Course Name- Nanophotonics, Plasmonics and Metamaterials

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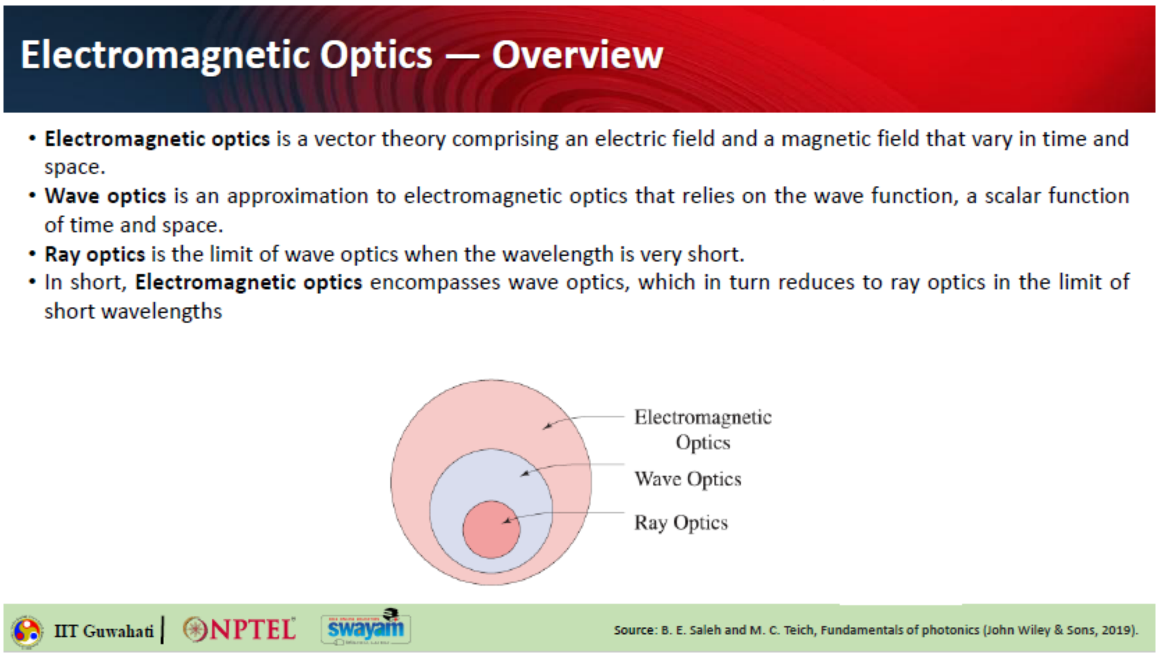
Week-02

Lecture -04

Hello students. Welcome to the fourth lecture of the online course on Nanophotonics, Plasmonics and Metamaterials. In this lecture we will look into the electromagnetic theory of light. So here is the lecture outline. We'll have a brief overview of the electromagnetic optics and then we'll look into divergence, curl and gradient operations, Gauss theorem and Stokes theorem, the constitutive relations.

Then we'll look into Maxwell's equation, the overview, then Gauss law for electric field, Gauss law for magnetic fields, Faraday's law, Ampere Maxwell's equation and then if time permits we'll look into wave equation and boundary conditions. So that's the picture of the great man James Clerk Maxwell who actually did this wonderful work on advancing the theory of light and proved that light is an electromagnetic wave phenomenon. He formulated a set of fundamental equations of enormous importance and that is why they belong to his credit because using these equations as you can see these equations are basically done by Gauss, Faraday and Ampere. But then the set of these four equations are able to describe the electromagnetic property of light and that is why he is the one to combine these equations to describe the electromagnetic property of light and that's why these equations bear his name.

Now let's look into the electromagnetic optics. So wave optics has a far greater reach than the ray optics. Remarkably both approaches can provide similar results for many simple optical phenomena involving paraxial waves such as focusing of light by lens and behaviour of light in graded index material and periodic medium. But wave optics offers something that ray optics cannot. Something like wave optics has the ability to explain phenomena such as interference and diffraction which using ray optics you cannot.



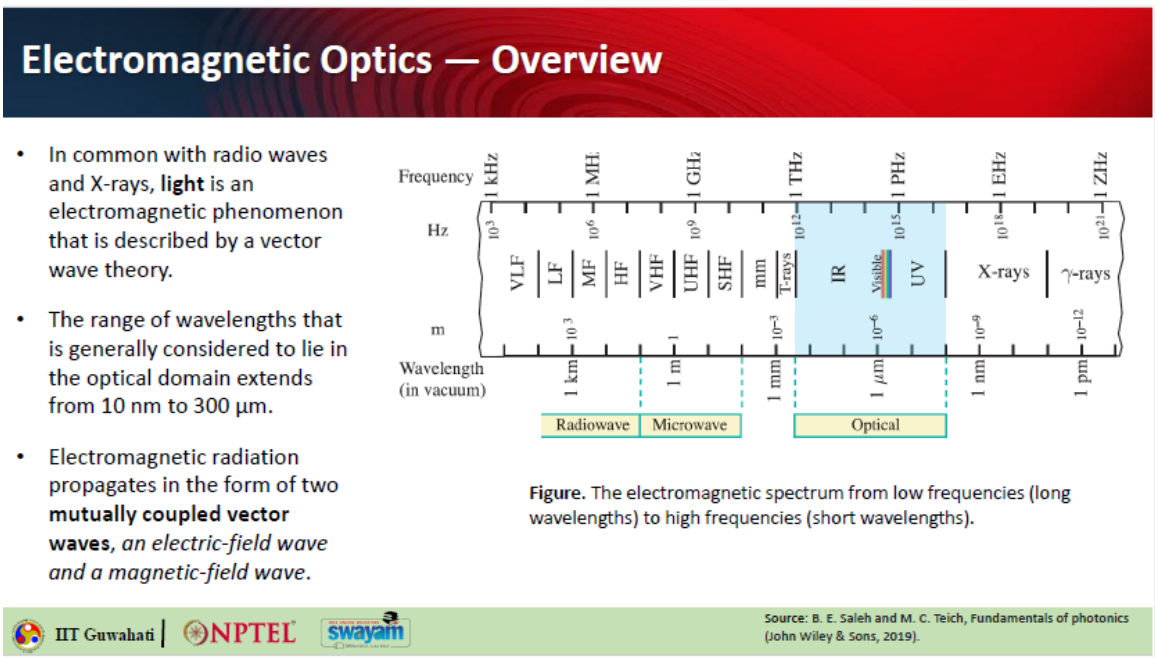
However ray optics sometimes also face difficulty in explaining the beam splitting. That is like the division of light using a beam splitter. So these are different areas which are all part of electromagnetic optics. But then wave optics can explain certain phenomena. Ray optics are able to explain certain phenomena and they are not good at certain different aspects.

So these are overall pictures of where electromagnetic optics, wave optics and ray optics are lying. So you can actually see that you know you can say electromagnetic optic is basically a vector theory that comprises both electric field and magnetic field that varies with time and space. If you look into wave optics which is basically a sub domain of electromagnetic optics. There you can say that it is an approximation of the electromagnetic optics that relies on wave function. That is why we call it as wave optics.

And you can describe it as a scalar function of both time and space. Then let's look into ray optics which is again another limit of the wave optics where the wavelength is very short. In short, you can say that electromagnetic optics encompasses the wave optics which in turn reduces to ray optics in the limit of short wavelengths. So there are situations where light has to be treated as vector. Then there are situations or limits in which light can be easily treated as a scalar function.

Or they can be in the short wavelength range. They can be described as rays. So why we need all these different approximation to try to explain different optical phenomena. So that is how you can actually get a broader picture of what do you mean by electromagnetic optics, wave optics and ray optics. Now this particular electromagnetic spectrum we have already discussed in our first module.

But still I am just putting it here just to give you a recap that when we say optics we do not actually mean only the visible. We are actually discussing you know wavelengths starting from ultraviolet to IR. So this is the optical wavelength range that we talk about. So starting from say 10 nanometer to several millimeter, not millimeter like fraction of millimeter. So that will correspond to the infrared boundary.

