Adaptive Signal Processing

Prof. M. Chakraborty

Department Of Electrical & Electronic Communication Engineering

Indian Institute of Technology, Kharagpur

Lecture No. # 39

Singular Value Decomposition

In today's and tomorrow's lecture, we will consider one topic which is very useful, very important thing in the domain of statistical signal processing and adaptive signal processing. You can call it and treat these lectures as a kind of appendix to our main lectures that ended in the last class. This particular topic, I repeat, it is very important in signal processing is called singular value decomposition.

Singular value decomposition, that is SVD of a matrix and pseudo inverse of a matrix; this is very useful in communication controls signal processing. I thought that will be a good occasion to introduce this topic to the students, but before I get into this; I have to derive certain properties of matrices and all that, which I will do today. In the next class, I will use these results to get into SVD theorem and its extension to pseudo inverse. We have already done some exercise on vector space theory.

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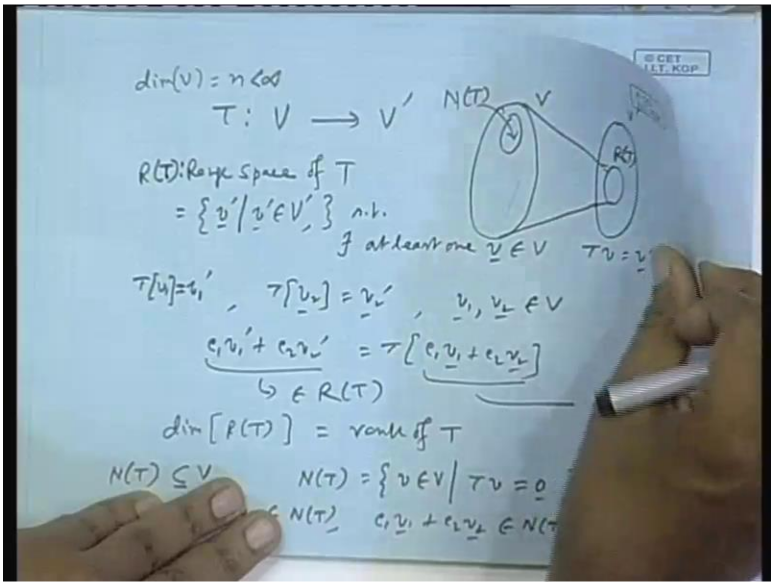

Suppose, I consider a finite dimension vector space V and a linear operator T, which maps V to this say, another space V prime. That is, there is V and this is V prime towards some field; field is either real or complex, and T maps it to some domain within V prime. We call it R. R is called the range; R is the range space. In fact, R will be vector space we will shortly, see range space of T.

Then, what is R? R consists of all vectors of V prime for which, there is a vector in V. So, that T working on that vector V, gives you this vector nr; that is R consists of V prime. V prime is element of this, and so, that there exists at least one v element of V with TV v prime; loosely read this. This is the R is vector space, because if you take say, 1 vector v 1 prime which is say, T T 1 equal to v 1 prime and again say, TV 2 equal to say, V 2 prime. V 1 v 2 element of v and of course, v 1 prime and v 2 prime element of v prime, then C 1 v 1 prime plus C 2 v 2 prime; this is nothing, but T of c 1 v 1 plus c 2 v 2, but this element, this is an element of V.

That means this is an element of R, because R consists of all maps of V. That is any vector R consists of all those vectors of V prime for which, there is a source in V. That is if we pickup any vector v prime from R, we must find at least 1 vector in small v and in capital V. So, that T working on v gives you that v prime. Now, you understand here, that V 1 prime is this part of R and V 2 prime belongs to R, then any vector linear combination of them also, belongs to R, because you find another vector as an element of V that is c 1 v 1 plus c 2 v 2 on which, if T works, gives you this.

That means this is closed and therefore, this vector space this called range space. Dimension of this range space, dimension of R, in fact, I will call it RT. This range of T; you should call it RT dimension of RT is called the rank of the operator T. Similarly, there is another space that is important, that is called null space of the operator T; that is null space NT; NT is a subspace of V. NT consist of all vector elements of V. So, that TV maps to 0; that is T working on those vectors match to the 0 of V prime. If you collect all these vectors, then it also becomes a space, subspace and is called null space. It is a subspace, because if you take v 1 v 2 element of NT, then obviously, c 1 v 1 plus c 2 v 2 also, element of NT. Because, if you apply T over this vector, T is linear. So, you can apply T over v 1, which is 0; T 2 v 2 which is 0; and summation is zero, which means, c 1 v 1 plus c 2 v 2 also belongs to NT, which gives NT as subspace, which is called null space. Dimension of NT is called nullity of T, and we will show this very elementary result, that for any operator T which takes a vector space V, and maps into another, a sub set of another space; subspace of another space V prime.

If dimension of V is given to be say, n; that is V has dimension n; final dimension null, V has dimension n, then n is equal to the dimension of RT, that is rank plus dimension of NT, that is nullity. That is rank plus nullity is equal to dimension of the original vector space V. That is very important, it is not difficult to prove; we can prove it.

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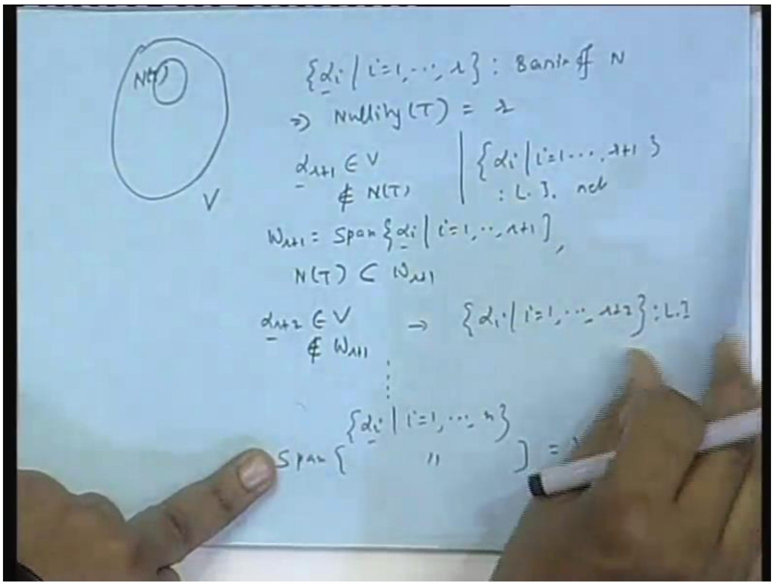

Suppose, you start with vector space V and there is this N. Let alpha i equal to say, 1, 2, dot, dot, dot, r; B, a basis of N; that means, nullity of T is r, because that is the dimension of r. Then, I can take 1 vector say, alpha r plus 1, element of V and not element of NT. That is from within V, but outside N; I would say NT; then, if I append alpha r plus 1 to this set, which is a linearly independent set, you have seen. You have seen this exercise. You can call it Wr plus 1 is the, I mean, you can say that alpha i; i equal to now 1 to r plus 1, is a so linearly independent set. This you have seen already, when we started discussing vector space, you have done this kind of exercise, that if you have this set of linearly independent set of vectors, consider the space spanned by them; that is NT; take any vector from outside, that append that vector to this. So, it becomes r plus 1 here.

