Adaptive Signal Processing
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Lecture – 20
Orthogonal Decomposition of Signal Subspaces

If a vector space has got an inner product defined on it, then it is actually called an inner product space which in a way general case of infinite dimensional spaces is also called, under some condition, Hilbert's space, by the way. He was a great mathematician in early part of nineteenth century, last century, Daniel Hilbert. He used to work; he developed that kind of frame work and worked extensively on that, his name, Hilbert's space. We will call it inner product space.

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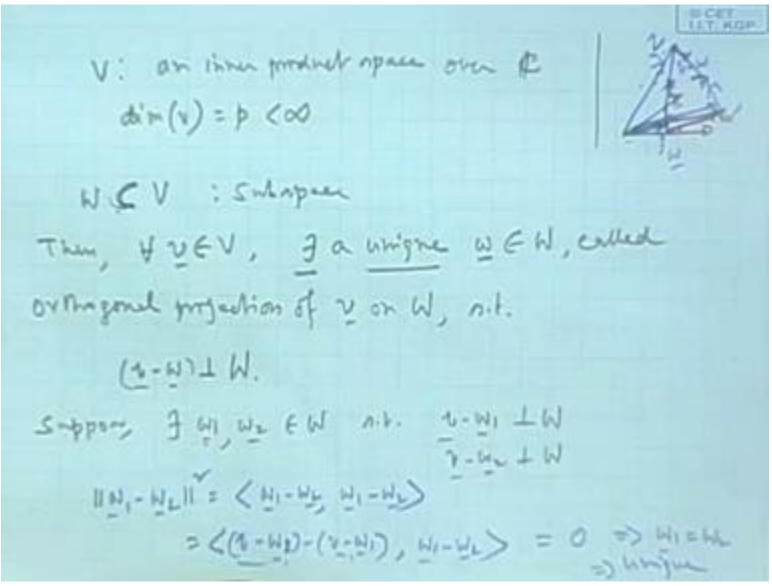

He had given you, V, an inner product space over complex split C or in general real C. In general, complex, or it could be real also. I am taking complex C. And, its finite dimensional, he is giving dimension v is equal to p; p is; that means, finite dimensional vector space, p dimension, fine.

Then consider, let w be a subspace, given you the notation for subspace, w be a subset; actually w is a subspace. It could be these with, I mean, it could be these are in original form that w equal to v, or w is a proper subset of v, both. But then for the time being, I am considering proper subset only. But this is a subspace. So, w I am writing this way. W is a subspace of v. It is not only the subset. It is given w is subspace of v.

Then, for all v vector element of v, and this v could be within w or outside w, but part of v. You understand? When I say for all v, element of V, V could be either within w or outside w which means also w, but within V. Then for all v, I am writing for all; actually it is I should write for each, but there is no these thing, notation for each. So, since it means for each v element of v, there is, for every v there exists a unique, remember unique, there exists an unique; both existence and uniqueness I need to prove.

A unique vector w is a vector element of W. So, given v you get only 1 w. In that sense, it exists and you get 1, but it is unique. What is this W? It is called orthogonal projection of v on W space; so that, it means what? I give the name. I am telling you given a v; that means a unique is W; and I give it a name, but what is this W? So, that v minus w, this vector is orthogonal to the space W. These are the notation; that this vector is orthogonal to the space W. You can call it as error vector or whatever; this is orthogonal to the space W.

That means for any, if you pickup any guy from W, take the inner product between that and v minus that w this vector, take any x element of w; v minus w comma x, inner product could be 0, for any x element of 0, this is what it means. It is unique; and then it has some property we will come to that. This is the crucial thing we now are adaptive filter course actually this project something. I wanted to arise these and its property and other things. For that only I developed all that vector space and dimension basis orthogonality and all those things.

Projection is a key thing because linear, I told you positively earlier the linear estimation is nothing but the problem of finding an orthogonal projection, then by inner context of optimal filtering it probably. But at that time as using the analogy of 3 d space and all those things not properly that mathematics; for that time I also wrote vector space and come back to that, these make a properly, I mean mathematical treatment, even math mathematical treatment to that. That is what it is now.

So, first I will prove uniqueness. Suppose it exist. How it exist? We will see later. It is very easy. Suppose it exist then given a v there is only 1 w. You cannot have w 1 and w 2 for; so that in both cases, v minus w 1 also orthogonal to W, v minus w 2 also orthogonal to W. If that be w 1 and w 2 has to be same. That is, this is the unique. That is suppose there exist w 1, w 2, element of w, so that v minus w 1 orthogonal to W, v minus w 2 orthogonal to W.

Then, I will prove that there is no other way, but w 1 and w 2 these 2 have to be same. That is this property at the error will be orthogonal to this capital W. This is possible only for 1 w; this w vector belonging to the W. You get, already in your mind, even in that, you know, I mean visualize the thing in 3 a dimension vector space; you know, that geometrical vector space given a vector you are projecting this to be perpendicular on a plane or whatever.

If it a point and that point vector is a projection you can have only 1. You can have 2 such, so that the error is still orthogonal to the this perpendicular to that plane. You understood what I am saying. And you, I will come back to that figure later; and may be here only. What I am saying is this. Suppose you have got w is your space span by these 2 vector that is the plane. And, there is your 1 vector v.