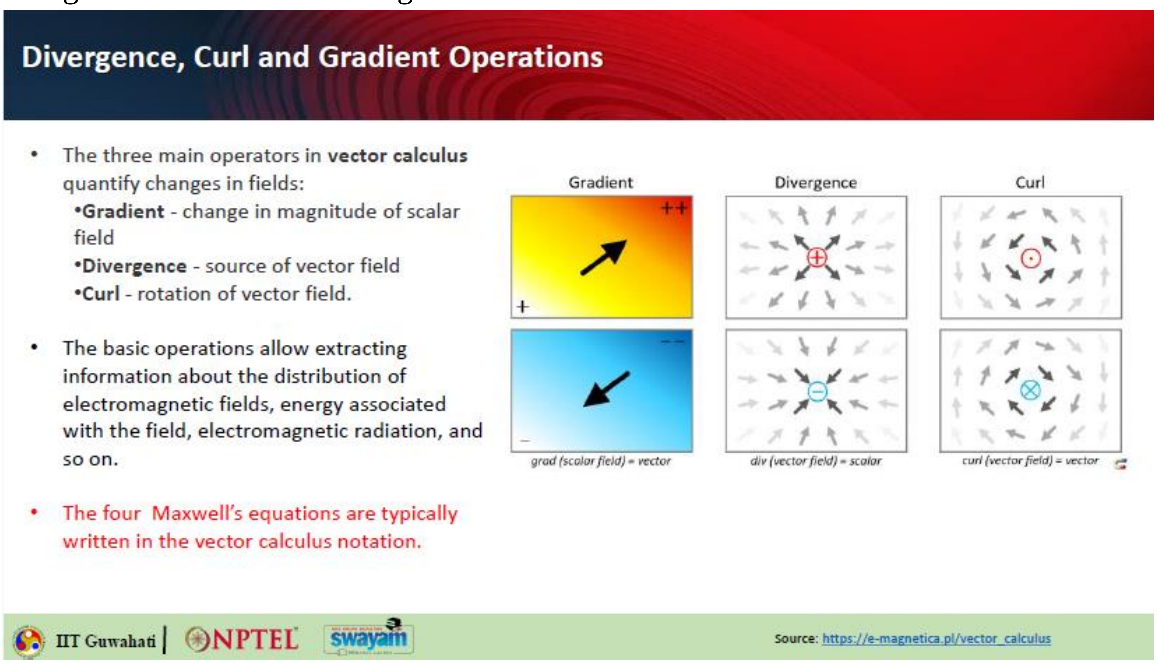


OK. So you can say several micrometer or thousand hundreds of micrometers. So this is the range over which the optical range is spread. And you can look into the bar here and you can see the frequency range over which this optical window is spread. So in common with the radio waves and the X-ray waves. So here you have radio and microwave waves.

On the other side you have X-ray waves. So electromagnetic phenomena here the optical range can be described using a vector wave theory. OK. Yeah. Here is the exact number 300 micrometer.

So if we say optical range it basically extends from as I mentioned 10 nanometer to 300 micrometer. So that is the range we are discussing. Electromagnetic radiation propagates in the form of two mutually coupled vectors. So one is electric field vector and other is magnetic field vector.

OK. So that is why we say light is an electromagnetic wave. Now before we go into understanding you know Maxwell's equation for electromagnetics there are some basic theories or concepts in vector calculus that we need to revise. So here I will quickly give you an overview of these operators which are very very popularly used in vector calculus. So these are gradient divergence and curl. So what is gradient? It is basically a change in magnitude of a scalar field.



So, gradient is basically the change. Divergence tells you the source of the vector field and curl is associated with the rotation of the vector field. So you can see from the diagram here that if there is positive charge here and then the positive charge is increasing in this region. So there is a gradient in this particular direction. Similarly, in case of negative charge you can say if there are more negative charge here and less here.

So, this is how the gradient is. OK. So always remember that the gradient of a scalar field is actually a vector. So the gradient that you are computing is of a scalar field.

OK. But the gradient itself becomes a vector because it will have the rate of change. OK. With this space. OK. And then it will also have a direction associated.

Divergence as I mentioned is basically you know, it tells you about the source of the vector field. So here you can see positive divergence. Here you can see negative divergence. OK. And the divergence of a vector field is basically a scalar.

And this is called curl is associated with the rotation of the vector field. So as you know that rotation can be either in counterclockwise or it can be in clockwise direction. So you can use your right hand thumb rule. So if it if the curl is in this direction in the direction of your fingers your thumb actually point the curl.

OK. So, the rotation of the vector field is in the direction of the fingers. The thumb gives you the curl. Similarly here the rotation is in this direction. So the thumb gives you the curl which is basically into the screen.

OK. In this case. So, the basic operations that allow extracting this information about the distribution of electromagnetic field then energy associated with the field and electromagnetic radiation and so on. So these basic operators like gradient divergence and curl can give you a lot of information out of the electromagnetic waves. And the four Maxwell's equation on which you can say the electromagnetism is completely dependent on. They are actually using all this vector calculus notations.

Divergence, Curl and Gradient Operations

Nabla ∇ operator

- In a 3D space, vectors can be split into orthogonal components, and partial derivatives can be calculated accordingly for each directional component.
- The del operator ∇ is a vector differential operator written as

$$\vec{\nabla} \equiv \boldsymbol{\nabla} \equiv \nabla \equiv \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Laplacian operator

- Nabla can be used in the "Laplacian" operator, referred to sometimes as "nabla squared" or "del squared", denoting effectively double differentiation:

$$\vec{\nabla}\cdot\vec{\nabla} \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} \equiv \nabla\cdot\nabla \equiv \vec{\nabla}^2 \equiv \boldsymbol{\nabla}^2 \equiv \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

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OK. So, one such important operator is called nabla. OK. Or you can also call it del as you'll see.

So, in 3D space. OK. The vectors can be split into its orthogonal components. So any vector will have its X Y Z components if we take a Cartesian coordinate system. So in this particular lecture we'll only focus on the Cartesian coordi nate system. As you know there are other coordinate systems as well like spherical cylindrical and there the equations for curl gradient divergence all these things will be different depending on the coordinate system you're choosing. So let us only focus here on the Cartesian coordinate system.

If required we'll go to the other coordinate systems depending on the applications we'll be discussing. So the vectors can be split into the orthogonal components and the partial derivatives can be calculated accordingly for each directional component. So the del operator is basically a vector differential operator. So here are the different notations that you can use for this del operator. And it is basically $\frac{d}{dx}$ or $\frac{\partial}{\partial x}$ you can say in $\hat{\boldsymbol{\imath}}$, $\frac{\partial}{\partial y}$ in $\hat{\boldsymbol{\jmath}}$ and $\frac{\partial}{\partial z}$ in $\hat{k}$.

OK. Similar to the nabla operator or del operator you can also have another operator called laplacian which is also based on nabla it is basically nabla square. So laplacian is also sometimes called nabla squared or del squared operator. So that is basically does what double differentiation. So they can be represented as you know. So, you have $\nabla \cdot \nabla$ or you can use them in bold phase.

So instead of using the vector arrows here you can actually look for bold phase the same thing shown here or you can simply write them as you know as these are like implied that this is basically a vector. Similarly the squared form also can be written in any of this form. That is fine. And they actually give you $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

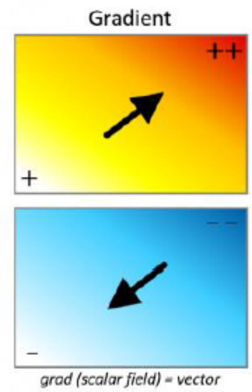
OK. Whichever way you call it. OK. Call it. So this is basically what this basically second order differentiation. So now let's look into a little bit of more details into the gradient divergence and cull. So as I mentioned that a scalar fields gradient is again a vector field. Reason is here there are two components.

Divergence, Curl and Gradient Operations

Gradient of a Scalar

- A scalar field's gradient is a **vector field** whose magnitude represents the rate of change and which points in the general direction of the scalar field's greatest rate of increase.
- If ∇ is made to operate on a scalar function F (such as scalar field), then the following notation for the **gradient** is used, with the **result being a vector**:

In a Cartesian system of coordinates	
(simplified notation)	$\text{gradient}(F) \equiv \text{grad}(F) \equiv \vec{\nabla} F \equiv \boldsymbol{\nabla} F \equiv \nabla F$
(full notation)	$\nabla F = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) F = \hat{\imath}\frac{\partial F}{\partial x} + \hat{\jmath}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}$



One is the magnitude. The magnitude actually tells you about the rate of change of that scalar field. And then there is a direction along which you know it is changing. So that is why gradient of a scalar field becomes a vector. So here are the two examples we have already seen.

So, I'm not discussing them again. Now if delta is made to operate on a scalar field as you understand you will get a vector. And this is given by this particular notation. So, you can write gradient of F or you can write grad of F or you can simply write you know ∇F.

OK. That will give you a gradient. So, same different variation in you can write del with a small arrow on top. That is the you know del operator. OK. Or you can write this way or you can write this way.

All this actually can't be the same meaning. So, how do you call it. How do you read it.

It is called grad F. OK. Gradient of F. OK. So you are basically taking help of the nabla operator for writing this. So when you do this gradient of f how it works. You have this del operator which can be written in this. So, as you have seen in the previous slide this one the del operator can be written as this.

So we just took it here and then F is what. F is a scalar field. So, you can simply apply the differentiation.

OK. And you will get this one. OK. So, the gradient may be in x or y or in all three. OK.