Then, again it is a linearly independent set, simply because, the new guy cannot be written as a linear combination of these fellows, because that new guy is already outside N of T. If you now consider Wr plus 1 to the space, the span of i equal to 1 to r plus 1 that is, then obviously, NT is contained in Wr plus 1 then you take another vector, alpha plus 2 element of V, not element of Wr plus 1, that is outside Wr plus 1, but inside V. If you append this, then again, append this to this set, then again, there is i equal to 1, dot,

dot, now r plus 2; this also LI, linearly independent. You consider this span of this Wr plus 2, so on and so forth. Finally, this process will stop, when r equal to, when you have got number of elements in this set, equal to the dimension of this vector space; that is equal to n. Then, you get a basis, I mean, this continues. So, finally, you get alpha i, i equal to 1, dot, dot, dot, n, and span of this is nothing, but V.

So, this way you get a basis of V, but again there is a nothing unique about it. Because, you are always picking any arbitrary alpha plus 1, alpha plus 2; just alpha plus 1 should be outside NT, inside V, so on and so forth. But there is no fixed choice. We have already done this exercise; there is nothing new on this, going to be first here. This is V. So, this way you can obtain the vector space, basis of giving the vector space V, one of the basis. Now, my claim is that, if I consider, I already know that if I apply T on alpha 1, I get 0; T on alpha 2, I get 0; dot, dot, dot, T on alpha r, I get 0. Because, alpha 1 to alpha r belongs to null space of T, but if I apply T on alpha r plus 1, or if I apply T on alpha r plus 2, or dot, dot, dot, if I apply T on alpha n, what do I get? I get definitely some vectors, belonging to the range space of T, and those vectors also, will be linearly independent; that is what, you can show easily.

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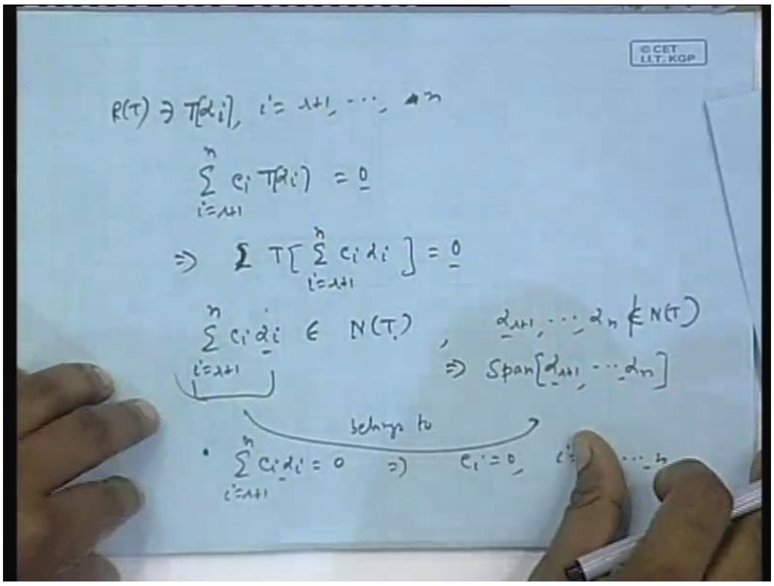

That is, if you consider T alpha i, i from r plus 1, dot, dot, dot, up to n, these vectors alpha r plus 1, alpha r plus 2, alpha n; they themselves are also linearly independent. Because, they are part of a linearly independent set, that is basis. Given a linearly independent set, any subset of it, is also linearly independent. If I do not consider from alpha 1 to alpha n, but consider from alpha r plus 1 to alpha n, then they are also linearly independent. On each of them, I apply T; I get T alpha i. T alpha i, it belongs to RT. Are these T alpha linearly independent? Answer is yes. That is, if you take this, may be, I put a bracket here. If you take this and sum, I have to r plus 1 up to n equal to 0. Is it the only choice, only solution, for this is, that C r plus 1 should be 0; C r plus 2 should be 0; dot, dot, dot, C n should be 0; that is all the coefficients should be zero and no other choice is possible?

If so, then it will be linearly independent, but this means, using linearity. It means, T working on C i alpha I, 0; that means, belongs to null space of T, because T working on this vector is giving you 0. This works on null space T, but at the same time I know, that alpha r plus 1 dot, dot, dot, alpha n, they are not element of null space. They are outside null space. So, the only, that is, if you consider; that means, if you consider span of only these people alpha r plus 1 to alpha n, this span, because this left hand side vector is a linear combination of alpha r plus 1 to alpha n. That means, this vector belongs to here, belongs to this span, but again, by this equation this belongs to NT; that means, this belongs to the inter section of NT and span of these.

The intersection between N of T and the span of this, is only at 0, because each of these vectors, I found, I told you that lies outside N of T and I consider that span. I call it, if I take that span, if I take NT, the only way, the only place, where they can interact without intersect is 0. Therefore C i, that means, this summation C i alpha i is 0, i equal to r plus 1 to n, but again, alpha is themselves are linearly independent. That means, C i equal to 0 for i equal to r plus 1 dot, dot, dot, n. So, that shows, that this vector T alpha i, i from r plus 1 to n, there are how many vectors; n minus r vectors. They are all linearly independent, though they remain in, they belong to R of T. They do not belong to V. They belong to R of T. They form a linearly independent set, fine, but what do you do with them?

We will show now, that these n minus r vectors, which are linearly independent and which belong to RT; they are, in fact, a basis; they constitute what? Basis of R of T, which means, dimension of R of T is nothing, but n minus r, because total number of elements is n minus r here.