This v if you take a perpendicular, then v minus this vector is this. This is v equal w vector. This is your v vector v minus w. This is v minus w. And since, I have taken to be 90 degree, v minus w is orthogonal to anybody, any fellow in this plane, in the space span by these 2. Is it not? But in this process do you, can you get more than 1 such projection vector. You can get only 1. Is it not?

If you call this w, you cannot get w 1 and w 2, 2 side vectors; so that for both of them, v minus w 1 having 90 degree, v minus w 2 also having 90; that is not possible; how many element there would be, you know. Is it not? But this is only visually it is coming. Actually the mathematical thing is here. Using those figures I remind you when I; I am not proceeding any angle, any meaning of the vectors. I am only using the basic motions of vector addition, scalar multiplication, and those few things about inner product orthogonality and all that.

I am not using any visual angle nothing. So, whatever we prove here that is the most general thing. Here, visually it is obvious that 2 such projections are not possible which hold 90 degree angle with the tip of the vector. But, mathematically the reason is here which follows. But, suppose, if it is true then I will show w 1 and w 2 has to be same, have to be same.

That means, if I take you gave me, give me 2 things, w 1 and w 2, and suppose they are not same, but still they hold the property, orthogonality property, then I said fine, if that be, so let me take the difference between these 2 projections. If at all 2 different projections exists, and take the norm square. Norm square means inner product with itself. But this w 1, w 2, I can always write as v minus w 2 minus, v minus w 1; and this is w 1 minus w 2, as it is. Now, w 1 belongs to W, w 2 belongs to W. So, w 1 minus w 2 belongs to W.

And this is the inner product. I can take out this component first. This inner product can be separated into 2 inner products; probably, these with these. What will that be? That has to be 0 because v minus w 2 orthogonal to W; but w 1 minus w 2 belongs to W because both w 1 and w 2 belong to W. So, that inner product will be 0; minus this one; inner product between these and these; v minus w 1 and w 1 minus 2 for that by the same digit that is also 0. Is it not?

Because v minus w 1 is orthogonal to the inter subspace w, and w 1 minus w 2 belong to W; so then this equal to 0. Is that understood? If norm square is 0, there is only possibility that the vector has to be 0 vector, by your axioms. So, that means, this is possible only when w 1 minus w 2 is 0 which means w 1 equal to w 2. So, if you take this, it is unique; implies unique; not too such w is exist; so that the v minus that w is orthogonal to the subspace W.

And then comes another thing which is basically generalized version of Pythagoras theorem here. That is I will show, and anybody if not knowing Pythagoras you also can show. That suppose, you are proving that v can be orthogonal, you project it, you can get only unique w; but suppose, I take up any other vector, so w prime also from the subspace; and I measure the 2 error vector; one is v minus w where w is the orthogonal projection like these; and I pick up another vector; say, this is w prime, I take these error also.

So, this is 1 error, this is 1 error; these 2 I compare. I compare who has the higher norm. And Pythagoras will come up, we will say this as, this norm is higher than these. That means orthogonal projection is such unique vector, so that the difference between the vector been projected; and that vector is orthogonal to the plane to the space, and that has the minimum norm square.

Whereas, you take any other arbitrary fellow in the space, find the difference between the v, and that fellow I will take the difference vector and find its norm square, that will always be larger than the one when you do the same with orthogonal projection of v on the space. That is why orthogonal projection is so important to us; it gave, it gives us that error which has minimum norm square or minimum norm.

There is a projection is the best estimate of v, when I am standing in, when I am situated in w space. Sitting in w space, the best estimate of v will be the orthogonal projection of v on w. Because other, if w prime I take, I find the error between v minus w prime. It will have larger norm than between v and its projection, orthogonal projection; whenever is a projection, orthogonal projection. It is very easy to, you know, prove this.

You have to find out the norm square of these vectors. That is any arbitrary w prime and v; v minus w prime is this vector; find norm square of that. That norm square will be what? You, from this point you connect this point. This angle is 90 degree. This norm square will be square of this plus square of this, Pythagoras. And therefore, norm square of the, this norm square will be larger than the norm square here; as well as these 2 points are different. Is it not? The Pythagoras will be proved very easily. And, again you see, using our basic axioms only I write the Pythagoras theorem. I am not using any geometry, the angle, and all that in all this proof. This is only for mental clarity.

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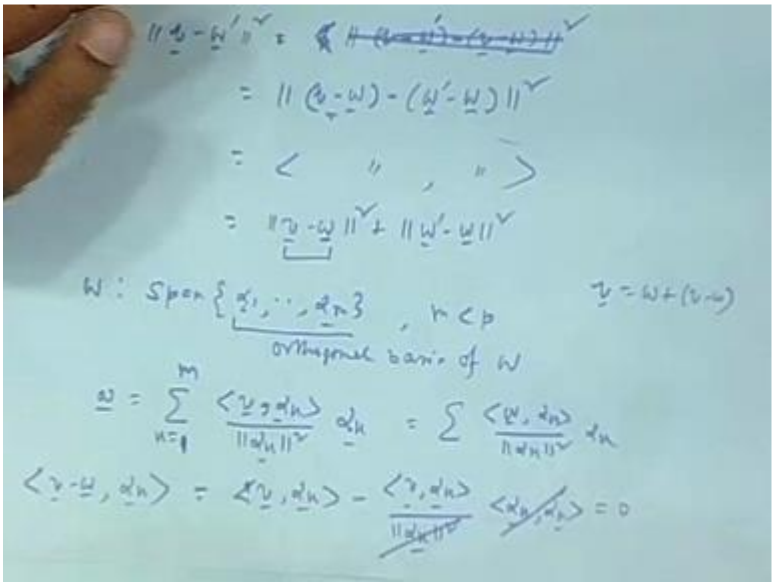

