Adaptive Signal Processing

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Lecture - 40
Singular Value Decomposition (Continued)

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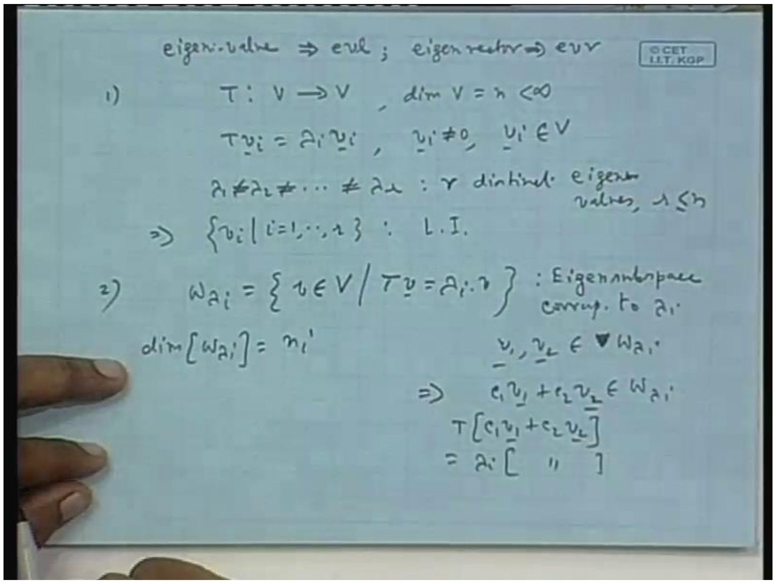

Let us start by considering what we discussed in the last class. Let us sum up. First, we have seen that suppose, number. 1; that suppose, that is given the T is a linear operator, mapping v to v and dimension of v is given to be some finite n, less than infinity. Then, if you have got some Eigen vectors, Tvi as lambda ivi where, vi not equal to 0 and vi belongs to v; that is, vi is a non 0 vector belonging to v such that, T working on vi gives you back vi only; just multiplied by constant lambda, then we say vi is an Eigen vector and lambda is the corresponding Eigen value for this operator T. Now suppose, we know we have lambda 1 not equal to lambda 2, not equal to dot, dot, dot, not equal to say lambda r; that is a set of r distinct Eigen values.

Eigen values in short, I will write as evl; Eigen value and Eigen vector, just evr; this is my notes, because Eigen value, Eigen vector, you know you have to write so many; too odd. So, evl and evr. Distinct evls and r less than equal to n; why less than equal to n? We will see very soon. Then, you have seen that the corresponding Eigen vectors, vi, i equal to 1 dot, dot, dot, r, linearly independent. That you have seen, that Eigen vector corresponds to this different Eigen value, distinct Eigen values; they are linearly independent, that we have seen last time. Now, secondly, we also have seen the concept of Eigen subspace corresponds to a particular Eigen value say lambda i. That is, suppose, I consider this say w lambda i as what; it consists of those vectors belonging to v such that, T working on v gives you nothing but the chosen lambda i times v.

Then, this is Eigen subspace corresponding to lambda i, remember one thing, when I say Eigen vector, it is strictly non 0. But, when I write W lambda i, I do not mention the condition that v is not equal to 0, because if it is a subspace; it has to at 0 vector; so I must highlight it. Why it is a subspace? We have seen last time, that suppose, v 1 and v 2 belongs to v, belongs to this W lambda i, then any combination, c 1 v 1 plus c 2 v 2 also will belong to W lambda i. Simply, because if we apply T what this vector, using linearity you will have c 1 T times, c 1 times Tv 1, which is lambda 1 v 1 and c 2 times Tv 2 which is lambda 2 v 2, then again, a linear combination of v 1 v 2, which means, this content is disclosed; that is if I take say T over this, this will be T plus time. Then, using linearity, T should be working on v 1 and T, should be on v 2 Tv 1 is lambda 1 v 1 lambda iv 1 Tv 2 is lambda iv 2, so we can take lambda in common and you get back the same vector.

That means, if v 1 and v 2 are two Eigen vectors, then c 1 v 1 and c 2 v 2 also, that is if v 1 and v 2 also belongs to W lambda i, then c 1 v 1 plus c 2 v 2 also, belongs to W lambda i, because T working on that, gives you the same vector itself; just multiplied by the same Eigen value lambda i. Therefore, if v 1 and v 2 are member of W lambda i, so any linear combination of them also a member of W lambda i, which shows this is an Eigen subspace. But this Eigen subspace, if I consider the dimension, you denote it by ni. This also you have seen, I am just writing down the main points of what we discussed last time.