Depending on that this partial derivatives will be computed and you'll get the gradient of f. OK. Now let us look into the other operator which is basically divergence.

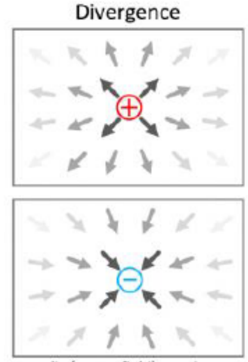
Divergence, Curl and Gradient Operations

Divergence of a Vector

- Divergence quantifies the magnitude only (no direction) of the amount of a vector field which "flows" out or into a specific region. In other words - the divergence calculates the amount of source (or sink) for a given field.
- If ∇ is made to operate on a vector function ***F*** (such as vector field), then the following notation for the **divergence** is used, with the **result being a scalar** (even though the input is a vector field).

In a Cartesian system of coordinates	
(simplified notation)	$\text{divergence}(\vec{F}) \equiv \text{div}(\vec{F}) \equiv \vec{\nabla} \cdot \vec{F} \equiv \mathbf{\nabla} \cdot \mathbf{F} \equiv \nabla \cdot \mathbf{F}$
(full notation)	$\nabla \cdot \mathbf{F} \equiv \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right)$

Divergence



div (vector field) = scalar

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The divergence quantifies the magnitude. There is no direction associated. So in that case you understood that if you take divergence of a vector field you will basically get a scalar field. That is why it is written divergence of a vector field gives you scalar field. So it tells you the amount of a vector field which flows out or into a specific region. So, in other words you can say that the divergence calculate the amount of source or sink for a given field.

OK. So, you can actually identify the source or sink from the divergence. If there is no source or sink what will happen? The divergence will be zero. Now what is the source of electric charge? Positive.

Electric field is a positive charge. OK. So, if you look into this particular diagram if you say there is a positive charge so the electric field lines are actually coming out of it. So, the electric field lines are originating from that particular source. So, this is called a positive divergence. On the other hand if you see that what is the sink of electric field lines that will be the negative charges.

Right. You just recollect your school physics you will see that the electric field lines are basically originating from the positive charge and they are coming back and entering the negative charge. Alright. So here you can see that you know if you have a standalone negative charge you will see the electric field lines will come into and converge into that particular negative charge. So this particular phenomena is nothing but negative divergence. So how do you actually write this thing? So, as I mentioned once again let me tell you that when you take the Dell operator OK or the Nabla operator and you operate your vector field.

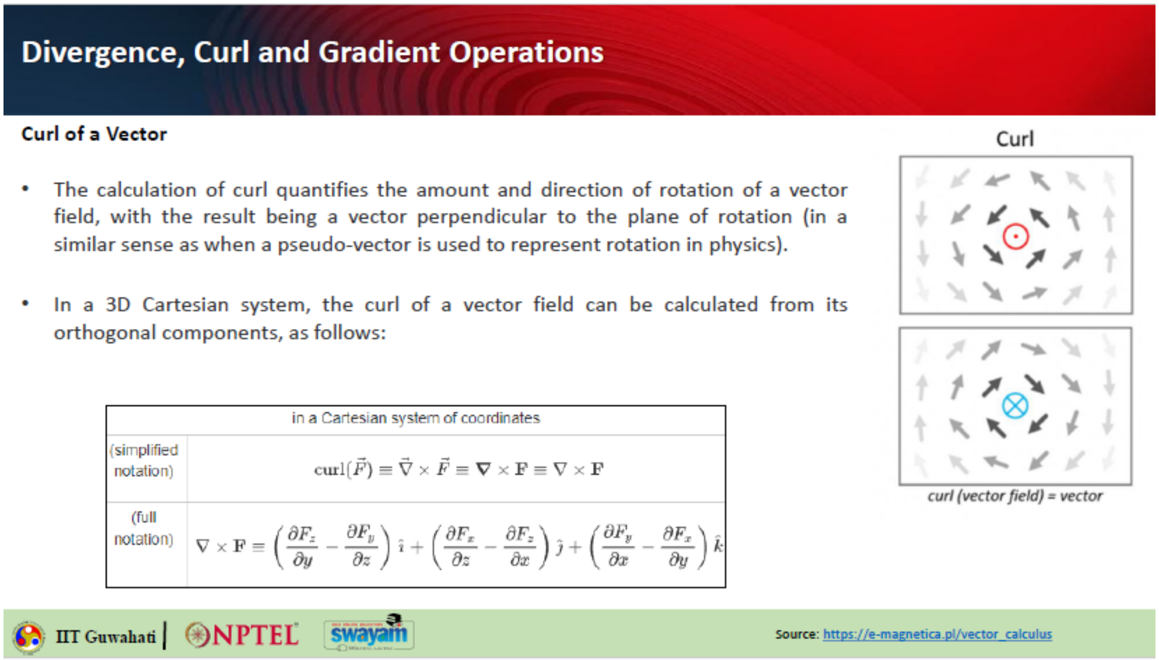
So, this time ***F*** is a vector field. So that is why ***F*** is shown as bold. OK. So, it's a vector field and you want to calculate the divergence and we know that the divergence of this vector field will be a scalar. OK. So, how do you notify it or what are the notifications? You can simply write divergence of ***F*** vector.

OK. Or you can write it in sharp form. Div ***F*** vector or you can write $\nabla \cdot \mathbf{F}$. So, again del can be given as this Nabla operator with a vector arrow marking or you can use the bold face or you can simply write this dot ***F***. As you see ***F*** is always maintained in boldface because otherwise you have to write this with the arrow to tell that this is basically a vector field. And when you have this vector field OK this vector field will have its component along x y and z and that you can also write delta in terms of $\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$.

OK. And then you can multiply the corresponding vectors and you will find this is what you get. So, this basically is basically the scalar field. And that is how we have come to this conclusion that when you compute the divergence of a vector field it comes out to be scalar. So, this is the scalar quantity.

Clear? So, now let's move on to the third operator which is curl operator. The calculation of curl quantifies the amount and the direction of rotation of a vector field. Now whenever you will see there is a rotation of electric field or magnetic field there is a curl associated with it. So how do you quantify this curl and what is the direction of this curl? OK. So curl can be actually represented as I mentioned before. So, either the rotation is in counterclockwise or it is in clockwise direction.

So, accordingly the curl will also have this direction upwards or downwards. So, curl is the result being a vector. So, again it's a vector perpendicular to the plane of rotation. So, this is the plane of rotation. This particular computer screen is the plane of rotation in which the vector fields are rotating.



OK. So in this case the curl will have the direction either out of it or into it. So how do you calculate curl in a 3D Cartesian system? So you can write the notation can be curl of the vector f. OK. So curl is always calculated of a vector field.

So the notation is this nabla operator with a vector sign. So, this is what you can say $\vec{\nabla} \times \vec{F}$. OK. Or you can write it this way. OK. And then when you take the cross product of the two vectors this is how the field can be computed.

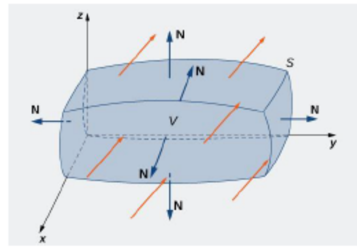
OK. So, it will be $\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$. This will be the component along $\hat{i}$. Then $\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}$ will be the component along y direction or you can say it is $\hat{j}$. And $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$ that will be the quantity along z direction or $\hat{k}$ unit vector.

So this is how you can compute divergence curl and gradient of any field. OK. Now there are two important theorems that will also be very much useful. So let us quickly cover those theorems because these are related to this vector fields and they will be important in understanding the Maxwell's equation. So one such theorem is called Gauss theorem or divergence theorem. Now what does this theorem states? You look into the picture here. So, this theorem states that the flux of a vector quantity at outward through a closed surface S is equal to the integral of the divergence of the function that is enclosed in the enclosed volume V.

Gauss's Theorem or Divergence Theorem

- The theorem states that the flux of a vector quantity outward through a closed surface **S** is equal to the integral of the **divergence** of the function in the enclosed volume **V**,

$$\iiint_V (\nabla \cdot \mathbf{F})\,dV = \oint\!\!\oint_S (\mathbf{F} \cdot \hat{\mathbf{a}})\,dS$$



- Therefore, if the given volume does not contain a source (or sink) of the vector field then the net flux through that volume must be zero **(i.e. all flux entering the volume must also leave that volume).** It is possible to find such volume that will entrap an electric charge, because **each electric charge represents an electric monopole.** → Leads to Maxwell's First Equation
- But it is not possible to find a volume which entraps a magnetic charge, so the magnetic field is "divergenceless" and thus there are **no magnetic monopoles.** → Leads to Maxwell's Second Equation

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So, if you take the flux OK. This is the flux in a closed surface. So closed surface are shown using this kind of circles on the integration. So you are actually doing a surface integral here and that is equivalent to the divergence of the function. So you are calculating the divergence of the function in an enclosed volume V OK. That actually covers this particular closed surface.