Here, see I am taking a v minus w prime, arbitrary w prime, and norm square of that. That will be what? V minus w prime, I can write as this norm square of v minus w minus, v minus w prime minus this; v minus, sorry; this will be very simple thing. This vector is nothing but summation of these 2 vectors; that is why I am writing. If you reverse the sign, w minus w prime, it becomes sum of this; is it not?

But actually, algebraically I am just writing v minus w prime like this; v minus w where w is the orthogonal projection of v on W; W is W; and this is the difference. And inner, norm square is inner product with itself. Is it not? These with itself. So, there will be, if we expand it there will be 4 inner products; these minus these, again these minus these. Direct terms will be v minus w with itself, first; that is this one.

And, another term will be these with itself; and 2 cross terms. Cross terms will be 0 because v minus w, w is the orthogonal projection. So, v minus w is orthogonal to anybody in W. And, this is lying in capital W; this difference because both each of them lies in W. Is it not? W prime lies in W; from that only I took w prime; so does w. So, difference lies in W. So, this vector is orthogonal to this vector. So, cross terms will be 0. Is it not?

So, this is the inner Pythagoras thing, you see. I put in use on 2 sides, and this is the orthogonal projection. So, this is the error for when I am orthogonally projecting v on w, finding that projection, taking that error. So, that error norm square is always smaller than the error when you are not orthogonally projecting, but you are taking arbitrary w prime, because this is, this plus this. Orthogonally these 2 are same. This is a finite projecting quantity. It is a positive quantity. So, total is larger than these. Is it not? This is that Pythagoras actually.

That is why orthogonal projection is so important to us, because the corresponding error vector has the minimum norm square or minimum norm. So, error will have minimum norm means minimum strength. So, minimum norm solution, minimum error solution will be orthogonal projection, that be the best estimate of an external fellow v when I am staying in the subspace w.

But then comes a question of existence. Everything I assumed, that I assumed that exist. If exist then its unique, then minimum norm, all those things I proved. But existence is the required. Existence is very easy. W is the subspace of V. V is the inner product space. There is a inner product defined. So, obviously, by GramSmith procedure I can have a orthogonal basis or any number of orthogonal basis, or orthonormal, even orthogonal basis say for w.

There is atleast 1 orthogonal basis of w exist; and therefore infinitely many exist. That we have proved that day. See, I did not do you all those things for fun. I did that thing with something in mind. So, there exist an orthogonal basis of w; call that basis vectors to be alpha 1, alpha 2, upto, dot, dot, dot, say, alpha m, where m is less than p in this case because I am taking a proper subspace. But, suppose w is a span of m less than p given orthogonal basis. So, W given, its 1 orthogonal basis given because orthogonal basis exist we have seen. Now, you have been giving that external vector v. V could be in w, outside w, but v.

To start with you can assume, let v to be outside because v is already inside; you see, projection of v is v itself. Because then only the error will be having minimum norm; and norm that will be error will be 0. If v is already in w, its projection will be v only. Projection is unique. And I find 1 vector v itself, so that the error vector is the minimum possible norm; that is 0. That means, that has to be the orthogonal projection because orthogonal projection has the minimum norm and projection is unique. So, that must be the orthogonal projection. Is it not?

So, that is the vector real case. That is why it is in vector, you take small v to be outside w, and in V. Then it makes sense really. If you already have v within w what is the point in estimating it. So, this is w giving to v. Now, you have been given a v. Now, I say you form this vector as linear combination of these elements, some linear combination, so that you get something belonging to W after all. Orthogonal projection also belongs to W, so that will be some linear combination of these, orthogonal, but what kind of coefficients?

I say, suppose you take the coefficients like this (v, alpha k), k equal to 1 to m, alpha k. That is v is that external vector; v is inner product. You take it inner product of v along any of these vectors, divide by the norm square. That is a coefficient, multiplied by the corresponding vector. See, you get something belonging to W only represent the linear combinations of these ones. But you can; I will now show that the error between v and this vector that is orthogonal to entire W because that is orthogonal to each of the basis vectors. You understand my logic?

I will find out v minus w error. If that error is orthogonal to each of them, then obviously, orthogonal to any vector of w because any vector is a linear combination of these. Is it not? So, I will just show, say, v minus w, say, with alpha k is 0, for all k, k equal to 1 to m. I will show this; and it is just not very easy. You replace w by these. So, 1 inner product is between v and alpha k. Write that separately, v and alpha k. Another is, minus with this.

Remember, all alphas are mutually orthogonal. So, in this inner product only one will survive. That is, when you have got k th term in a summation; and then that time what?

Alpha k with alpha k, and this is a coefficient. This coefficient is what? This coefficient; please see one thing. Only that will go out, but this is in the second coordinate and I am dealing with complex case in general.