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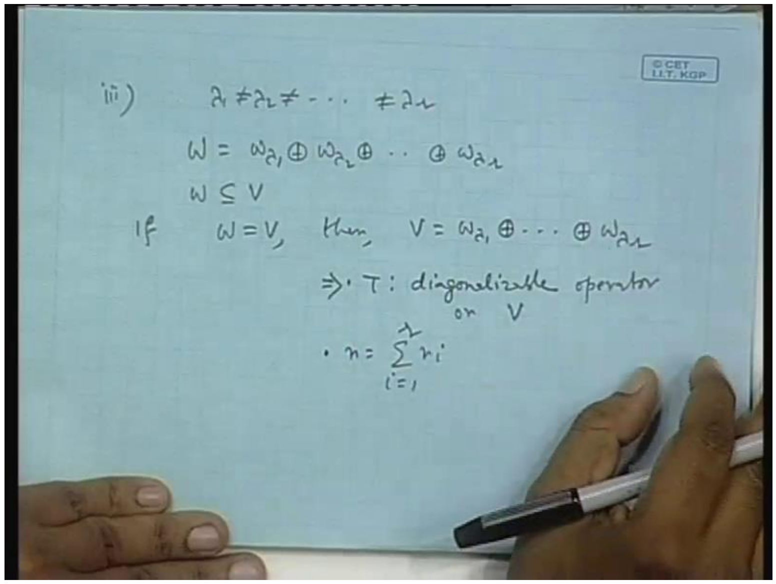

Number 3; suppose, we have got distinct Eigen values, lambda 1 up to lambda r. We have Eigen subspace W lambda 1, Eigen subspace W lambda 2, dot, dot, dot, Eigen subspace W lambda r. Suppose, I form W as the direct sum; it has to be direct sum, because as you know, as I have shown, that any vector belonging to W lambda 1, any vector belonging to W lambda 2, they form a linearly independent set, isn't it? Therefore, when I am summing, when I am considering the combination, it has to be direct sum. This means, if you consider one basis of W lambda 1 on basis of W lambda 2, dot, dot, dot, 1 basis of W lambda r, it happened all those basis vectors, then we get a basis of W. The real thing, I mean, it is a basis, because the real thing set also linear independent, and spans W, fine. This is we know. If clearly, W has to be, I mean, within v.

If W is equal to v, then we have v is equal to W lambda 1 dot, dot, dot, W lambda r, then we say T is diagonalizable; number 1. Why diagonalizable? I will come to that, but we just take T as a diagonalizable operator on v, and clearly, dimension n, that is n, that is dimension of vector space v will be equal to dimension of W lambda 1 plus dimension of W lambda 2 plus dot, dot, dot, plus dimension of W lambda r, which is ni; summation of, is i equal to 1 to r. These two points follow. This is one point; this is one point; these two points follow. We just give a name T diagonalizable operator; why we say so? To understand that we come back to the basic property of operators, as a diversion from here.

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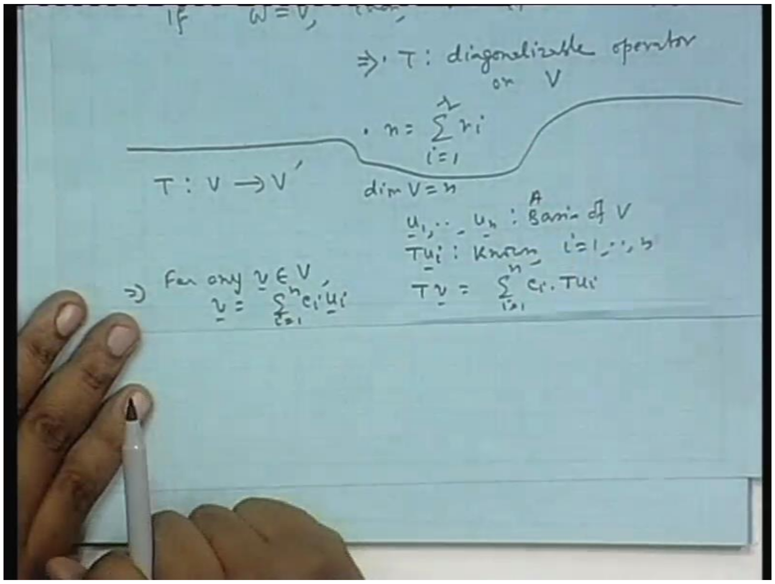

Suppose, we are considering, that is we are proving, we are presenting, some other result which is relevant here, and that I am showing here. Suppose, I am considering any linear operator say v to any v prime, then my claim is, T can be defined simply by taking up any basis of v, if dimension of v say equal to n. You say any basis say u 1 dot, dot, dot, say un, a basis; a basis of v, then my claim is T can be defined perfectly, if you are giving the action of T on each of the basis vectors. That is, if I know what is Tu 1, if that is given to me; if I know what is Tu 2, if that is given to me dot, dot, dot, if I know what is Tun, mind you, Tu 1, Tu 2 ,dot, dot, dot, Tun, they all belong to u prime, because that is why the vectors from v are mapped to. If these are known, that is, if Tui known, i equal to 1 to n, then for any v element of v, we can write v as some linear combination of this basis vectors, i equal to 1 to n.