So, here you can see this is the closed surface and this is the particular volume V. OK. So, in this equation the V denotes the volume, F denotes the analyzed vector field, S is basically the surface that is covering the entire volume. So, that is *S* and $\hat{a}$ is nothing but the normal vector or this is the unit normal vector. So, from this theorem what do you understand? That if the volume, if this particular volume does not contain a source or a sink OK. It means the net flux through that particular volume must be zero. That means there is no flux originating or terminating at a particular place inside this volume.

So, all the flux that is entering this volume must also leave this volume. OK. So, it is possible to find such volume that will entrap an electric charge because each electric charge represents an electric monopole. So you can actually have monopoles in electric charges whereas if you try to find monopoles in magnetic charge that is not possible. OK. We will come to that later on. But this is the main understanding of Gauss theorem that if you want to take the surface integral over a closed surface for a particular vector field that is equal to the divergence of that function in a volume made out of this you know surface closed surfaces.

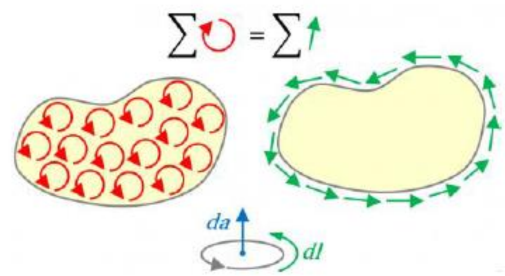
Now it is not possible to find a volume that interrupts a magnetic charge. OK. That is why the magnetic field is divergence less. So, if you take f as a magnetic field you will see that the divergence of the magnetic field is basically zero because you are not able to find any one particular pole separated out. So, there is no magnetic monopole and that is why the divergence of magnetic field lines is always zero. So, the first one that you have discussed here will lead to the first Maxwell's equation and this one that magnetic monopoles does not exist that will give rise to the second Maxwell's equation.

Stokes Theorem

- Stokes' theorem states that the surface integral of the curl of the vector field **F** over an open surface **S** is equal to the closed line integral of the vector along the contour enclosing the open surface.

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint_C \mathbf{F} \cdot d\mathbf{l}$$

- In other words, the circulation of a vector around a given boundary is equal to net curl over the whole surface of the patch limited by that boundary.



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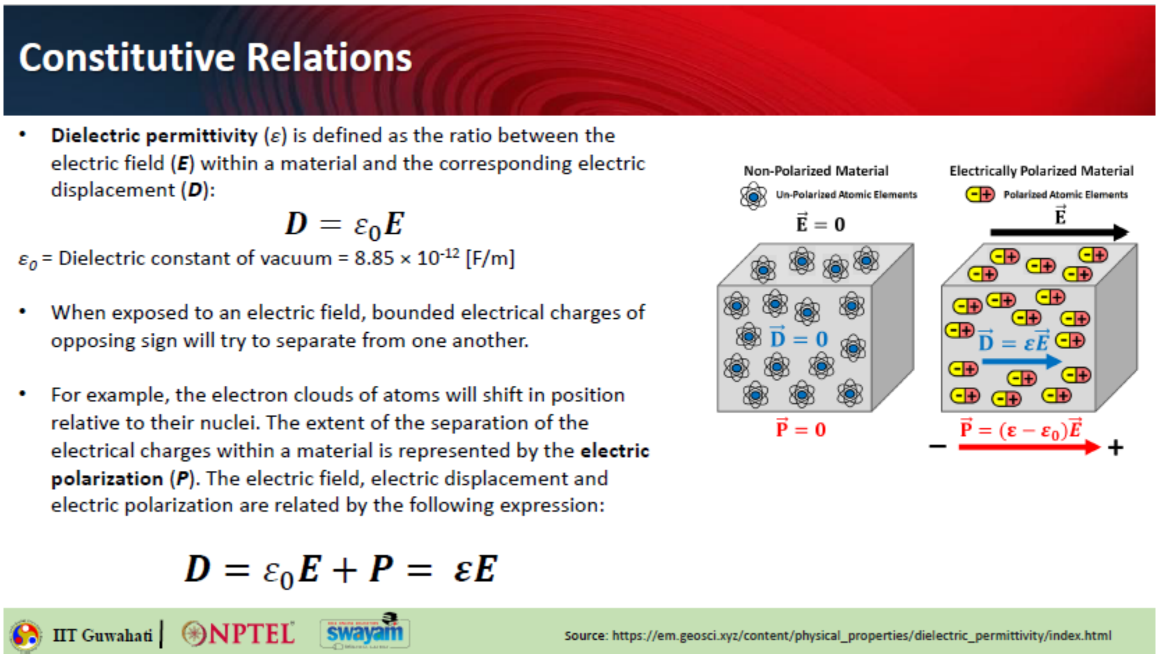
So this was one important theorem called Gauss theorem or divergence theorem. The second important theorem is the Stokes theorem. Now as you can see in the picture here the Stokes theorem actually states that the surface integral of the curl of the vector field ***F*** over an open surface. So, let's assume this is an open surface and there are many curl of the vector field. So, if you compute the overall curl of this vector field in this open surface that will be equal to the closed line integral of the vector along the contour that is enclosing this open surface. So, this was the open surface and if you take this particular line that is you know that is basically along the contour which is enclosing the open surface.

So the line integral of the vector. So the line integral of the vector along this contour that will be same as all the curls that are actually there in this open surface. OK. So if you take the curl and you sum up all the curls or you integrate all the curls OK over this particular open surface S that is equivalent to this one. So, it's very nicely shown pictorially you can understand the sum of all the curves is basically sum of this line integral and what is *d*a? *d*a is nothing but now this is the unit vector that is normal to the surface. So, in other words you can say that the circulation of a vector around a given boundary is equal to the net curl over the whole surface of the patch limited by that boundary.

So, with that we understood the basic theorem and the basic operators that will be needed for dealing with Maxwell's equation. Now let us try to understand the constitutive relations. And before we do that OK we have to understand that Maxwell's equation OK talks about the electric field and magnetic field and when there will be this electric field magnetic field interacting with matter the permittivity and permittivity these two factors will come into the picture. Now what are these pictures? These terms permittivity and permeability you must have studied in your school days these are basically the measure of how electromagnetic field or how a matter actually interacts with electric field and magnetic field.



So, that is we'll go into each of this. So, let's look into the first one which is the dielectric permittivity. So, dielectric permittivity is basically a diagnostic physical property which characterizes the degree of electric polarization a material can experience when it is subjected to some external electric field. So, permittivity is related to external electric field. So, how do you define the dielectric permittivity? It's defined as a ratio between the electric field within a material and the corresponding electric displacement.

So, we'll explain this with using this particular diagram. So, when there is no electric field all these are basically unpolarized atomic elements you can see the electrons with their electrons and all those things. The displacement is zero, polarization is zero because there is no applied electric field. As soon as there is an electric field applied to this material you will see that the electron cloud is trying to you know repel this electric field and they move out from the nucleus. So, what happens you know there is a slight space region with more electrons and there is a region where there is a deficiency of electron that can be denoted as minus and plus.

Deficiency of electrons means you can mark that as positive. So this is how you know the polarization of each atomic element takes place. So, each of these will get polarized. And because of that you can write down that the polarization is proportional to the applied electric field. And if you try to find out what is the value of that polarization that we'll see using the permittivity. So, first of all the definition is clear that you know dielectric permittivity is basically the ratio of electric field.

So it is $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$. So, what is ε_0 ? It is basically ***D/E***. So it is the displacement field over the electric field that gives you epsilon naught which is vacuum permittivity.

OK. So, the value for vacuum permittivity is well known. It's $8.85 \times 10^{-12} [\mathrm{F/m}]$. What is F? Farad. So, when we expose to the electric field we have seen that you know the bonded electrical charges of opposite sign they try to separate from each other and that results into you know the polarization of the material which is called electric polarization. And you can write ***D*** in the presence of a material it becomes $\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$. So, this is the extra term that is coming when your electromagnetic field is interacting with the material.

OK. And ***P*** can be written as $(\varepsilon - \varepsilon_0)E$. What is epsilon? That is the permittivity of the material. ε_0 is vacuum permittivity. When you add this two up you get ***D*** is proportional to ε_e in a particular matter. On the other hand if you look for magnetic field. So when exposed to an applied magnetic field the collection of individual magnetic dipole moments within most material will attempt to reorient themselves in the direction of the applied field.

And this will generate some induced magnetism. OK. Which contributes towards the net magnetic flux density inside the material. These are all known factors but still I am covering them quickly. The degree in which the induced magnetism impact the magnetic flux density depends on the materials magnetic permeability. So, let us define permeability. So, permeability is basically the ratio between the magnetic flux density b within a material and the intensity of the applied magnetic field ***H***.