This is the coefficient coming with what? This is coming with w. So, this will go out as it is; v alpha k by, alpha k square. This is a coefficient that goes out; and then alpha k with itself. Alpha k with itself means again another norm; and obviously, this is the alpha k norm square. That and this cancels. These 2 cancels; and these 2 are same. This is 0. So, then this is the actual form of that. It not only shows, it exists. Uniqueness I have already proved. Now, exists I have proved that such a orthogonal position exist; and you now got the specific form of the projection.

That take any orthogonal basis, it could be anywhere; either these are in terms of beta 1 or whatever. Whatever you choose, can be the inner product with the external vector v with them divide by the norm square. These are the coefficients by which it linearly combine the chosen basis. That will be the always getting at the same omega, same w. That is the projection. These are from the form of the projection.

Student: Voice not clear.

No, I know that, suppose I am throwing you this thing. Suppose I do this I know because of linearly I have experience, I know. But suppose I do this, then it exist. Actually know this is you will see; each coefficient is what you know. Each, some coefficient times are alpha 1, some coefficient time alpha 2. Each is projection of v along alpha 1. Projection of v along alpha 2, we will see later. And, all of them I am linearly combining. It comes easily from that thing.

Now, so existence, uniqueness, all the thing done; I know the specific form also. Instead of v you could also, instead of v you can also, v is the external fellow. This coefficient as, I mean you can also write as w comma alpha k; simply, even if this is not there. W is something belonging to; at this is a very small w. Forget our projection and all. Small w is part of capital W. Is it not? So, that means, in terms of this basis you can write w as a linear combinational basis; this way we have already seen earlier.

That you can always write w as a linear combination of this which combiner coefficient will be w inner product with these by norm square, but that is same as this; another, from another angle you can see; v is what? V has got 2 components. One is w component. Another is the error component. That is, v is what? W plus v minus w. So, if each inner product you replace v by these w and v minus w, w with alpha k comes here; v minus w is orthogonal to entire w and therefore, to each of the alphas. So, that inner product goes. Instead of v you can also put w here. Remember this. You will see many times I will be playing this kind of tricks and taking some active error filters and all that related. I have not told about linear operation. Is it not? I have not told about linear operation. So, I, let us tell about something about linear operation.

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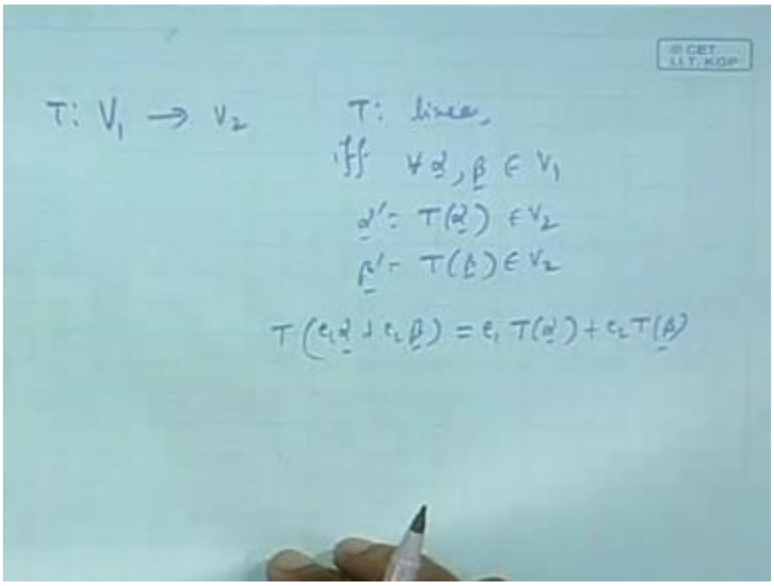

Suppose, there is a vector space v 1; there is v 2. And, I define a linear operation between v 1 to v 2. That means what? To each element in v 1, take each element, map it to some v 2. That is all. 1, 2 may need not possible. You take every 1 to you will get only 1; many to 1 possible. When many to 1 possible then actually it is not any invertible operation because, you know, out of anything here you can get more than 1 solutions. So, those are the thing; but 1, 2 is not possible; many to 1 may it possible. These operator will be called linear operator.

If T linear; if for all alpha beta, element of v 1. Suppose, you have, you know T alpha; T alpha equal to say alpha prime; alpha prime belonging to v 2. T beta, you call it beta prime belonging to v 2. Individual mappings you know. But, suppose instead of alpha, beta, I get c 1 alpha plus c 2 beta; T working on that. If T is linear then this would be c 1 times, T alpha plus, c 2 times T beta. Then it is linear; but, if and only if.

You understood what am I saying is a linear operation. That is it means if instead of giving each vector separately it will be linear combination, T working on that is simply T you apply on the individual components and combine them in a super position hold. That is linear operator. Orthogonal projection is a linear operation.