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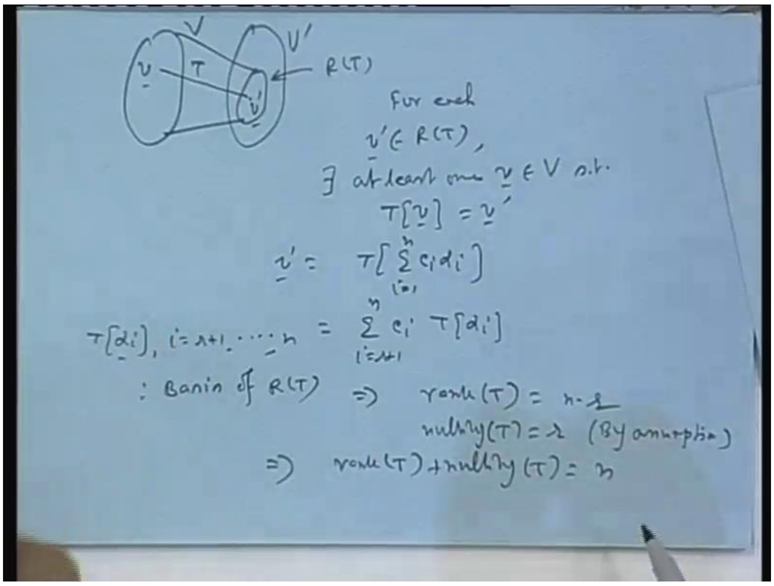

To show that, they form a basis is not difficult. That is, if you take this R of T, if you take vector say, any vector, this is V prime; this is V, if you take a vector say, V prime here. There must exists at least 1 vector v, so that, T 1 V; T working on V, gives you V prime, that is, for each V prime element of R of T, there exists at least 1 v element of V, so that, T working on v gives you v prime. But I already have a basis of this vector space V. So, I can write this vector as a linear combination of those basis vectors, alpha i and i will now go from 1 to n, v prime is equal to this. Now, using linearity I can apply T on each of the alphas, but from alpha 1 to alpha n, they belong to the null space. So, T working on them will give you 0. So, essentially, what you get is C i T alpha i, i equal to r plus 1 to n. Earlier, I have shown that T working on alpha i from r plus 1 to n; they constitute a basis of RT.

They constitute, they are linearly independent set. Now, I will show not only linearly independent, any vector v prime belonging to RT, can be written as a linear combination of those linearly independent vectors. This proves that T alpha i, i equal to r plus 1 dot, dot, dot, n, is a basis of R of T; that means, rank of T which is the dimension of R of T. This is nothing, but total number of elements in this basis, which is n minus r. What was r? Nullity of T was r, by assumption; we talk that to be r. This implies rank T plus nullity of T is equal to n, which is the dimension of original vector space V. This is a very important result. This one result will need, since, I do not know your linear algebra background; I have to do all these things by myself. Otherwise, I would have skipped this. What is the implication of this null space?

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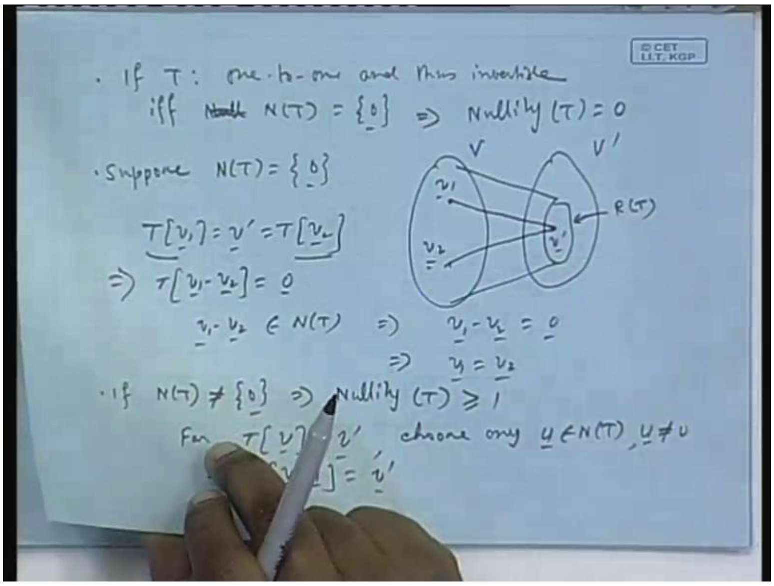

Suppose, I say that if T is 1 to 1, that is, it takes 1 vector, gives you 1, say, takes v and gives you v prime. And no other vector gives you v prime, that is v points to v prime and v prime points to v. Then, the relation is 1 to 1. Given v, I can get v prime, and given v prime, I can find out who generated v prime; that is v, uniquely. In that case, we say T is invertible. That is given v prime, I know it is mother; it is its origin; that is on whom, by working on T, by using T, I could get the vector. T 1 to 1, and thus invertible, if and only if null space, that is NT, consists of only 1 vector, that is 0, implying nullity of T is 0. This is very easy to show, that suppose, it is given that NT consists of 0 vector, and this is V; this is V prime; say, R of T. You take a vector V prime here.

Suppose, it is not 1 to 1, I have a situation like this. Then, there is at least a case like this, where there are two vectors beyond v 2, which under T map to the same vector. In that case, I will say that T is not invertible, because given v prime, I do not know, whether its inverse is v 1 or v 2; because more than 1 possibility exists. So, it is not invertible there. Suppose, we have a situation like this; that is TV 1 equal to v prime, equal to TV 2.

That means, if you take this and this; that means, T of using linearity v 1 minus v 2 is equal to 0, which means v 1 minus v 2 belongs to NT; that means, v 1 minus v 2 is simply 0 vector, because NT consists of 0 vector by assumption, which means v 1 has to be equal to v 2. So, in that case; that means, it is not possible to have 2 different v 1 and v 2. So, that T of V 1 gives you V prime; also T of V 2 gives you V prime; it is not possible. If it gives so, then v 1 and v 2 must be same.

So, given the null space consists of only 0 vector that is nullity 0, T definitely is 1 to 1. But, on the other hand, given that, if NT not equal to 0 vector, implying nullity T greater than or equal to 1; that is it has at least 1 vector, I mean, it was just, I mean, its dimension is not 0. That means, it has many vectors; many non 0 vectors. Suppose it is so, if it is only having 0 vector, T is 1 to 1; we have seen. If it is not having only 0 vector, that if its dimension is greater than or equal to 1, then also, is it that you can have T 1 to 1 and status? No, you cannot have, because for any TV, say, equal to v prime; choose any u element of NT, so that u not equal to 0.