OK. And provided that both the fields are sufficiently weak. So, you can write that ***B*** the magnetic flux density is proportional to $\mu_0 H$. What is μ_0? It's the permeability of free space and the value is $4\pi \times 10^{-7}\,\mathrm{H/m}$. OK. And then there is additional term which is $\mu_0 M$.

So, this is the magnetization of that particular material we are talking about. Now in most cases we deal with non magnetic material in the optical field. So, we can safely take M= 0. So, that brings this equation to a simpler form that ***B*** is simply $\mu_0 H$. The contribution from the material perspective for the magnetic field we are not considering most cases in the optical or nanophotonics domain.

Constitutive Relations

- When exposed to an applied magnetic field, the collection of individual magnetic dipole moments within most materials will attempt to reorient themselves along the direction of the field.
- This generates an induced magnetization, which contributes towards the net magnetic flux density inside the material.
- The degree in which the induced magnetization impacts the magnetic flux density depends on the material's magnetic permeability.
- **Magnetic permeability** (μ) defines the ratio between the magnetic flux density ***B*** within a material, and the intensity of an applied magnetic field ***H***; provided the fields are sufficiently weak.

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M} = \mu \boldsymbol{H}$$

μ_0 = permeability of free space = $4\pi \times 10^{-7}$ H/m

Note: For now, we will focus on materials for which

$$\boldsymbol{M} = 0 \Rightarrow \boldsymbol{B} = \mu_0 \boldsymbol{H}$$

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OK. Because we deal with again non magnetic material. Fine. So, let's look into the Maxwell's equation now. So Maxwell's equation is nothing but it actually describes the electromagnetic field by the two related vector fields. One is electric field and another is magnetic field. And this fields electric and magnetic fields are both function of R and T that is space and time. Now after the myriad of researchers carried out for fundamental reasons behind the source of electromagnetic field and the relations between electric and magnetic field by the pioneer scientists like Ampere, Coulomb, Faraday, Gauss. The revolution in the electromagnetic field could happen when James Clerk Maxwell he could propose this set of fundamental equations in 1865.

Maxwell's Equations — Overview

- An **electromagnetic field** is described by two related vector fields that are functions of position and time: electric field ***E*(*r*, *t*)** and magnetic field ***H*(*r*, *t*)**.
- After the myriad of researches carried out for fundamental reasons behind the source of electromagnetic field and relation between electric and magnetic fields by pioneer scientists **Ampere, Coulomb, Faraday and Gauss**, the revolution in the Electromagnetic Fields happened when **James Clerk Maxwell** proposed a set of fundamental equations in 1865.
- The Maxwell's equations are valid for both static and dynamic electromagnetic fields in a media.

Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} \quad \text{(Gauss' Law)}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \text{(Gauss' Law for Magnetism)}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{(Faraday's Law)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampere's Law)}$$

E = Electric field vector
H = Magnetic field vector
D = Electric flux density
B = Magnetic flux density
ρ = charge density
J = current density

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So before that the individual laws of electric field and magnetic fields were existing. But the field of electromagnetism came after the integration of these equations by Maxwell. So Maxwell's equations are valid for both static and dynamic electromagnetic field in a media. So these are the Maxwell's equations. So, divergence of ***E*** is nothing but rho V that is a charge density in a volume divided by epsilon.

Maxwell's Equations — Overview

- Gauss's law for electric fields: While the *area integral of the electric field* gives a measure of the net charge enclosed, the *divergence of the electric field* gives a measure of the density of sources.
- Gauss' law for Magnetism: The *net flux* will always be zero for dipole sources.
- Faraday's law: The line integral of the electric field around a closed loop is equal to the negative of the *rate of change of the magnetic flux* through the area enclosed by the loop.
- Ampere-Maxwell equation: This gives the total magnetic force around a circuit in terms of the *current through the circuit*, plus any *varying electric field* through the circuit (that's the "displacement current").

Maxwell's Equations

Integral Form	Differential Form
$Q_e(t) = \oint\!\!\oint_S \vec{D}(t) \bullet d\vec{s} = \iiint_V \rho_v(t)\,dv$	$\nabla \bullet \vec{D}(t) = \rho_v(t)$
$\oint\!\!\oint_S \vec{B}(t) \bullet d\vec{s} = 0$	$\nabla \bullet \vec{B}(t) = 0$
$V_{emf}(t) = \oint_L \vec{E}(t) \bullet d\vec{\ell} = -\iint_S \left[\frac{\partial \vec{B}(t)}{\partial t}\right] \bullet d\vec{s}$	$\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t}$
$I(t) = \oint_L \vec{H}(t) \bullet d\vec{\ell} = \iint_S \left[\vec{J}(t) + \frac{\partial \vec{D}(t)}{\partial t}\right] \bullet d\vec{s}$	$\nabla \times \vec{H}(t) = \vec{J}(t) + \frac{\partial \vec{D}(t)}{\partial t}$

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On the other hand divergence of magnetic field is zero. Reason is that there is no magnetic monopole. If there is a monopole present then only you can have a positive divergence or negative divergence but then that is not there.

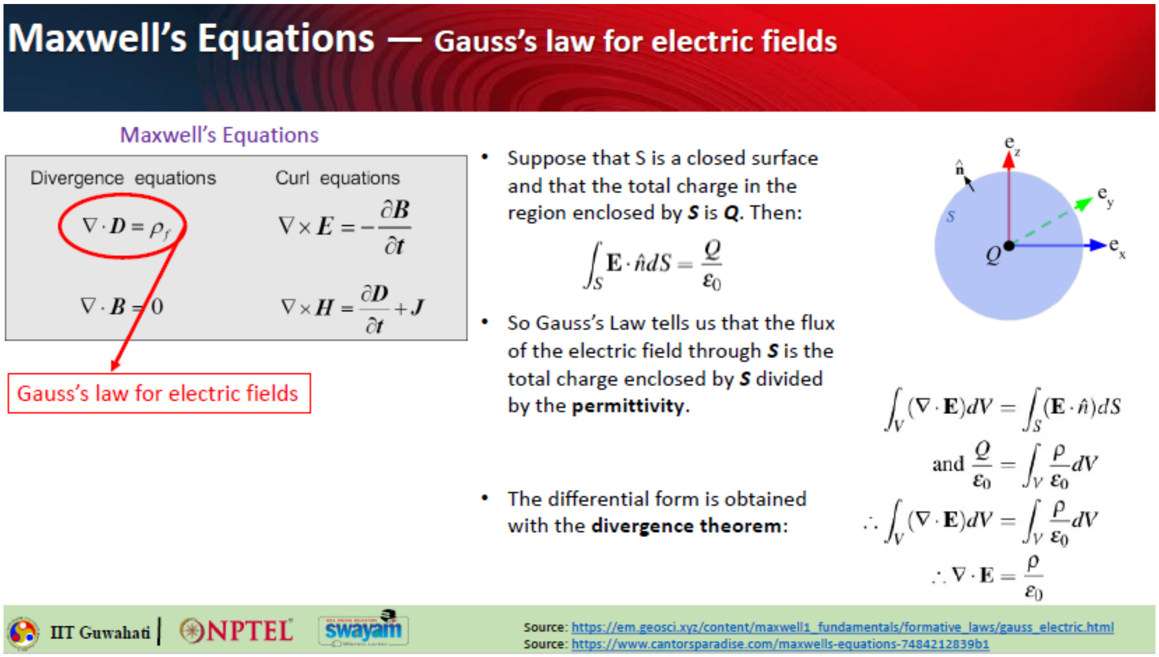
So the divergence is zero. $\nabla \times \mathbf{E}$. So the first two are called Gauss law. This is called Gauss law for magnetism. This one is Faraday's law which is $\nabla \times \mathbf{E}$ is related to the time varying. So, if there is any time varying magnetic field that will have some circulating current.

So that is given by $\nabla \times \mathbf{E}$. So $\nabla \times \mathbf{E}$ is $-\mu \frac{\partial \mathbf{H}}{\partial t}$. That is called Faraday's law. And we also have Ampere's law. So this is basically the corrected Ampere's law or we can say this is based on Ampere's law. So here we have $\nabla \times \mathbf{H}$. So there is a magnetic field circulating.

Which is basically dependent on the current density plus the change in the electric flux with time. ε_e is nothing but ***D***. So these are the terms that you have to keep in mind. So, you have seen where we are writing all of these in terms of ***E*** and ***H***. But we also know from the constitutive relationship that ***D*** equals ε***E***. That means the electric flux density or the displacement field is basically ε***E*** and the magnetic flux density that is ***B*** is μ_0***H***.

So here in this equation if you put μ and ***H*** together you basically get ***B***. So, these equations become $-\frac{\partial \vec{B}(t)}{\partial t}$. And this equation becomes $\frac{\partial \vec{D}(t)}{\partial t}$. So Maxwell's equation can be written in both integral form as well as differential form. So, we will only focus here on the integral form because of the you know it's easy to write and also, it's easy to describe.