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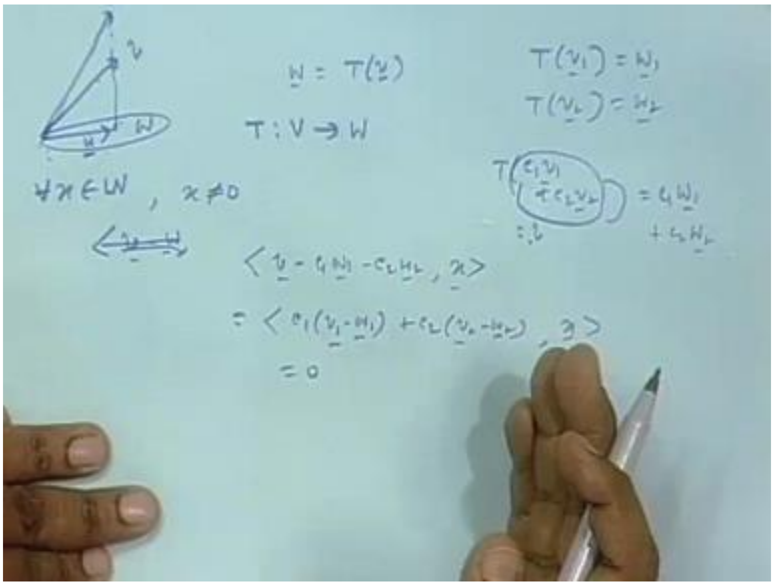

That is suppose you have got this space w. You have got a vector v which could be in w also. Both v and w they are in the bigger subspace of V. Here projecting orthogonally, we are getting a vector w. This unique, it exists, everything I know. Now, suppose w, I write as an operation on v, that is working with T working on v will give you w. It is an operation after all; linear or not linear, I do not know.

I, if instead of v I use another vector, I will get another projection. Capital W space is constant. You keep taking different values of v. Different vectors you keep getting different projections on w itself. So, that means, projection is an operation which takes any choice of yours as v and gives you something in return from w. Is it not? So, this an operator relation. It takes v as its operand and gives you something from w as the output. It is an operation.

I am saying this is linear; that is, v 1 projected on that; suppose, it is given here some w 1. This is trivially seen actually T v 2 w 2. That is v 1 is 1 vector. Project you get w 1. V 2 another vector. Project means subspace remain same. The subspace w, I am not changing. V 1 is something I am projecting, I am getting w 1 out of w. V 2 is another fellow I am projecting, I am getting w 2 out of w.

Then I would say instead of v 1, v 2, if I have c 1, v 1 plus, c 2, v 2. If I project this vector I should get, this might clear, I should get just these. So, T actually is crime, what? T is a projection, T is an operator v to w. Is it not? T, you see; what does T take? T takes any vector from the first, from the vector space v. That could be in w; that could be outside w, but it is always v. But what it gives you? It is always part of w only. It takes any fellow from v. Any fellow from V and gives you something from w.

It is, obviously, you can see at many to 1 transformation because w is constant in these. Suppose you take, physically if you take a line from here, you take another vector, these 2 projections will be same. Geometrically you can see, you can abstract any abstraction only you can see. It is many to 1 transformation. That is through this vector if I add anything along these, this error, any multiple of the error I add, I get this fellow. This vector plus error times, some constant. It is equal to this; will give you this.

That also you will have the same projection; you can find out anyway. So, it is a many to 1 transformation. My only thing is, only contestant here, that is a linear transformation. That I say instead of v suppose I start with v 1 I get some kind of projection w 1; another v 2 I take, I get w 2. Instead of v 1, v 2 if I have a linear combination c 1, v 1 plus c 2, v 2 outside. Project that, that will same as c 1 w 1 plus c 2 w 2. That is seen very easily.

I will just maintain the thing, fact, that, make use of the fact that projection is unique and that projection error vector is orthogonal to that space. Is it not? So, question is if these be the vector v, is it that v minus these that error is orthogonal to w. If that be then obviously, this is the projection of v on that because projection is unique. If you call this vector v, linear combination, then v minus this quantity; it is a vector in w. I do not know whether it is projection or not.

So, I take error. If that error is orthogonal to W, then obviously, it is the projection of v on W because error is orthogonal, and projection is unique, and that is very easily seen; V minus w for all x, take any x, element of w any vector of w; v minus w, sorry; v minus this quantity; c 1 w 1, c 2 w 2 with x; x is any vector on w. Is not that this error is orthogonal to any vector of w? So, I took any vector, any x.

If you can take x nonzero also, otherwise it becomes trivial. If you take x with 0 then irrespective of anything else the inner product is always 0. Suppose, I take nonzero x out of w, then is it that this error is also 0, obviously. Because, what is v? V is c 1 v 1 plus c 2 v 2. Put that here. So, you can write c 1 times v 1 minus w 1 plus, c 2 times v 2 minus w 2 comma x. And now separate out, v 1 minus w 1 with x to be 0 because w 1 given to be the projection of v 1 on w; same with the other; so inner product is 0. Is it not? We trivially seen. So, it is a linear operator.

If you, otherwise you can see its specific form. This was the form. Is it not? This was the form. Instead of v 1, v here. Suppose you put real thing vector c 1 v 1 plus c 2 v 2, you can spread out c 1 times inner product v 1 with alpha k, c 2 times inner product of v 2 times alpha k. So, in that way you get 2 different projections because this is linear in the first coordinate. So, orthogonal projection is a linear operation.

Remember this. 2 vectors projected on the same space. 2 vectors, each one is projected on the same space, you call them w 1, w 2. Instead if they are added and project it, projection will be nothing but w 1 plus w 2. Please understand this.

Another important portion now I come to that is direct sum decomposition of vector spaces. Again, I will come back to the just vector space and not inner product; and then I will come to again the special case where vector space also has been given a inner product and it is called inner product space. So, I again come back to the generalization of just a word I obstruct vector space, no inner product, nothing.