How to compute Tv? If I know Tui, it is very simple. Tu will be working on this and using linearity, it will be nothing but ci T working on ui. If this ui Tui is known, for any vector I can find out what its map will be. This you will be known; that means, any operator actually, can be defined perfectly, just by, I mean, just by defining its action on the elements on the basis vectors of any basis of v, that we choose. If I know them, then action of T on any arbitrary vector say v here, can be easily worked out, using these known values. So, this is the elementary result of linear operators. Now, using that and using all these results, I will try to explain what do I mean by diagonalizable operator. But remember, it is a diagonalizable operator, if we have this, that the entire vector space is v, is given in terms of direct sum, of the individual Eigen subspaces corresponding to the distinct Eigen values of T. Then, I will call T as a diagonalizable operator.

Actually, it is quite simple, that suppose, I consider a basis of W lambda 1, a basis of W lambda 2, dot, dot, dot, a basis of W lambda r, and append them. Then, that will be a basis of the original vector space v. On that basis, if I apply T on each of the basis vectors, you know the result will be, can be written down very easily, and that is what takes us to diagonability.

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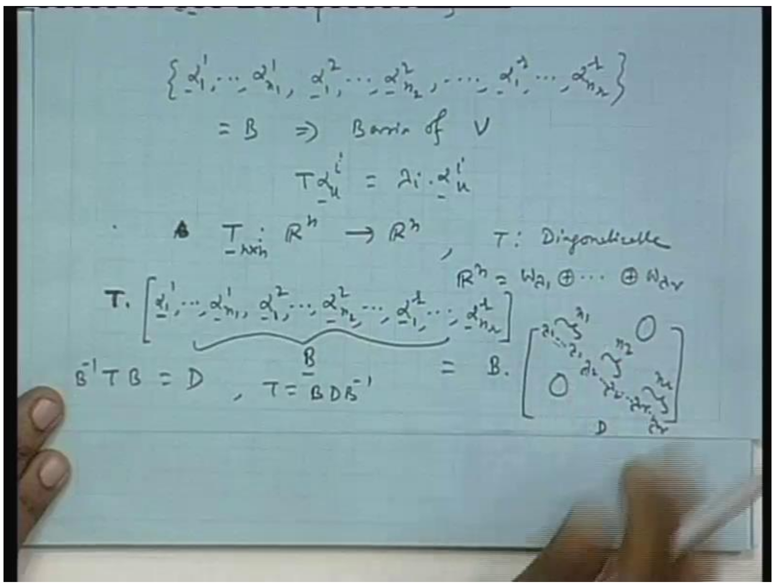

That is, suppose, some elements I am considering the i th Eigen subspace, that is w lambda i; that Eigen subspace corresponds to lambda i. If I pick up vectors from here, alpha i and k, and k will be say from 1 to n i; n i is the dimension of w lambda i, and if this is then a basis of w lambda i; w lambda i has dimension n i; there are n i components here, right. Alpha i 1, alpha i 2, alpha i 3, dot, dot, dot, alpha i in the subspace n i; so n i components, they are linearly independent. They form a basis of w lambda i, so I am calling it a basis of w lambda i. These are for i equal to 1, that is Eigen value 1; i equal to 2, Eigen value 2 dot, dot, dot, i equal to r, Eigen value r; Eigen value lambda r; lambda 1 lambda 2 up to lambda r. Then, obviously, given from that direction, decomposition result, that is from this result, we can write down, that if I append this, that is, if I append these basis vectors; that is, first, start with lambda 1 dot, dot, dot, alpha 1 up to n 1. This is a basis of w lambda 1. Then alpha 2 1 dot, dot, dot, alpha 2 n 2 dot, dot, dot, then alpha 1, I mean, alpha r 1 dot, dot, dot, alpha r n r.

Then, this is you can call it as B, this is a basis of v, right. This is basis of v, therefore, if I have to define the operator T, I can simply take its action on each of the basis vectors, and give you the results, then the entire operator T is defined and given any arbitrary vector v; I can write down its action of T on that, simply in terms of these results, that is, if I know T working on alpha 1, 1; t working on alpha 1 n 1; dot, dot, dot, I mean, then I know I can write down the action of T, that is given any arbitrary vectors small v element of v. What is t v? That I can easily write down, that follows from just discussion we had, a few minutes back, but what is the action of T on them? Pick up any one; T of alpha say particular one, say i k. This will be nothing but the same vector multiplied by the Eigen value, lambda i. So, this definition is very simple; that is T working on a particular vector, here, basis vector gives you the same basis vector.

Just like, you know, if they are like position vector in a space T, working on a position vector does not change the direction. Direction of the vector remains same, only its length; that is the magnitude changes. That is the factor of attenuation or amplification, you can say gain factor of lambda i; that is the beauty of Eigen vectors. That means, T, action of T on each of the basis vectors here, is very simple. I do not need any other expression; just the same basis vector itself, just multiplied by the corresponding Eigen value. This results in diagonally, then it is called the, then we say the operator is diagonalising, actually. Now, why and how diagonalizable; what is diagonal here? To understand that, we have to now, take a special case of operator as the matrix, that is, so long we are discussing in a very abstract way; abstract vector space and abstract operators and all that.

