k_z and k_x ok.

Now for the sub wavelength object placed at the inner boundary wave propagates along the radial direction. So when they propagate along the radial direction they gradually compresses its tangential wave vectors and that is how they give you magnified images at the outer boundary because it it the radially they go out ok and you can get a magnified image. And once that magnified image is larger than the diffraction limit it can be easily resolved using far field. So here is an example of such hyperlens in application. So you see if you simply have air and there are two objects which are just separated by 80 nanometer. So these are sub diffraction limited line sources and only air you are able to do the far field imaging but you cannot resolve you cannot say that there are two sources you think that they are like like only one source ok.

But when you have first you have this kind of hyperlens put on top of your these two line sources. So they diffraction freely they go and then they create a bigger image and when they are above the diffraction limit they can be imaged using the conventional technique. So they can propagate in air and give you the information that there are basically two sources right. And these are simulation results here the radius of the hyperlens at the inner and the outer boundaries are taken to be 240 nanometer and 1200 nanometer. The permittivity of the metal and dielectric that is used for making this hyperpolic metamaterial or hyperlens is taken as this 2.

3 minus 0.3 I and the dielectric one is simply 2.7. And the filling ratio is taken to be 50 percent and this is how it shows that you are actually able to do very good imaging using hyperlens. So with that we will stop here and we will start the discussion of tunable photonic metamaterial based devices in the next lecture. In case you have any doubt regarding this lecture you can always drop me an email mentioning MOOC in the subject line. Thank you.