But they actually convey the same meaning. These are just different forms of writing the same equation. So let's start with the first one that is Gauss law for electric field. So it says that while the area integral of the electric field gives a measure of the net charge enclosed, the divergence of the electric field gives a measure of the density of those sources. So here you can see the divergence is giving you the charge density. So from this also you can see the integral form. So here you see the divergence sorry the displacement field integrated over a closed surface is basically giving you the charge that is enclosed.



You can also write it in terms of the density charge density over volume. This V stands for volume charge density. So you call it rho V. And if you integrate it over the volume so you will get nothing but the charge. So this one and this one then needs to be equated. So this is volume integral, this is surface integral.

So how do you actually change from surface integral to volume integral? You can take the help of the theorems that you have studied. So we will come into that. But let me quickly give you an overview of Maxwell's equation. So the second one is the Gauss law of magnetism. So it tells you that the net flux will be always zero for dipole sources.

In magnetic field the poles cannot be separated so they always remain as dipole. And there you will see that the divergence of the magnetic flux density is zero. Faraday's law actually gives you that the line integral of the electric field around a closed loop is equal to the negative rate of change of the magnetic flux through the area enclosed by that loop. So, you can see it from here or you can also write it as the $\nabla \times \boldsymbol{E}$ is nothing but $-\frac{\partial B}{\partial t}$. And finally the fourth equation or the Ampere Maxwell equation, it tells you that the total magnetic force around the circuit in terms of the current through the circuit plus any varying electric field through the circuit.

That is basically the displacement current. So, you can actually see it here that $\nabla \times \boldsymbol{H}$ and these are all time varying and space dependent. So $\nabla \times \boldsymbol{H}$ is nothing but $\frac{\partial D}{\partial t} + J$ that is the current flowing through the circuit plus there is something extra which is the electric field displacement over time or the rate of change of the displacement field. That is basically the displacement current.

The time rate of change of any field is basically this one is current. So let's look into the first equation in more details. So, we can write $\nabla \cdot D$ is nothing but ρ_f. So this is also called as Gauss law for electric. So if we assume that there is a S that is the closed surface. And the total charge in this region enclosed by this closed surface S is Q.

You can write you know $\mathbf{E} \cdot \hat{n}$, n cap is the unit vector normal to the surface. When you do this surface integral you will get it as $\frac{Q}{\varepsilon_0}$. So the Gauss law basically tells you that the flux that you are getting or you can say the flux of the electric field through S is basically the total charge enclosed by this closed surface S divided by the permittivity. So this is the flux that is equal to the total charge divided by the permittivity. Now if you try to this is the integral form.

So if you try to convert this into the differential form you can take help of the divergence theorem. How to work with that? So this equation is basically a surface integral. OK. So, from surface integral you can use the divergence theorem and you can say the divergence of that particular field vector field over the volume will be same as this particular surface integral.

So, now this is in terms of volume. OK. Again, the right-hand side of this equation $\frac{Q}{\varepsilon_0}$ you can write it as $\int_V \frac{\rho}{\varepsilon_0} dV$. Now if you take these two together. OK.

So, here you can see you can actually find out that these are both volume integrals. So the quantity this quantity and this quantity must be equal. So that way you can obtain $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$. Simple. So, that way from every integral form equation you can actually use those theorems that you have studied and the vector identity is we can come to the differential forms. Similarly looking at the second Gauss law for magnetic field.

Maxwell's Equations — Gauss's law for magnetic fields

Maxwell's Equations

Divergence equations	Curl equations
$\nabla \cdot D = \rho_f$	$\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \cdot B = 0$	$\nabla \times H = \frac{\partial D}{\partial t} + J$

Gauss's law for magnetic fields

- Gauss's law for magnetism states that **no magnetic monopoles exists** and that the total flux through a closed surface must be zero.
- $\nabla \cdot \boldsymbol{B} = 0$ is derived from $\int B \cdot dS = 0$

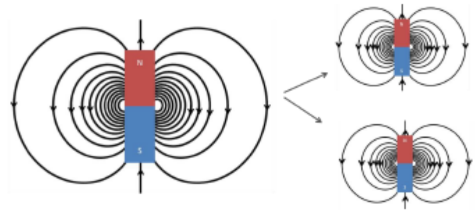


Fig. When a bar magnet is cut in two, you get two bar magnets

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So, Gauss law for magnetism says that no magnetic monopole exists and that is why the total flux through a closed surface must be 0. So $\nabla.\boldsymbol{B} = 0$ and that can be derived from $\int B \cdot dS$. So, if you take a closed surface OK with area S so this is the total flux OK through that closed surface and that is 0. And this is happening because if you take a bar magnet and try to cut into two parts hoping that you will be able to separate north and south pole that doesn't work. The small magnet also becomes it also have its own north and south pole.

Maxwell's Equations — Faraday's law

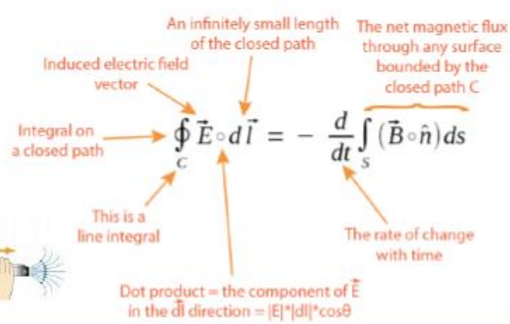
Maxwell's Equations

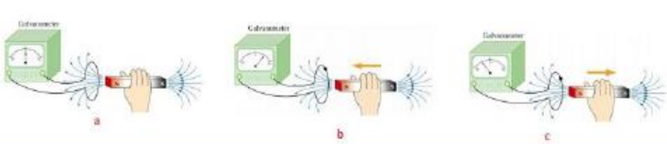
Divergence equations	Curl equations
$\nabla \cdot D = \rho_f$	$\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \cdot B = 0$	$\nabla \times H = \frac{\partial D}{\partial t} + J$

Faraday's law

Faraday's Law of Induction: Integral Form

- The line integral of the electric field around a closed loop is equal to the negative of the rate of change of the magnetic flux through the area enclosed by the loop.





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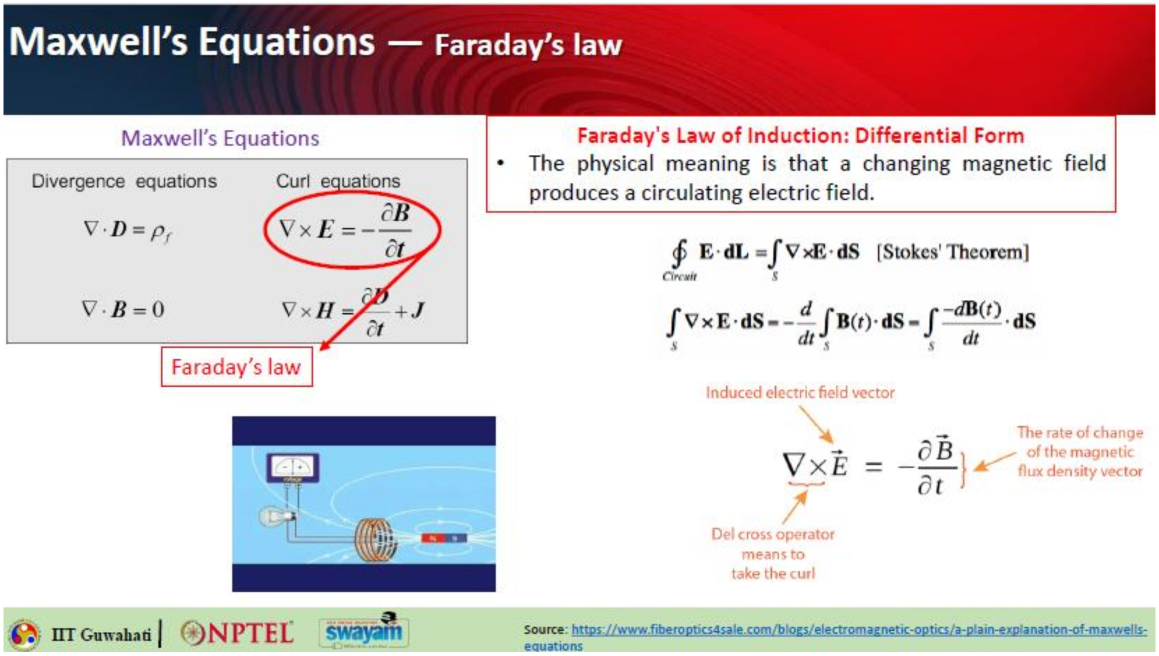
So, if you can if you break it infinitely small size also there also there will be two poles present. So, there is it is not possible to have any magnetic monopole.