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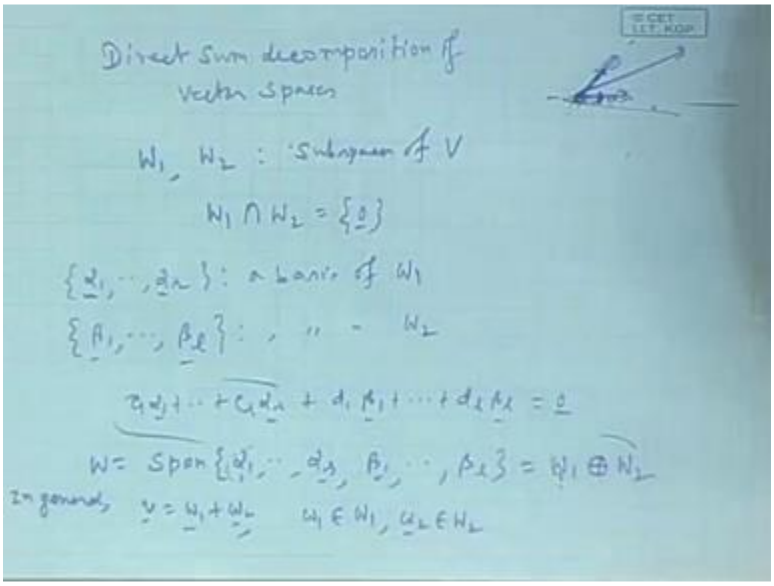

And then I have direct sum decomposition. Suppose w 1, w 2, subspaces of v. Our strategy is same. V is the vector space. Dimension p and all that over complex field c. W 1, w 2, are 2 subspaces such that there is no intersection, but they have to intersect at one point because that is the 0. 0 is common and unique. So, that give that w 1 intersect w 2, only at the minimum possible thing that is 0 vector. Like, you know, x axis and these axis. This line is one subspace. This line is another subspace. This is span by this fellow. This is span by this fellow. Is it not?

These 2 subspaces, they are subspaces of the 2 d plane, 2 d space. This plane of this you know. It is a subspace, but there are they intersect only at 0 vector at the origin. Something like this. Then suppose, alpha 1 dot, dot, say, alpha r be a basis, no orthogonal basis here because no inner product, a basis of w 1. Beta l a of basis of w 2. That means this is a linearly independent set; this a linearly independent set.

Then my claim is if you append the 2 that also the linearly independent set. Physically it is, because these fellows are outside the space span by these. So, if you append anybody that forms a linearly independent set. But mathematically this set union this set, is a linearly independent set; you can easily prove that. Suppose you form a combination like this - c 1 alpha 1 plus dot, dot, plus c r, alpha r, and then d 1 beta 1 plus dot, dot, dot, plus d l beta l equate to 0 vector.

If this is so; if this possible that you have non zero solutions for these; suppose you have non 0 solutions for this for all, that means this part is negative of this part. When you write this, if it otherwise. Suppose I have only these, I can always take these on the left hand side, right hand side or take this the right hand side. So, this summation is equal to this summation.

But these 2, that means, this is something; but left hand side is a linear combination of alpha 1 to alpha r which it belongs to w 1, right hand side is a linear combination of beta 1 to beta l, so it belongs to w 2. But if this is then some vector, if left hand side equal to right hand side, that means this would imply, it will belong to that both w 1 and w 2. Left hand side belongs to w 1; right hand side belongs to w 2; we know.

But, if you can equate that 2, that means, they belong to both w 1 and w 2. But what belongs to both w 1 and w 2? Only 0. So, this part is a 0 vector. But if this is 0 vector alpha 1 to alpha 2 is a linearly independent set. So, only value possible is c 1 to c r is 0. If this part is a 0 vector, beta 1 to beta is a linearly independent set. See the way I had proved.

These are the things we should to develop, you know, this mathematician logic; or, this is just logic, no kind of, you know, which was a thing. No analogy, no histolytic discussions. See the proof; the strictness, correctness of the proof. If this is so these equal to these with a negative sign. That means this is something, which belongs to both w 1 and w 2. So, these and these have to be the 0 vector. If this is 0 vector, all the coefficients must be 0 because alpha 1 to alpha l linearly independent, same applies there. So, this is the only solution for this is c 1 to, these are methodical mathematical proof c 1 to c r and d 1 to d l 0.

Obviously, here you see, if you append that 2, here this is span by one fellow, here this is span by one fellow. These 2 fellows are linearly independent; if you append that 2, they are linearly independent. If you consider the span of that union set that will be a bigger vector space than w r and w 2 that will contain w 1 and contain w 2; if you call w to be span of an, after this can be a basis because they are linearly independent. So, I am considering the span of these.

Dimension of the space will be l plus r. So, this w then I will write as this way; this a direct sum. This called direct sum. It is a direct sum of w 1 and w 2; it will mean what? Any vector of w. Please read this. This is key to the remaining part of the adaptive filter course; this direct sum decomposition thing. Earlier I said about orthogonal projection, and very soon I will generalize, I will come to inner product space; and I will take as very specific case of such direct sum decomposition called orthogonal decomposition.