And T V plus u will also give you v prime, because T u is 0; because u belongs to null space. Therefore, v and v plus u, two distinct vectors, give rise to, after mapping, give rise to the same vector in RT that is v prime, which means obviously, T is not invertible. So, if NT consist of 0 vector; 1 to 1. If it is not, it cannot be 1 to 1. That means, T is 1 to 1. If and only if, the null space consist of only 0 vector and nullity 0, which means, rank should be same as the equal to n; that is the dimensional original vector space v. Then only t is invertible. This, we are doing in case of using this abstract notion of linear operator and all that, but in the case of matrices, because you have to now come to matrices.

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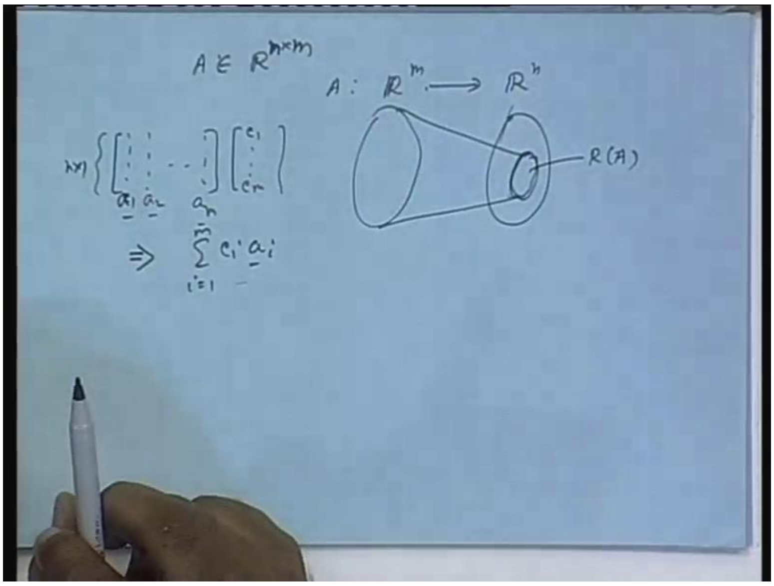

You consider matrix A belonging to say, real matrices; m cross n. What does this do? A, it takes say, n cross m, it takes R to the power m; that is real valued vectors of length m; it maps to R to the power n. How? Because you take the vector, then apply, I mean, pre multiply the vector by A, you get another vector of length n. So, that is the relation and that is of course, linear relation, we all know. So, it is a linear operator working on R m give you R n. This is a special example of that case. In this case, what is RA? A is the operator; what is RA? Now, if you see a matrix, it has got a column, say C 1, another column, not c 1; call it A 1; another column say, A 2 dot, dot, dot, another column say, A; how many columns, m columns?

If you multiply these, if you take a vector from R n, say, C 1 dot, dot, dot, C m, then, this is nothing, but a linear combination of these columns. These columns are of length n; n cross 1. These columns belongs to Rn. That means, what is the output vector, that is, after this mapping, after this multiplication, you get a vector from Rn. What is that vector is nothing, but linear combination of some specific column vectors, belonging to Rn. What are these column vectors? Columns of this matrix; that means, Ra is nothing, but Ra is what is given by, each element of Ra is given by linear combination of the columns of a; that means, the column space.

What is the column space of this matrix? Set of all possible linear combinations of the columns of this matrix, that is same, as the range space of this matrix. Because, in the range space, if you pick up any vector, that is nothing, but some linear combination of the columns, because when you multiply a vector by a matrix, by this matrix, you get nothing, but linear combination of the columns, as shown here. So, this range space consists of nothing, but the column space of the matrix, I repeat. Because, any vector of range space is linear combination of certain columns, and this columns are nothing, but the column vectors of this matrix.

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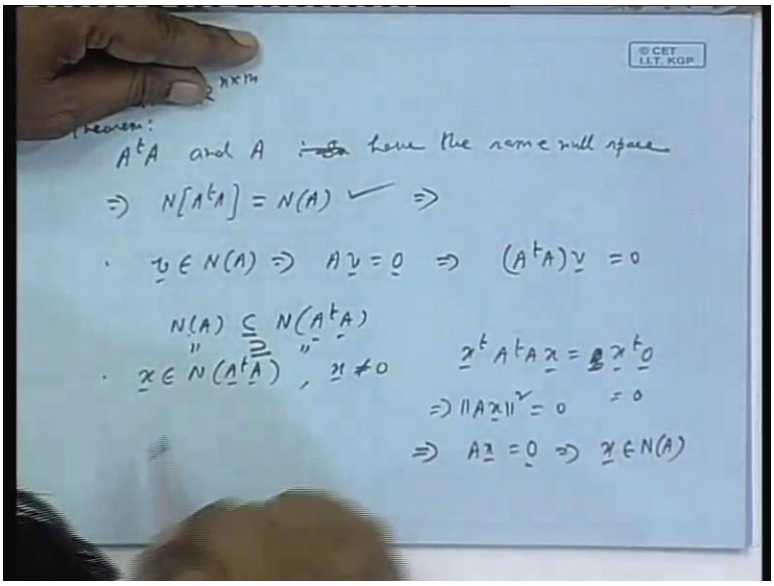

Now, some results, relevant to SVD, you consider such matrix A, belong to say as before, R n cross m. Then A transpose A and A; they have the same null space; this is theorem. That is null space of A transpose A is same, as null space of A. This can be proved easily. Suppose, you take a vector say, v from null space of A; that means, Av equals to 0. V is of length m; 0 is of length n vector, because A is n cross m, but then, this implies A transpose A, if it works on v, that also is 0; because Av is 0, isn't it?

That means, if v belongs to NA, then v belongs to N of A transpose A. This means NA, this is very trivial, is contained in this, because any vector belonging to NA means, A working that is 0 and in that case, A transpose A working on that also is 0; that means, that vector also belongs to the null space of A transpose A. But you have to prove the other way also, that LHS is containing RHS, and then RHS also containing LHS. This is all we have to prove and this is not difficult. You pick up, firstly, LHS and RHS. These two subspaces have to have one thing in common; that is 0. At 0 at least, they have something in common; they are same.