The third one is called Faraday's law. OK. So, this is one of the first two equations that connect ***E*** and ***B***. OK. So, ***E*** is a conservative field that you have to keep in mind in the absence of a magnetic field or you can say the magnetic field is constant in time. So, electromagnetic induction was first independently discovered by Michael Faraday in 1831 and then by Joseph Henry in 1832. And Faraday was the first to publish his results of the experiment. So, this is known as Faraday's law of electromagnetism.

So, if you think of this particular equation which is known as Faraday's law there you are we are able to connect ***E*** and ***B***. OK. So what happens in this particular equation. Let's look into the integral from first. So it says that the line integral of the electric field around a closed loop. So, this is how you can write it is equal to the negative rate of change of the magnetic flux through the area and closed by that particular loop.

So, if you take this is the magnetic flux and this is the negative rate of change of the magnetic flux through that particular loop. So here is a picture snapshot that also shows you the same thing. So if you have a magnet and this is a loop that is measuring the amount of field lines going into it. So when you move it towards OK there is a rate of change in the magnetic flux lines or magnetic field lines. So that will rotate the needle of the galvanometer to one direction.

And if you take it away from this particular loop it is recorded in the galvanometer in the opposite direction. OK. So this actually tells you the law of magnetic induction electromagnetic induction. So, let's see how do we get this particular differential form equation from the integral form.



So, what is there in the integral form. It's a closed loop integral line integral you can say $\mathbf{E} \cdot dl$. OK. So, let's write $\mathbf{E} \cdot dl$. So, if you use $\mathbf{E} \cdot dl$ you can also write it in terms of surface integral by taking the curl of that particular field.

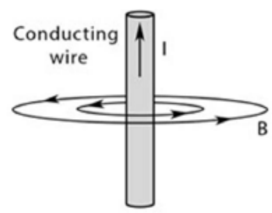
And that is what we learned from the Stokes theorem. And when you write this so it becomes $\oint_{\text{circuit}} \mathbf{E} \cdot dl$. OK. And you are integrating over the surface now. So this is same as the right hand side here which is already having a surface integral. So, you can now compare the quantity inside the integral and you can write that this side $\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$. So, this is how we are able to get this differential form.

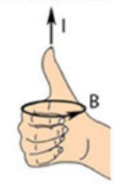
OK. So the physical meaning is very simple that a changing magnetic field will introduce circulating electric field. OK. So changing electric field means it is changing with time. OK.

So a time varying magnetic field will introduce a $\nabla \times \vec{E}$ means circulating current. Curl means the electric field going this way. So there is a circulating current. So this is Faraday's law. The next one let's look into the fourth Maxwell's equation that is coming from the Ampere's law.

Ampere's Law - no time dependence (Incomplete)

- Suppose you have a conductor (wire) carrying a current, I. Then this current produces a Magnetic Field which circles the wire.
- Ampère had shown how to make magnetism from electricity.
- **Right Hand Thumb Rule:**
  Thumb points in the direction of the electric current and fingers curl around the current indicating the direction of the magnetic field.





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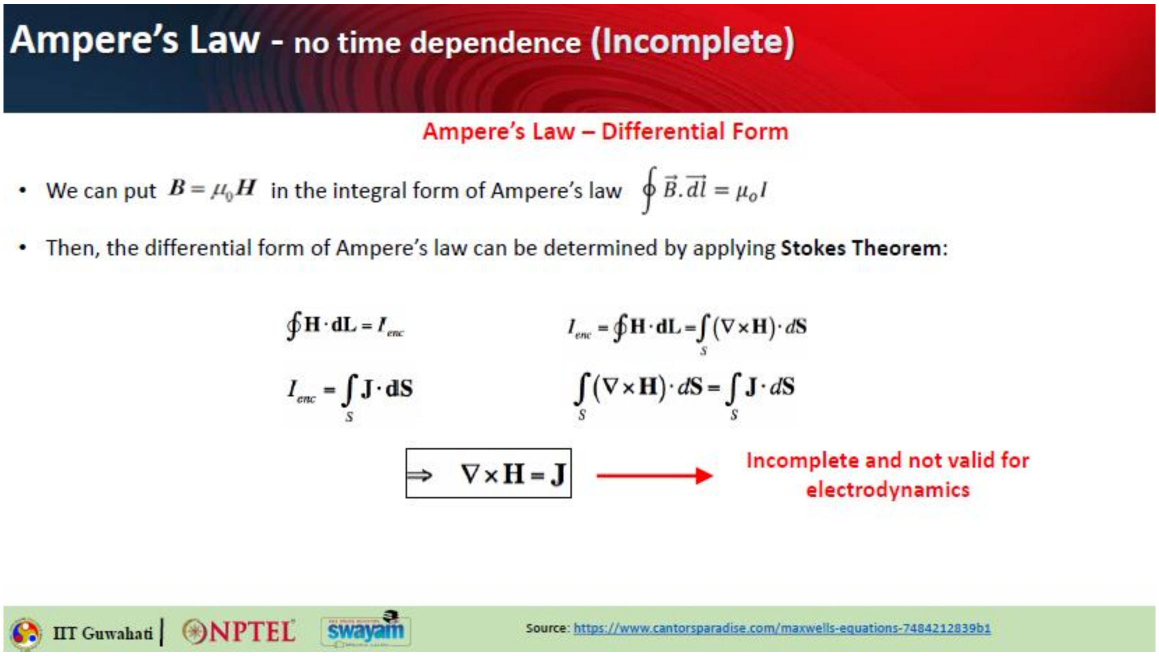
So, Ampere's law does not have any time dependence. So, it simply says that if you have a conductor carrying current I. OK. So, when the conductor is carrying current I it will produce a magnetic field that will circle the wire. OK. So, again you can take it this way.

So, I is the right-hand thumb rule it follows. So if thumb points to the direction of current flow the fingers they will point the direction of the magnetic field. OK. So Ampere had shown how to make magnetism from electricity. So that is actually a big big discovery because you are able to get magnet fields or magnetism by current flow. OK. Now if you try to look this same thing in the integral form it is written in terms of the Biot-Savart law.

So from Biot-Savart law we know that now the magnetic field due to a long straight wire can be written as $B = \frac{\mu_0 I}{2\pi r}$. OK. So, I is the current, mu naught is the vacuum permeability and $2\pi r$ is basically the you know r is the radius of this particular rod. Then because B and dl both are in the same direction you can take their dot product and that comes out to be also B dl.

OK. So, when you take the line integral. So, how do you get this line integral of this small element dl along this circumference you will get $2\pi r$. OK. So finally you can write. OK. I'm just skipping the steps and finally you can write that the line integral of B is nothing but $\mu_0 I$.



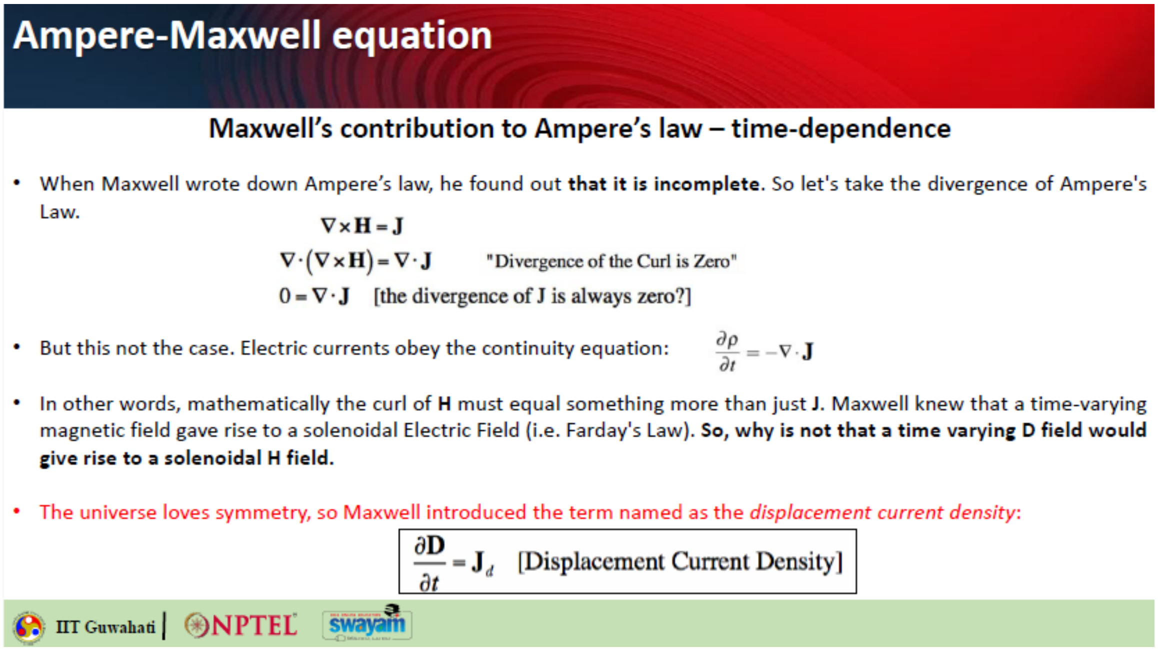
OK. So, it means the amount of magnetic field that is being generated is proportional to the current that is flowing in the wire. So, you can also put this particular relation that you have learnt that B equals $\mu_0 H$ in the integral form of the Ampere's law and that will also help us to get the differential form of this Maxwell's equation. The fourth one. So, you can write you know in this equation if you write $B = \mu_0 H$ you get $\mu_0 \vec{H}\cdot\vec{dl} = \mu_o I$.