If we consider the special case of matrices, then we will show, we will see that matrix operator under such case, will become, I mean, it can be converted into a diagonal matrix, equivalently a diagonal matrix. You can see this; that is, suppose, we consider A or why A; T, because T is a matrix. It takes you say Rn to Rn; that means, T is n cross n. It takes n into 1 vector; column vector; maps in to r n. So, it is a square matrix. So, I am pursuing suppose, T, and in the same way, I mean, T has got distinct Eigen vectors; distinct Eigen values; lambda 1 to lambda r; r is less than or equal to n, and T is diagonalizable; it is given that t diagonalizable. That is, Rn can be written as A; that is, Rn that is vector space can be written as w lambda 1, which is the vector space; the Eigen subspace corresponds to Eigen value lambda 1. It will consist of column vectors of length n cross 1 each, which is an Eigen vector of T, corresponding to Eigen value lambda 1.

Everything same as before, only thing is, we have given a specific structure to the operator and to the vectors and all that, and vectors and all are column vector type of length n, and operator is a matrix, square matrix of size n cross m. I am given the T is diagonalizable, which means the vector space Rn can be written as a direction, become as a direction of the various Eigen subspaces, corresponding to distinct Eigen value; this is given to you.

I will again, use the same basis notation for w lambda 1, w lambda 2, w lambda r, dot, dot, dot. Then, suppose, I form, I multiply T with a matrix from like this. I just place these basis vectors; now each vector is a column vector only; that is the only difference. Each vector is an Eigen vector. They have a specific structure, that is, they are not abstract symbols, but they have a specific structure; that is, each of them is a column vector of size n, is a real value vector, you can say because I have taken r though, it could have been complex also. Just the difference is, they have specific structure; there is a column vector; otherwise, I mean, property wise everything is same. Suppose, I put the column vectors, side by side, in this order only, say alpha 1, 1 is a column vector, then alpha 1 n 1; column vector, then alpha 2 1; column vector, dot, dot, dot, alpha 2 n 2; column vector, dot, dot, dot, alpha r 1; column vector, alpha r nr is a column vector; you can call this matrix B.

Then, T working on this column vector is nothing but since this is an Eigen vector, is nothing but this Eigen vector itself, multiplied by lambda 1. T working on alpha 1 n 1 will be nothing but alpha 1 n 1, multiplied by the same Eigen value lambda 1. T working on alpha 2 1 means, it will be alpha 2 1, multiplied by the Eigen value lambda 2, so on and so forth. So, resulting thing also will be matrix of those column vectors. You can easily check and you can call this B. If you can easily check, that this will be B times a diagonal matrix D, where initially, it will be lambda 1 dot, dot, lambda 1; how many times? n 1 times. Then, lambda 2 dot, dot, lambda 2; how many times? n 2 times; dot, dot, dot, lambda r, dot, dot, lambda r; how many times? nr times and 0 0; diagonal matrix. That is why, n 1 is called the multiplicity of lambda 1; n 2 is called the multiplicity of lambda 2, dot, dot, dot, n r, is called the multiplicity of lambda r.

Before that, I will come to that again. Consider this matrix B. It is a square matrix. It is because total number of column vector is same as the dimension of the vector space. That is very important. If the vector space v, had dimension n, which I have told. Since, we can have like this, direction decomposition now, if you add the dimension of the respective Eigen subspaces, that should be equal to n. That means, if you count the number of columns, that is equal to n, that is very important. That is why, it is m cross n matrix, and the columns are linearly independent; they form a basis of B. That means, we have got a square matrix, which is having full column rank, and therefore, it can be invertible, going by your last day discussion. If it is invertible, then we can write B inverse, multiplied by B inverse on both sides; B inverse TB with this matrix D. Alternatively, with T is equal to BDB inverse, either form. This is called the diagonalization; T and D are called similar matrixes. Given T, we diagonlized T, by this method; we get diagonal matrix. With the diagonal matrix, the diagonal matrix D consist of only diagonal non 0 diagonal entries; not non 0; some Eigen values could be 0 also. But it mainly consist of, this is diagonal, and other is diagonal matrix, diagonal entries corresponding to the Eigen values of T.

Each Eigen value, I mean, lambda 1 occurs n 1 times, where n 1 is the dimension of w lambda 1. Lambda 2 occurs n 2 times, where n 2 is the dimension of w lambda 2, dot, dot, dot, and the n 1 is also called the multiplicity of lambda 1; that is, the number of times lambda 1 occurs in this diagonal matrix. In general, lambda i has a multiplicity ni means, it occurs ni times in this diagonal matrix, and what is ni? ni is the dimension of w lambda i, that means, dimension of the Eigen subspace w lambda i, indicates the multiplicity of the Eigen value lambda m.