Now, pick up a vector x belonging to the null space of A transpose A, and x is not 0. Because, at 0 they are common. So, no point of taking up 0s here. This means A transpose A working on x is 0; that means, 0. Now, if I premultiply this by x transpose, here also, I get scalar 0 here. But, left hand side is Ax norm square, in the Euclidean norm sense, equal to scalar 0. This is possible, only if this is the 0 vector, you know norm square is 0 and norm is 0, only if vector is a 0 vector; this is 0 vector; which means, x belongs to null space of A. Then, we prove that this null space of A contains this also. That means, we proved this.

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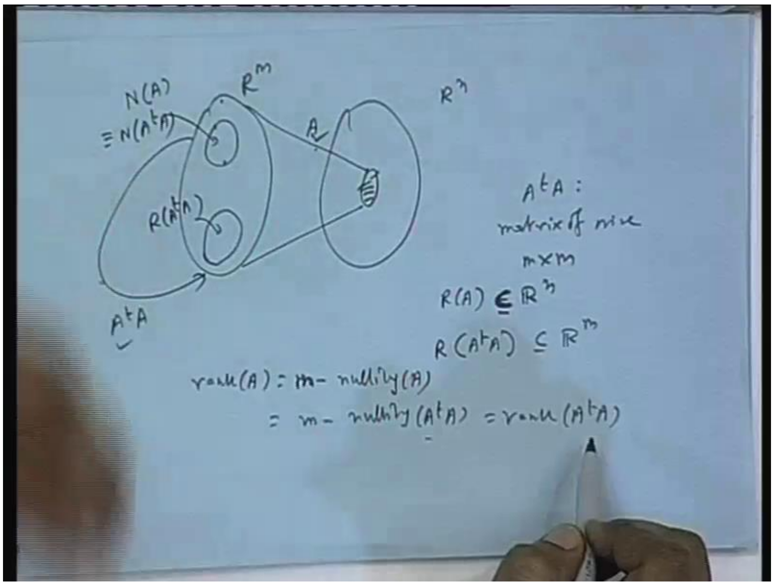

What is the implication of these? That, we had a situation like this. This was under map A, and this was under A transpose A, mind you, A transpose A is a matrix of size, what; A was m cross n. So, it will be m cross m. That means, it takes a vector from R n and gives you the vector, I mean, generate some vector from R m only. That is, while I know that range of A is a subset belongs to R n, range of A transpose A belongs to Rm. But null spaces are same. That is, if there is a space, this is n of a equivalently, n of A transpose A. So, under A, Rm is mapped to Rn, and there is a null space. Under A transpose A, you can show that this is mapped to itself; A transpose A map to itself. But this operator and this operator; both have the same null space this. Now, that means, what is rank of A? Rank of A is the dimension of this range space, and rank of A transpose, this is R of A transpose A.

Rank of A means dimension of this range space. That is nothing, but n minus, that is, sorry, m minus; m is the dimension of r m; m minus nullity of A. What is rank of A transpose A? That is the dimension of this space, which is equal to again, dimension of the original space, that is same as m, minus nullity of this, and nullity of this, is same as nullity of A. That means m minus nullity of A transpose A; that is rank of A transpose A. That means, A and A transpose A; they have the same rank.

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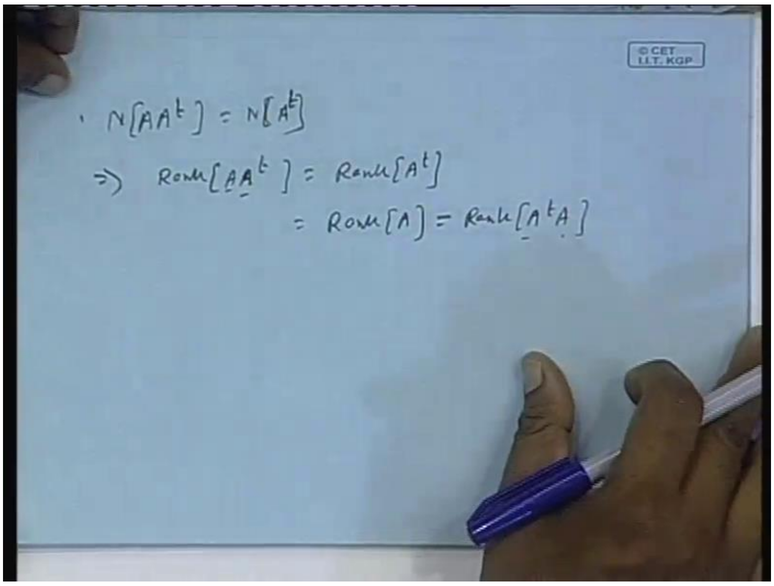

Similarly, by the same, I will not do it, you can similarly do. AA transpose, N of AA transpose is same as N of A. This can be proved implying rank of, sorry, this is A transpose; means, rank of AA transpose is same as rank of A transpose, and those who know elementary matrices, they know rank of A transpose is same as rank of A. We have seen only rank of A is nothing, but rank of A transpose A. This shows that A transpose A and AA transpose; they have the same rank. Now, remember one thing. A transpose A and AA transpose, both are symmetric matrices; real hermitian matrices. That is symmetric matrices.

So, we have, they can be diagonalised. They have got a set of orthogonal Eigen vectors. Their Eigen values are real and non negative; they can be 0 or positive, this we have done on plenty of occasion in this course. Remember that both these; though AA transpose is of size n cross m, and this is of size m cross n. So, Eigen vector is of different size. But this also has set of Eigen vectors, A transpose A, m number of Eigen vectors, which are mutually orthogonal or orthogonal one, and corresponding Eigen values are real and non negative. Here also, we have got n number of Eigen vectors which are mutually orthogonal, and the corresponding Eigen values are real and non negative, mind you. In fact, we will show them, that rank of such symmetric matrix is also given by number of non negative Eigen values.

If Eigen values repeat, I will count it, if it repeats twice, I will count it as 2, and that way, I count number of non negative Eigen values , but remember these. Now, again I will be going to the Eigen values, Eigen vectors side and we all know what they are, but we have to do quickly.