That is central to the entire adaptive that will come out. Everything is wasting on that. This will be denoting as this way notation only. What does it mean? Any vector of w is what? What does it mean? First, w 1 and w 2, whenever you see they intersect only at 0 direct sum is mutual. So, intersect only at 0; 0 vector.

And, by these I am constructing a larger subspace which will take the basis of these and these as the union to form from its own basis; which means any vector of w will be what? In a linear combination of these means there will be 1 component with a combination of this fellows, another component combination of this fellows. So, in general, any vector of w will be a summation of a vector from here and a vector from here.

That is in general; so remember it is not set union of w 1 and w 2. Please understand that. This is not that w 1 is the subspace, w 2 is a subspace; you are taking the union of that; please note; this is the difference. It is not union of w 1 and w 2. You are adding not appending; adding a component from here and here. It is not union. What is that direct sum of these 2, these 2 d plane? Because these 2 basis together, these 2 form together larger basis; they span these. And therefore, any vector in these will be a linear combination of something from here and something from here. And these are not some component from here, some comp in general.

Of course that will also take care of those cases where w 2 is 0; and therefore, v is w 2, w 1; that is only these axis. W is a direct sum means, it will also contain w 1 means, those linear combination for which the coefficients here are 0. So, all elements of w 1 are part of w because that time you are taking 0 element from here; 0 vector and some element from here adding, there part of w, vice versa. All elements of w 2, adding 0 from here, give you the same element, that also a part of w.

But, why we are not adding with 0? We are basically adding something from here something from here, they are non zero. So, there is a typical form of element of w. Are you understanding? Please see, this is not set union of w 1 and w 2. If you do know in mind, does not work, always think carry this one, whenever you get this sign plus, first tell your mind that this means they intersect only at 0. Next I will take a basis of these, basis of these; I will form a union of the basis; that is the space, that space is indicated by this fellow, this notation.

So, there any vector will be having 2 components, summation of 2 components, addition of 2 component, in general; a component coming from here, a component coming from here. Sometimes the component, this component could be 0. So, you get the component here only. So, basically that time you are taking that segment of w which is nothing but w 1. Like this 2 d plane, if it is in the 2 d plane, if you working along this street, this line, it will be hitting upon only these vectors. They will not represent; they will be present only at the origin.

So, 0 plus this, will give you this, that way; or, it can be on this line, 0 times these, here plus these, will give you the same thing there; those are part of these. But, in general, any vector will be, if you write it, it will be having one component in this direction, one component in this direction. Now, suppose, I have got a inner product defined on these, I am not considering linear, here there is no inner product; only linear independence dependence was basis.

Now, I am giving one more this thing, you know, constraint; there is, not constraint, one more thing, one more tool I am giving that is it is not only. So, general it has got also additional thing given to it that is inner product. So, when you are having inner product, all those thing are projection and orthogonal, everything comes up.

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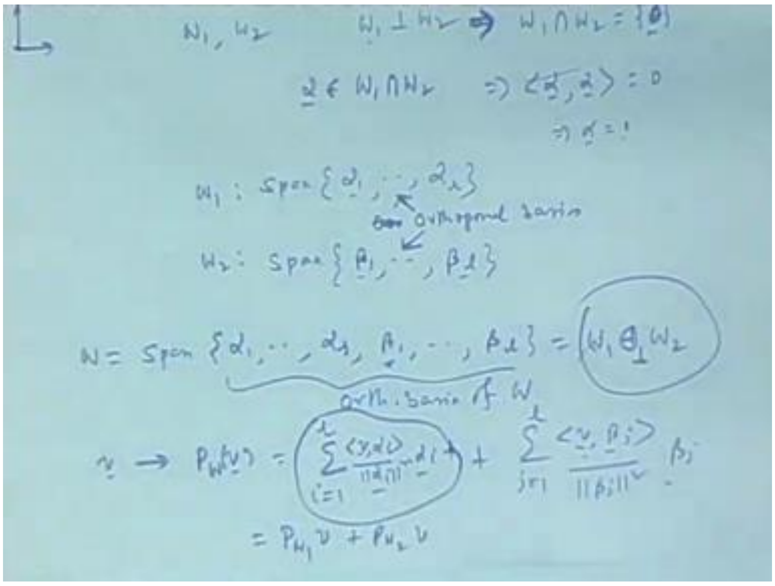

There you see one thing, suppose w 1 and w 2 are such that it is given that w 1 is orthogonal to w; that means any vector of w 1, any vector of w 2, they are mutually orthogonal. Then my claim is they intersect only at 0. Here I directly get that w 1, w 2 such that they are intersecting at 0. Here I am saying then if they are orthogonal, then it shows that w 1 orthogonal to w 2 is nothing but 0. Can you prove this? It is very easy. 0 vector; one line proof.

Student: Voice not clear.

But how? What will you do? Suppose, there is an alpha that belongs to these, that belongs to this space; that means alpha belongs to w 1, and alpha belongs to w 2, and w 1 and w 2 are orthogonal. So, that means, alpha with itself is 0 because alpha belongs to w 1 also, alpha belongs to w 2 also. Please see the way I prove. There is no space logical difference anywhere. Alpha belongs to only thing, I am not writing the way instead taking book, most of the thing are conveying that will waste too much of time.