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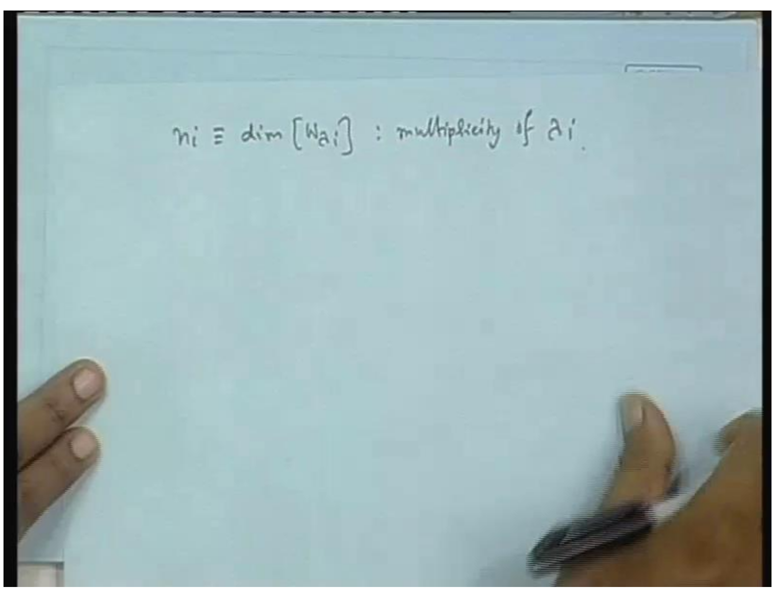

That means, ni which is dimension of w lambda i, also is multiplicity of lambda i. All these basic results are very important in signal processing, you know, though our purpose is to taking you finally to SVD and all that, it is important that we do all these. Next, now consider, now we will generalize further, I mean, we will consider the case, where T is an operator from v to v of course, and therefore, T has distinct Eigen values. But instead of taking from lambda 1 to lambda r, say for a change, let us take from lambda 0 to lambda r; lambda 0, lambda 1, to lambda r. They are distinct, so lambda 0 not equal to lambda 1, dot, dot, dot, not equal to lambda r, out of these, one of the Eigen values I will take to be 0 value; that is, it takes and others are non zero.

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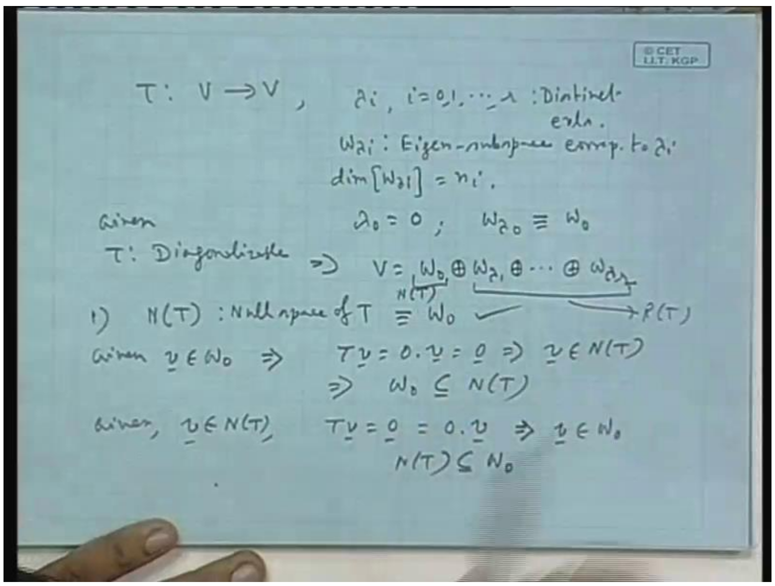

That means, I will consider T from v to v lambda i, i equal to 0, 1, dot, dot, dotl; I start with 0; distinct evls; Eigen values. W lambda i Eigen subspace corresponding to lambda i. Dimension of w lambda i is ni, and we have the additional condition that lambda 0 is 0. Also, w lambda 0, lambda 0 is 0; we then denote as w 0. Given T, diagonalizable, meaning v is w 0, then w lambda 1, lambda r. This is what is given. Then, our claim is this T operator which takes v to v; its null space will be given by w 0; that is, this much and its range space will be given by this. That is, first result; null n of T, there is null space, in case you have forgotten the notation, null space of T is identical to w 0. How? To show it, suppose you take v element of w 0. That means, Tv is equal to Eigen value 0 times v, which is 0 vector; this it means v is element of NT also, means w 0 is contained in N. On the other hand, this is given this. Again, given v element of NT, we have Tv is vector 0; vector 0 is 0 times vector v, because we know the scalar 0 times any vector will give the 0 vector, that was the very first thing in vector space, I assume, I mean, vector space theory. That was done in the class. That means, because Tv 0 means Tv equal to 0 scalar times v, that is the Eigen value zero.

That means, v is an Eigen vector, of course, I am taking v to be non 0 here. If this is only 0, then NT and w 0, they are very simple; this result, they consider only 0 vector, however that case. So, v is an Eigen vector, corresponding Eigen value 0; this is an Eigen vector corresponding to Eigen value 0; that is why, I am writing, that is v belongs to w 0. v is part of NT, v is part of w 0 also. So, it is NT contained in w 0. This implies this. This is w 0 is the null space of that operator T, and now, we have to show that range space is given by this; this is given by this RT; this is NT and this is RT. This is a proof that it is RT; RT is the range space.