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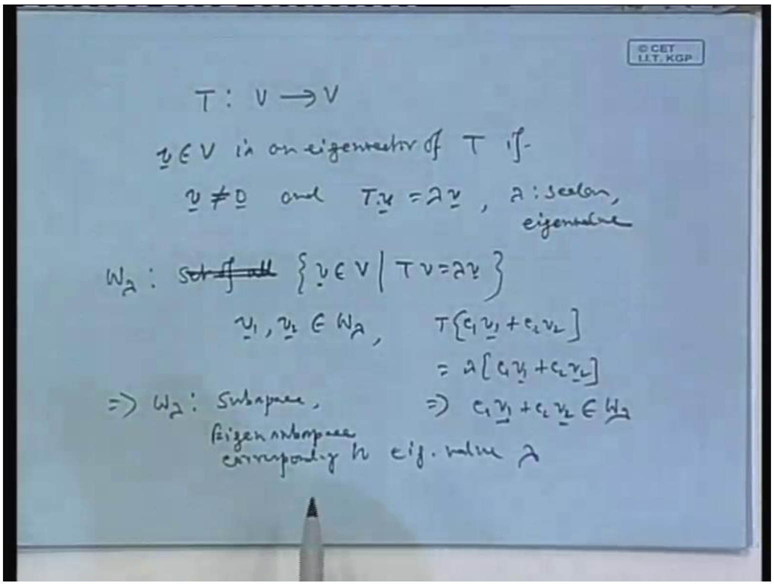

Again, I will to come back again to this abstract notion of operators and all that, but this time T is working on v to v, and v is a finite dimension of vector space. Then, small v element of V is an Eigen vector of T, if v firstly, is not 0 and Tv is some scalar times v itself, that is you work on, you use T 1 v, you get a vector in the same direction. Just it is either amplified or attenuated by a factor lambda. Lambda scalar belonging to the field is called Eigen value. Then w lambda, suppose, set of all that is mathematically, you collect all elements of v. So, that T v is lambda v, then w lambda is not only a subset, we will see it is subspace; it is called the Eigen subspace, belonging to lambda. How?

Just simply, if you see, if v is the Eigen vector, 2 v is also the Eigen vector, 3 v is also the Eigen vector; it is scalar multiple, because T on some constant times v, constant will go out, the constant times T v, that is constant times lambda v, which lambda is into constant times v. So, that is an Eigen vector. Remember that Eigen vector is not unique for a Eigen value lambda. For each Eigen value lambda, there are many Eigen vectors possible. Suppose, I consider all such Eigen vectors, belonging to the same Eigen value lambda, and I give it a name w lambda. Then, this is subspace, because if you take v 1 v 2 element of W lambda, then c 1 v 1 plus c 2 v 2, this also belongs to W lambda. Because, if you apply T on it, you can apply use linearity c 1 into T v 1, which will give you lambda v 1 and c 2 in to T v 2, which will give you lambda v 2. So, lambda you can take common. Again, you get c 1 v 1 plus c 2 v 2; that means, this fellow implies c 1 v 1 plus c 2 v 2 belongs to W lambda.

That means, W lambda subspace, not just a subset; it is a subspace, called Eigen subspace, corresponding to Eigen value lambda. Now, we will show that if we have two distinct Eigen values, well as distinct Eigen values; lambda 1, lambda 2, dot, dot, dot, say, lambda r.

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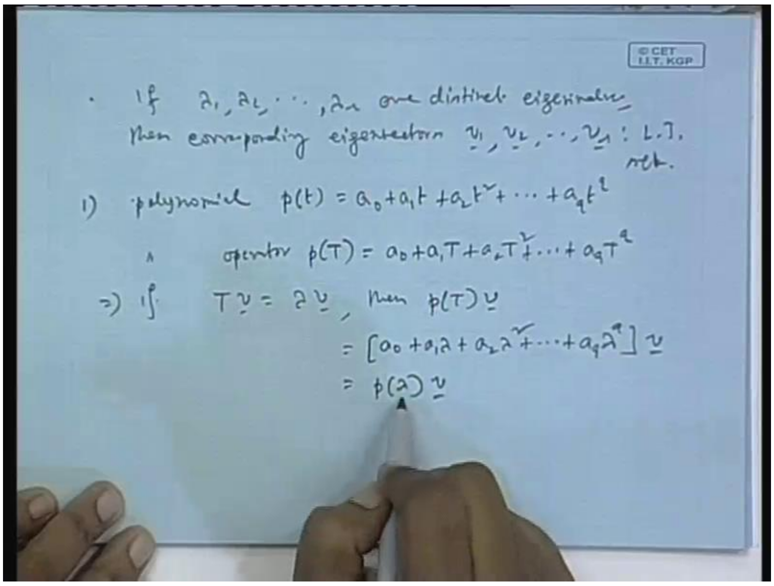

The corresponding Eigen vectors are linearly independent. We will show that, but before that, that is we want to show, that if lambda 1, lambda 2, dot, dot, dot, lambda r are distinct Eigen values, then corresponding Eigen vectors v 1 v 2 dot, dot, dot, vr form a LI set; linearly independent set, we have to prove it. But before we prove it, some properties 1- suppose, consider a polynomial; polynomial p of say, anything, say t; just for the polynomial, something like this, a 0 plus a 1 t plus a 2 t square dot, dot, dot, say anything, aq t to the power q.

Then, you found the polynomial operator p capital T, that is wherever, you have got small t, variable coming; just replace it by capital T, a 2 T square. T square means, T followed by T. That is T square working on a vector means, first one T on it, whatever comes, again one T on it, so on and so forth, and aq t to the power q. Then, you see one thing, if there is a vector Tv equals to lambda v, then instead of T, if I apply this operator; polynomial operator p T on v, what I get is a 0 times v. So, a 0 plus a 1 times

Tv, Tv is lambda v. So, a 1 lambda, then a 2 T square v; T square v means, first Tv which is lambda v, again T on that; means, lambda square v.

So, a 2 lambda square plus dot, dot, dot, plus a q lambda to the power q v; I repeat again, pT, you replace pT by this expression. a 0, what we want, v means, a 0 v a 1 into Tv means, a 1 into lambda v; lambda comes here; a 2 square v means, a 2 T working on Tv; a 2 T Tv, Tv is lambda v, take lambda out. Tv again lambda v. So, you get lambda square. So, a 2 lambda square and v dot, dot, dot. So, you get these. This is nothing, but same polynomial which you started with, but instead of T, we have got lambda. So, p lambda v. If you have got your operator in a polynomial form, that works on a Eigen vector, resulting thing is the Eigen vector, multiplied by a scalar where, you in case of the operator, you replace the Eigen value. This is one property.