So μ_0 you can cancel out from both sides. You can simply write that the line integral of H is nothing but the current enclosed or simply I. OK. And I can also be written as what is J? There is a current density. So density times the area that will give you the current. So, I can be written as H. dl and H can be written it can be converted into the surface integral by taking the curl of it.

So this equation can be now equated to this surface integral. So here also you have surface integral here also you have surface integral. You can equate these two quantities and you can write curl of H is nothing but J. That means when you have a current density you are actually or current flowing you are actually getting a magnetic field lines around it. But then this is an incomplete equation and it is not valid for electrodynamics.

It is good for electrostatics but not for electrodynamics. So, this is where only this form is called the Ampere's law. But then why did Ampere's law went wrong and how it became incomplete. The first thing was that it didn't have any time dependence. So, Maxwell brought in time dependence in this particular equation and that was the biggest contribution of Maxwell.

So Maxwell wrote down the Ampere's law and he actually found out that it is incomplete. Because when you take the divergence of the Ampere's law. So, Ampere's law is curl of *H* equals *J*.



So, if you take the divergence of this two. So, you are taking divergence on both sides. So, you are getting this equation on the left. Now divergence of a curl is zero. That we all know. So, it means divergence of this *J* is becoming always zero. So that is that the case all the time. But that is not the case all the time because electric currents they obey the continuity equation.

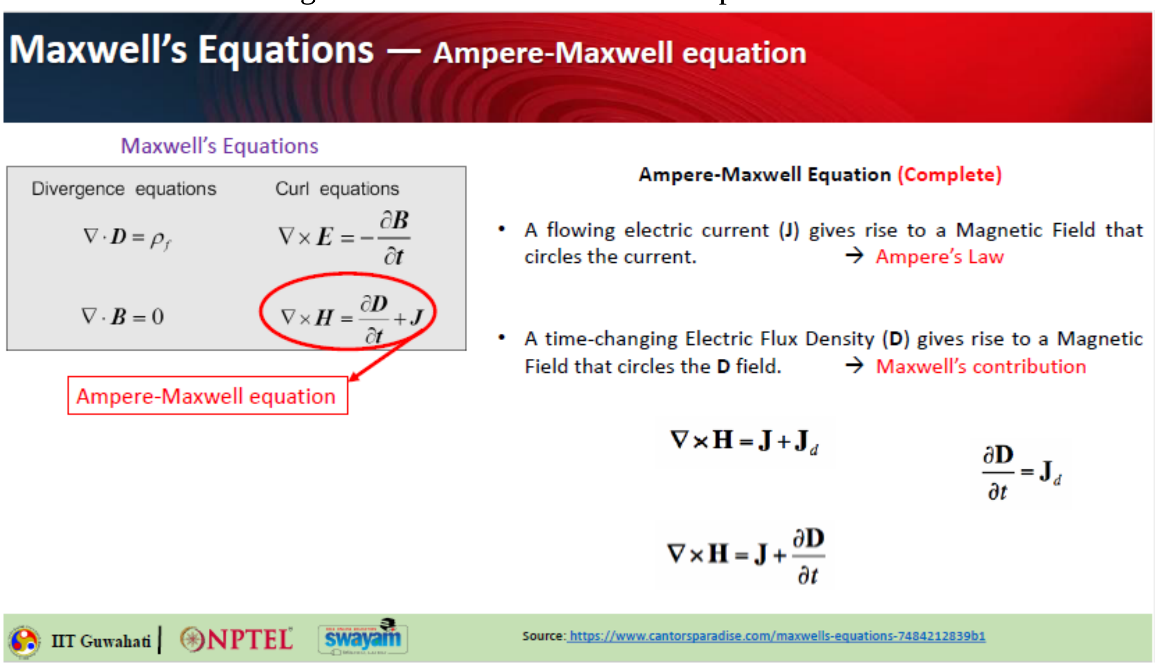
It means if there is some change in the charge density rho. That will also affect your current density. So, you can also relate it like $\frac{\partial \rho}{\partial t}$ is nothing but $-\nabla \cdot \mathbf{J}$. So mathematically, you can say that the $\nabla \times \mathbf{H}$ is not only just this particular quantity *J* plus it has got some extra thing. And that is some time dependent thing. So, Maxwell knew that a time varying magnetic field can give rise to solenoidal current that he has seen from the Faraday's law.

Then he thought that why not you know a time varying d field can give rise to a solenoidal *H* field. So, this is actually the beauty of nature because nature loves symmetry. And that is how Maxwell was able to introduce this new term called displacement current density which he named as J_d. So, $\frac{\partial \mathbf{D}}{\partial t}$ is the rate of change in electric flux density. And that is given as the displacement charge density.

So, when you add that term to this current density this equation is complete. Then in that case it will be able to explain all the phenomena in the electrostatics as well as electrodynamics. OK. So, this is the Ampere Maxwell equation in the complete form that a current a flowing current that is J_d give rise to a magnetic field that circles the current that is fine.

That is the pure Ampere's law. And then you also have a time changing electric flux density D that also gives rise to a magnetic field that will circle the D field. OK. So, this is basically the Maxwell's contribution.

So, you can write $\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d$ where $\mathbf{J}_d$ is nothing but $\frac{\partial \mathbf{D}}{\partial t}$. OK. Or you can simply write this one. So, here is a complete summary of Maxwell's equation. So, as you can see here on this particular column shows electrostatics or electromagnetics and here it is time varying. So, it is the dynamic one. So, in electrostatics and electromagnetics we assume electric and magnetic fields are independent of each other.



And in the time varying or the dynamic theory we assume that they are coupled to each other. Maxwell's equation these are all given in the integral form. So, in electrostatics you have seen this very well that the surface integral of the electric flux density is basically giving you the enclosed charge and that remains same even in the time varying nature also. Similarly, when you see that the electric field line integral is zero but in the time domain or time varying nature you will see that can be related to the rate of change in the magnetic flux density. Similarly the third equation that remains unchanged that you have already discussed.

But the fourth equation again the magnetic field lines they are not only equal to the current. But they are also having a contribution coming from the displacement charge density. So, this is how the modification in Blue they are showing the modifications that have taken place in the dynamic case. In the differential form this is how the differential equations look like. So, these are all based on there are two divergence and two curl equations as we have seen.

Maxwell's Equations — Static vs Dynamic

Table: Comparison of Maxwell's equations for static and time-varying electromagnetic fields.

	Electrostatics / Magnetostatics	Time-Varying (Dynamic)
Electric & magnetic fields are...	independent	possibly coupled
Maxwell's eqns. (integral)	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{encl}$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{encl}$
	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{s}$
	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$
	$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{encl}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{encl} + \int_S \frac{\partial}{\partial t} \mathbf{D} \cdot d\mathbf{s}$
Maxwell's eqns. (differential)	$\nabla \cdot \mathbf{D} = \rho_v$	$\nabla \cdot \mathbf{D} = \rho_v$
	$\nabla \times \mathbf{E} = 0$	$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$
	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
	$\nabla \times \mathbf{H} = \mathbf{J}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}$

Note: Differences in the time-varying case relative to the static case are highlighted in blue.

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So, $\nabla \cdot \mathbf{D}$ is ρ_v that remains same for the time varying or the dynamic field as well. $\nabla \times \mathbf{E}$ is zero but $\nabla \times \mathbf{E}$ in dynamic electromagnetic theory is $-\frac{\partial}{\partial t}\mathbf{B}$. Divergence of $\boldsymbol{B}$ is zero that remains same but $\nabla \times \mathbf{H}$ was only $\boldsymbol{J}$ according to Ampere but then Maxwell added this new term which is $\frac{\partial}{\partial t}\mathbf{D}$. And that completes the electromagnetic theory. So, these four equations are popularly known as the Maxwell's equation and they can actually describe the phenomena of electric field and the magnetic field that is varying in time and space.

So, with that we will stop here today and in the next lecture we will cover the wave equation and the boundary conditions that we could not cover in this particular lecture. Thank you. So anything any queries any doubts you have you can email to me that should. Thank you.