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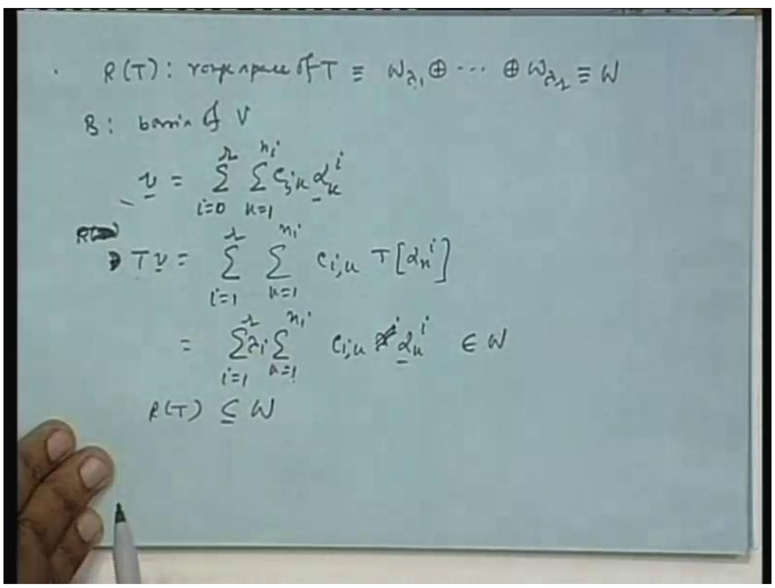

That is another point, we have to prove that R of T, which is same as range space of T. This is equal to w lambda 1, which is Eigen values which are non 0. I am considering only them, and they, respective subspace and bringing the direct sum. This is the range space that we have to prove. How to prove it? We know B as the basis of v, we have already seen it. In case, you have forgotten, what was B? We took the basis of each of the Eigen subspaces; alpha 1, 1 to alpha 1 n 1, then alpha 2 n 2, alpha 2 n 2, dot, dot, dot, alpha r to alpha nr; that was B. That means, given any vector small v of capital V, you can write it as a linear combination of these fellows. That means, any v can be written as linear combination of them, that we can write as a double summation. The vector is at the i th and i n subspace, alpha I and k and k will go from 1 to the respective dimension number of elements in the basis, and i will go from Eigen value 1, Eigen value 2, to Eigen value r; that is, the i will go from 1 to r. This is a linear combination. So, there will be combiner coefficient; there will be two indices i comma k. So, this is a typical description of a vector v, in terms of that, basis v chosen. In that case Tv, where will Tv be? Tv means T will work on this linear combination and using linearity of T, T will work directly on this vector, for i equal to, that I am sorry, it should be i equal to 0, because we consider the case where now, I mean, we have got the Eigen values from w 0, from lambda 0, which is 0 Eigen value, to lambda r.

Lambda 0 is actually having the value 0. So, you have actually, r plus 1 distinct Eigen values, as I did in the previous slide; that is lambda 0, which is actually 0, then lambda 1, lambda 2, dot, dot, dot, lambda r. That is why I am starting i equal to 0, going up to r. Now, when T works on this, first consider i equal to 0 case, but i equal to 0 case; i have got the vectors alpha 0 1, alpha 0 2, dot, dot, dot, alpha 0 n zero; they are all belonging to the null space, or T working on them will give you, I mean, they all have a 0 Eigen value associated. That means, when T works on them, they will get multiplied by this scalar 0, and they will become 0. So, that means, they do not contribute anything, otherwise, that means I can start at i equal to 1 not at 0, go up to r; k equal to 1 to ni cik; T directly working on, as I told you, i equal to 0 case is ruled out, because for i equal to 0, this is Eigen vectors actually have Eigen value 0 or equivalently, they belong to the null space so T working on them; they give nothing but 0 vector.

So, they are eliminated from this sum and here, I have got the same thing; same summation cik. But they are Eigen vectors so lambda i; you can even put lambda i here; lambda i same also ki. So, this is a linear combination of those Eigen vectors, of those basis vectors, for this space. If you call this to be equal to w, what is the basis of w? If we append the basis of w lambda 1 to data w lambda 2, dot, dot, dot, data to w lambda r, then you get a basis of w. This linear combination is nothing but is a linear combination involving those basis vectors. Because, for i equal to, start i equal to 1, k varies from 1 to n 1, so that covers the basis vectors of w lambda 1, just a linear combination; same for other lambdas also. That means, this is contained in w, so that means, rem, because Tv belongs to RT, that is if you consider the range space of T, any vector there, can be written as some kind of Tv. Tv belongs to, I know, RT and Tv belongs to, that is, this actually for any vector, I am not taking for, any vector belonging to the null range space, we can write it as some Tv; for any some v belonging to v, because range space only consist of action of T on the elements of v.

So, any vector, if you pick from the range space RT of T, basically, it can written as some kind of Tv, where v is some vector of v. But that vector, I write as a linear combination of this basis vector, starting at i equal to 0 up to r and all that. T works on that and that gives you this. That means, range space is contained in w.