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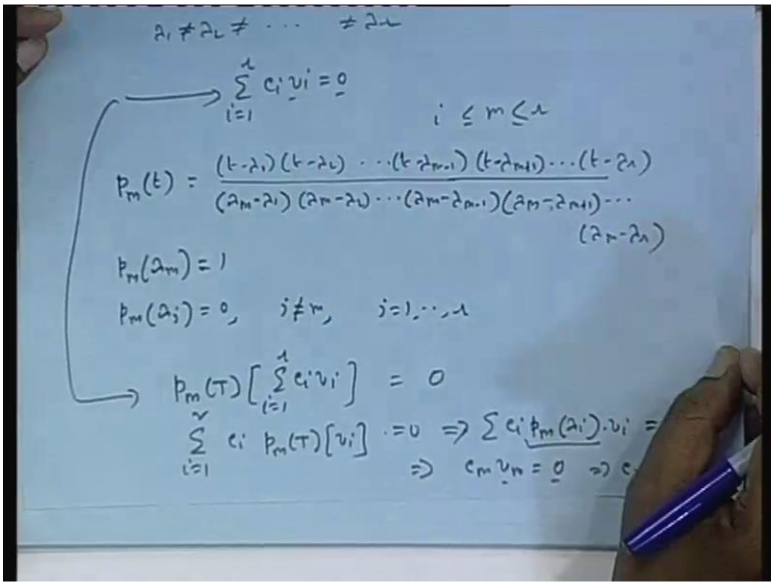

Another property is, another thing is that, we are given lambda 1 not equal to lambda 2; not equal to; dot, dot, dot, not equal to lambda r; corresponding Eigen values are Eigen vectors are v 1 v 2 dot, dot, dot, vr. You have seen it. I have to prove that, if ci vi equal to 0, that I have to show that each coefficient is 0; that is, v 1 to vr; these are linearly independent set; that is the objective of this theorem, I mean, that is the objective to prove actually. That shows that the Eigen vectors corresponding to distant Eigen values; they are linearly independent. So, I pick up those Eigen vectors here, v 1, v 2, up to r, i equal to 1 to r form a linear combination, equate it to 0. I have to show that, this is possible, only if c 1, equal to, c 2 equal to, dot, dot, dot, equal, cr equal to 0, and no other solution exists. Then, that will show this is linearly independent.

How to show that? So, I pick up from i equal to 1 to r, a particular case, say, m. Take m less than equal to r, greater than equal to i. I will show that cm equal to 0 and then, you can pick up m, to be anything, from i to i plus 1 to i plus 2 to r. So, that will show that each coefficient is 0. Because n is general thing, generally. Suppose I form a polynomial Pmt as this. Here, I take lambda m minus lambda 1, lambda m, these are all scalar numbers, lambda 2, dot, dot, dot, lambda m, I go up to lambda m minus 1. Then, I skip lambda, go for this, dot, dot, dot, lambda m minus lambda r. Here again, t minus lambda 1, t minus lambda 2, dot, dot, dot, t minus lambda m minus 1, and then, t minus, just follow, the denominator, m plus 1, dot, dot, dot, t minus lambda r.

What is the property of this? You know, if you replace t by lambda m, what you get? Lambda m minus lambda 1, lambda m minus lambda 1; cancels, lambda m minus lambda 2, lambda m minus lambda 2; cancels. All the terms cancel, you get 1. If you take any other lambda j equal to 0, for j not equal to m, and in general, j equal to 1 to r, but not equal to m. Then it is 0. So, I can always construct a polynomial like this. I picked up a particular index m, I am looking at m eth Eigen value lambda m, the corresponding coefficient here, is cn, Eigen vector is vm, I am focusing on that, keeping that in mind, I consider a particular polynomial.

Now, on this equation, if I apply PmT on that; right hand side, of course, is 0, because this polynomial operator working on 0 vector, will be 0 vector, because of linearity and this is a polynomial operator, I mean, whether you have the operator as T or a linear combination in a polynomial form, linearity remains. So, PmT can be applied individually on vi s. That means, working on vi equal to 0, but this means, I have already proved one property; this polynomial working on vi means, resulting thing is the vector itself, multiplied by a scalar value. What is the scalar value? Instead of T in place of T, put the corresponding Eigen value. That means, left hand side is ci Pm corresponding lambda i times vi, and this is equal to 0, but Pm lambda, you have seen, only when i equal to m; this is 1, otherwise, this is 0; that means, cmvm is equal to 0. So, one possibility is, either vm equal to 0 or cm is 0, but vm is an Eigen vector by definition; vm cannot be 0; that means, cm is 0 and m, you take as i or i plus 1 dot, dot, dot, up to r; each coefficient is 0, which proves that Eigen vectors belonging to different distinct Eigen values; they are linearly independent.

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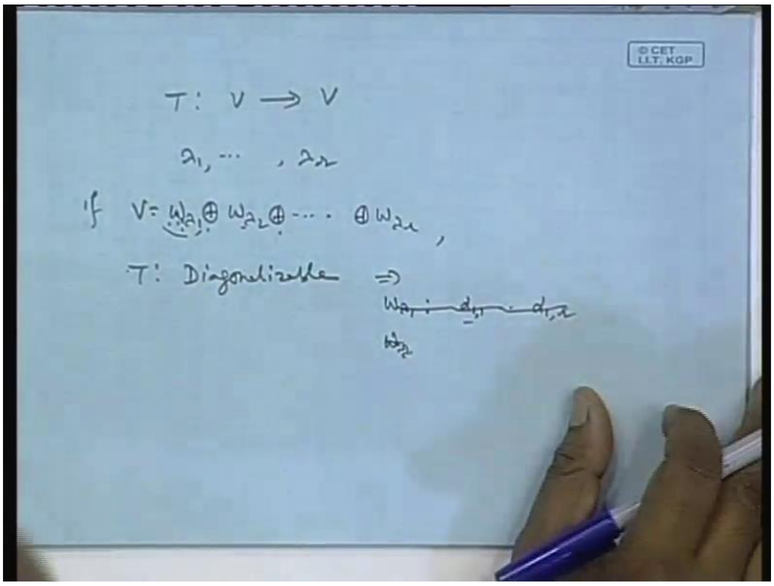

That means, if T is from v to v and you have got these distinct Eigen values; lambda 1 dot, dot, dot, lambda r, you form the corresponding Eigen subspaces. Eigen vectors becoming to w lambda 1, if you can find a basis of this subspace; if you find out basis of w lambda 2, you can find a basis w lambda r, so and so. Since, this basis, I mean, vector from here; w lambda 1, vectors from w lambda 2; they are linearly independent; that means, you can form a direct sum of them. If this direct sum appears to be same as V then, if this, then we say, T diagonalizable. This is because suppose, it is diagonalizable; w lambda 1, you have got a basis, alpha 1, 1 to say alpha 1, r; W lambda 2; what I want to say is this; you can form a basis if this is so; you can form a basis of v by taking one basis of w lambda 1, another basis of w lambda 2, dot, dot, dot, basis of w lambda, just appending them.