I am taking an alpha belonging to this. That means alpha belongs to w 1, alpha belongs to w 2; alpha, alpha inner product must be 0 which means norm square of alpha 0. This means alpha 0. So, that means, this has only 1 element, 0 vector. See if w 1 obviously, if you have these, if you have these 2 axis which are perpendicular each other they intersect. There is nothing common.

Student: voice not clear

This is proof. I am proving. This implies, and then I prove it. This implies, and then I prove. That suppose, I do not know this, so I pick up any alpha from here. That alpha belongs to w 1 also and w 2 also, but w 1 and w 2 are orthogonal. So, alpha with alpha must be 0. Then norm square of alpha is 0. Then alpha is 0. So, only fellow lying in the intersection is 0.

Then, if you take w 1 as the, say, for span of, I mean you have got say, span of as before alpha 1 dot, dot, alpha what, l 2 r. But this is given to be now; not just any basis orthogonal basis because you have given inner product not orthogonal basis. Orthonormal also, but I am taking the general case of orthogonal. Both these are orthogonal basis.

After I writing in twice, this is also orthogonal basis, this also orthogonal basis. Then, I form the union of the 2 basis; obviously, w 1, w 2 intersect at 0 we have proved as before; and now, we form a basis; we form these. Is it an orthogonal set? If you take any 2 between here they are mutually orthogonal, any 2 here orthogonal. But, any alpha from here, beta from here, they are also orthogonal because alphas belongs to w 1, betas belongs to w 2, and you have given me w 1 and w 2 are orthogonal. So, it is an orthogonal basis.

If you consider the span of these if it is a span of these and call it w then this is orthogonal basis. This is an orthogonal basis of w. Then this w, I will write again as before; you see if in otherwise this is a basis, and I as before I can write it like this. Because, any vector of w will be linear combination of this which means component from this side, component from this side.

I told you, whenever you have this, imagine that they are intersecting at 0; and you are appending the 2 basis; you are forming the larger basis; span of that is what is meant by these. So, any vector of that span space, this space is d. In general, I have 2 component, 1 from here, 1 from here. Here, I have got the extra thing to, you cannot put a this sign here, to indicate that these 2 are intersecting at 0 because these 2 are mutually orthogonal.

And therefore, you can take this orthogonal basis orthogonal basis. Orthogonal basis always exists. If you append them, you get a larger orthogonal basis. Take the span of that, that is meant by these. Now, if you have a vector v, you have to project it on w. P w is an operator v. I am, please understand this notation; actually, I suddenly brought this notation from somewhere, p w; what I mean is, I have told you that projection operation is a linear operation. Is it not?

That time as I am using the notation T. T working; or instead of T, you know, I am bringing p. P w v stops the projection; and projecting on whom? W. So, P w is an operator it takes a v, P w v. So, what that it is means actually? There is a vector v, I am projecting it on the subspace w; but that formula I know. Please understand; that formula I know. What is the projection?

It will be some say orthogonal basis. Please see this; this total is orthogonal basis of w. So, v with inner product; v and inner product between v and any of this fellow, divided by norm square of the fellow, multiplied by the corresponding fellow, summation. So, you will get 1 set from alphas, 1 set from betas. Then will be 2 summations. One say, i equal to 1, 2 or another say j equal to 1 to a. And, this time you will have v, alpha i, by norm square alpha i.

Student: Why we are not taking orthonormal basis?

This is more general. In orthonormal this will become 1, but I want to be more general, so that this expression will show off. That is all. You can take orthonormal. That is not problem; there is no difference, ok; times alpha i. If you take an orthonormal I will not detect any marks, it is. ok. But I do not need any equation that things are valued only for orthonormal basis.

When I prove assuming orthogonality that makes my life simple, this task become 1. And this side again, v with beta j divide by; this is the projection, but you see the beauty, what is this component? This is the projection of v only on w 1. Orthogonal projection of v on w 1. What is this? Orthogonal projection of v on w 2. That means this is nothing but P w 1 v plus P w 2 v. This is possible only when the 2 subspaces, w 1 and w 2, are mutually orthogonal.

Please note this. This will play tremendous role in all the adaptive filters things we do. Overall projection is the summation of the individual projections where the 2 subspaces are mutually orthogonal. So, that is all for today.

In next class, I will take you to again this random variables. And, that time I will be dealing with inner product space or Hilbert's space, you can say of random variables. Random variables, they are all defined, get vector spaces involving them. What is meant by addition, subtraction, inner product, all that? Then, I will get into what is called linear prediction, estimation, all those things.

One filter will be developed in that process. One structure called lattice filters, linear prediction lattice filters, and all that. Finally, you will make it adaptive.

Student: Voice not clear.

Of course, otherwise not; because if that orthonormal then only I pick alpha 1 to alpha r and beta 1 to beta l, they will become orthogonal basis. Since they are orthogonal basis, overall projection can be maintained like this. But, I am picking up one component here, and another component here. And, now I tell you without doing this, taking the explicit form, please take the exercise.

Suppose, you do not know this form, this form may explain very easy; suppose you do not know the form, you only know that w 1 and w 2 are 2 orthogonal, I mean 2 subspaces which are mutually orthogonal to each other; v projected on w 1 was giving you small w 1, v projected on w 2 subspace was giving you small w 2, then proof, directly that v projected on this thing will be summation of w 1 and w 2, without taking to this form.

Thank you very much.