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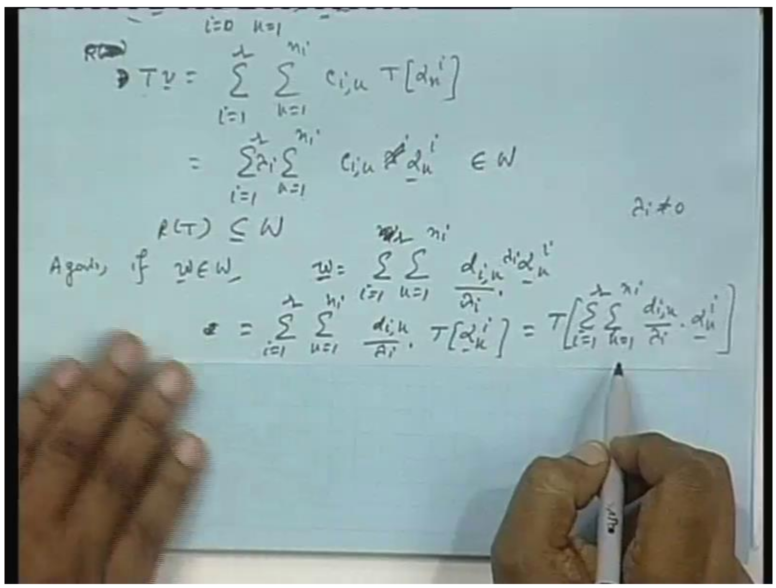

On the other hand, again, if v belongs to w, v will be a linear combination of those basis vectors. v should be i equal to 1 to ni, k equal to 1 to, sorry, i equal to 1 to r, k equal to 1 to ni, may be, some other coefficients, di comma k, alpha ik; that is just the basis vectors of these subspaces and linear combination. Because, w is nothing but the direct sum of this distinct Eigen subspaces. Therefore, any vector belonging to w, you can write like this. You can call it v, you can call it even w; that is better, vector w. There is any vector, can be belonging to capital W, can be written like this. But you see, this I can also write as; this I have written as, I can simply divide it by lambda i, because lambda i not equal to 0; 0 Eigen values can be separated out; they are i starts from 1. So, I can divide by lambda i and multiply it by lambda i. Therefore, I can write it as i equal to 1 to r, k equal to 1 to ni, dik by lambda I, T working on this will give you the same vector, alpha ik, but multiplied by the corresponding Eigen value lambda i and then using linearity. This is nothing but T can be taken outside the summation; this scalars times this vector.

But what is this vector? That means, this T working on some vector, so that means this thing belongs to RT.

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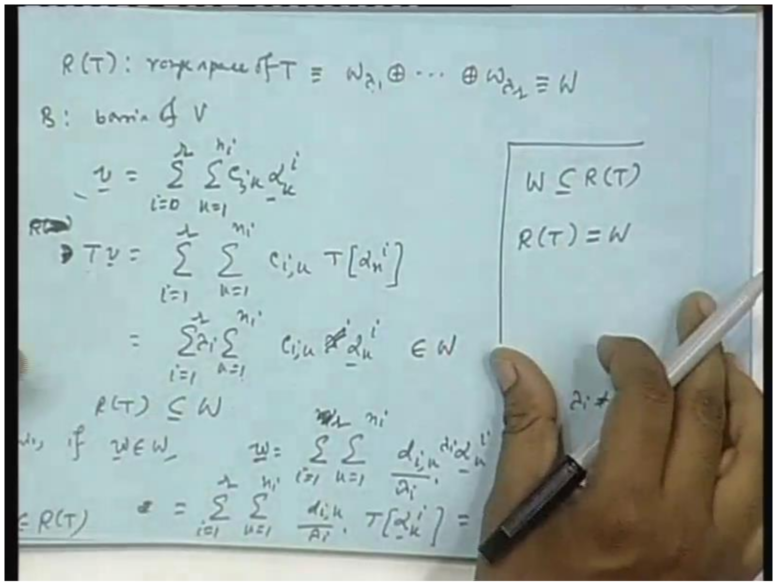

This means, w also belongs to RT; a subset of RT, which means RT is same as this w. This is the range space of T. This also shows, if this is the range space; what is the dimension here? Dimension is n 1 plus n 2; if you sum up the dimension of the respective subspaces; here, you got n 1; here, got n 2 dot, dot, dot, nr. What is the dimension of w? n 1 plus n 2 plus dot, dot, dot, plus nr. So, that is the dimension of w, means dimension of the range space, means rank.

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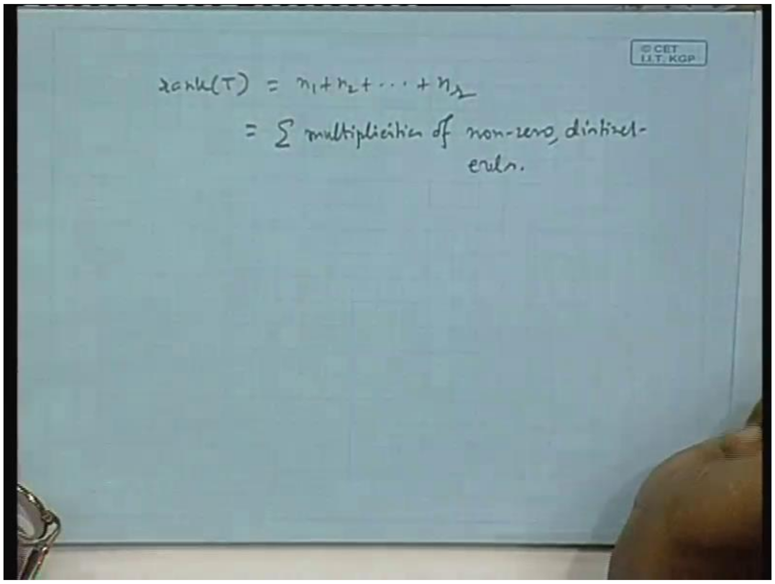