That will form a linearly independent sum, because it is direct sum, because I told you; Eigenvectors (( )) to distinct Eigen values, they are linearly independent; you are picking at basis already from w lambda 1, already from w lambda 2; just appending the basis. So, you get a basis for v. If any vector in that basis, we apply t; this vector is an Eigen vector. So, you get nothing, but that vector itself multiplied by the corresponding Eigen value; that means, this kind of operation, that means, if you have, I mean if you take any vector small v, belonging to the capital v, write it as a linear combination of this basis; capital T works on that. So, what you have to do, simply capital T work on each individual basis vector, and the corresponding same, since, each individual basis vector is an Eigen vector, some Eigen value or other; you will get nothing, but that Eigen vector coming back multiplied by the corresponding Eigen value. Now, when I come to matrices, then this will be further clear, but before I go to matrixes, just one more thing, I want to do so.

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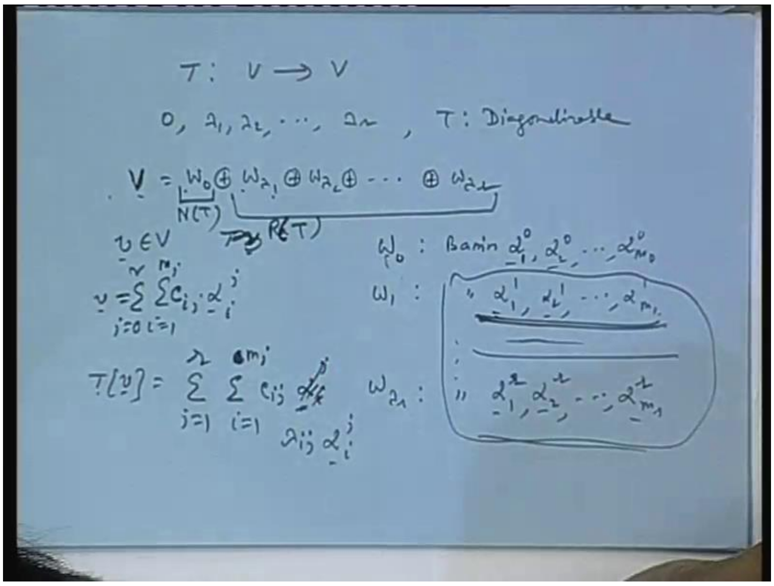

That suppose, T takes v to v and you have got lambda 1, you have got suppose 0 0 Eigen value and then, lambda 1, lambda 2 dot, dot, dot, lambda r and v is given to be, that is also, T is given to be diagonalizable; that is v is nothing, but w 0 direct sum w lambda 1; direct sum w lambda 2; direct sum dot, dot, dot, direct sum w lambda r; this is given to you. In that case, if you take any vector, small v element of V; T working on v; that will be what? That will be given by a linear combination of, that first, we have to find out the basis of w 0. Suppose, this Eigen value is w 0; this has got a basis say, say alpha 0 1 alpha 0 2 dot, dot, dot, alpha 0; total number is something, may be, m 0.

W 1 has a basis; alpha 1, 1; alpha 2, 1; dot, dot, dot, alpha 1 m1; then w lambda r has a basis; alpha r 1, alpha r 2, dot, dot, dot, say, alpha r mr. Obviously, you have got how many elements here; m 0, you have got m 1, you have got mr. So, m 0 plus m 1 plus dot, dot, dot, plus mr; so, v equal to the dimension of v, because if you append the basis, all these basis, they form a basis for v, that you have seen. Because the operator is diagonalizable, because you have got the, I mean, therefore, you have got the direct sum decomposition of v in terms of this. But there is a difference between Eigen subspaces w 0 and the other one; Eigen subspaces corresponds to w 0, and the other one corresponds to this thing, but non 0 Eigen values. Now, if you pick up any v element of v, you can write v as a linear combination, in terms of these basis vectors.

C ij alpha ji, i equal to 1 to mj and then, j equal to 0 to r. For j equal to 0, you have got these ones; alpha 0 1, alpha 0 2, dot, dot, dot, alpha 0, up to m 0, and likewise. This is nothing, but this is notation and geometry. Actually, I am doing nothing, but I am linearly combining them. If I apply T over v, then T can be applied directly on this vector. So, remember, for j equal to 0, you have got alpha 0 1, alpha 0 2, dot, dot, dot, alpha 0 m 0; T working on each of them will give rise to the same vector, but multiplied by 0 Eigen value. So, they give rise to 0. So, I do not have to consider them. That means, I have to consider from here to here; that means, Tv is nothing, but from j equal to 1, I have to start, to r and then, i equal to 1 2 mj cij alpha ji times, because T working on, times, the corresponding Eigen value. Eigen value will be just a minute. There are two indices; lambda ij and alpha ji.

So, this actually shows that range space of T, any vector belonging to the range space, that is nothing, but of this form Tv, is nothing, but a linear combination of whom, this Eigen vector corresponds to the non 0 Eigen values. These Eigen vectors form a linearly independent set, because I mean, within w 1 itself, it was forming a basis, means, they were linearly independent; within wT itself, other one was forming a basis of w 2, it was linearly independent so on and so forth. When I appended them, it was still linearly independent, because they corresponded to, I mean, Eigen vectors corresponded to different Eigen spaces; they are linearly independent. That means, I got a set of linearly independent vectors; one set here, I mean, it consists of Eigen vectors, I mean, this dot, dot, dot, this all appended.

I got a set of linearly independent vectors, which are Eigen vectors, corresponded to the non 0 Eigen value. So, that any vector of range space is a linear combination of them. That means, range space is actually is nothing, but the all the span of the, what, actually, range space is nothing, but this part; range space is nothing, but this part, because any vector belonging to the range space, is a linear combination of these vectors; that is the basis vectors of w lambda 1; basis vectors of w lambda 2; dot, dot, dot, basis vector of w lambda r. It is a linear combination of them. They are all linearly independent. So, this part corresponds to range of T; R of T, and this part corresponds to null space of T. Because any vector from there, if you pick up, if you apply T over there, you get 0. All right. So, I stop here. In the next class, I will use this result and get into SVD.

Thank you very much.