That means, what is the rank of T? n 1 plus n 2 plus dot, dot, dot, plus nr; that is multiplicity of non 0 Eigen value, say 1 plus, multiplicity of another non zero Eigen value, 2 plus dot, dot, dot, multiplicity of another Eigen value. That means summation of multiplicity summation of multiplicities of non 0 distinct epls; epls means Eigen values; this is the rank. Find out what are the distinct Eigen values, and which amongst them are non zero; just pick up the non zero distinct Eigen values and find out their multiplicities; add the multiplicities; that will be the rank of T.

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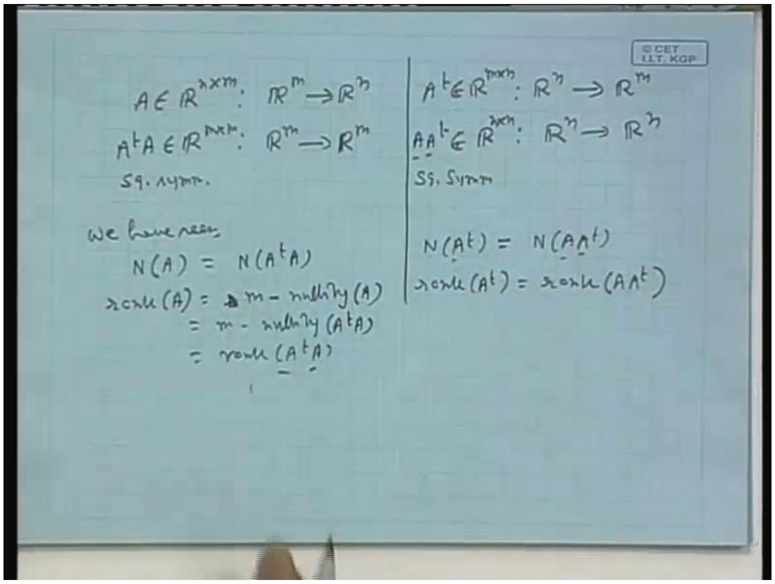

Then we now come closer to that SVD, take this case of matrixes; that is we now take a matrix A of size n cross m; that is this belonging to this form n cross m, means it is taking vectors of length m and multiplying it, and given the vector of length n. That means it is mapping Rm to Rn. On the right side, I take the case A transpose, which is Rm cross n, by the same token, it is taking Rn and mapping to Rm. Then, you consider this, A transpose A; obviously, A transpose A is a square matrix of size m cross m only. He takes Rm, but you should get Rm only, so square matrix, like the operators we consider earlier, v to v; Rm to Rm. Similarly, on this side, A A transpose n cross n, this maps Rn to Rn itself. These are all square matrix and symmetric matrix; square symmetric; real symmetric matrix; Hermitian matrix; actually, square. In fact, they are all, we know they are actually, non negative definite. That we can easily see, we have done it many times in this course.

Any matrix in this form, is a symmetric matrix and non negative definite, because if you take any non 0 vector x, multiplied by x transpose, and pre multiplied by x transpose, multiplied by x, x transpose A transpose Ax; you can write it as norm square of Ax, which always real and non negative. For such matrices, this form that the Eigen values is real and non negative. Also, we have seen one more thing, that if you consider two distinct Eigen values, the corresponding Eigen vectors are orthogonal or will be orthonormal. You have seen it in the case of correlation matrixes, and that time, you proved it. Eigen values, Eigen vectors corresponding to distinct

Eigen values; they are actually orthogonal and they can be normalized; that is each of it norm 1 and therefore, you can say they are orthonormal.

Now, this is given to you, we have also seen, that null space of A, that is on subspace of Rm, on which if he works, gives the 0 of Rn; that is the same as null space of A transpose A. You have proved that in the previous class and therefore, we have seen rank of A; rank of A and nullity of A together is m, that is rank of A is m minus nullity of A; and rank of A transpose A is again, is what? Rank of A transpose A plus nullity of A transpose A is equal to n, so rank of A transpose m minus nullity of A transpose A, but null space of A transpose A are same; they have the same dimension, same nullity. Therefore, the two ranks are same; that is, in case you have forgotten, m minus nullity; that is the dimension of the null space of n A; but null space of A and null space of A transposeA, they are same. So, m minus as well as nullity of A transpose A, because the two spaces are same, so the dimensions are same. But this is equal to rank of A transpose A, which you have seen in the previous class. In the similar manner, you can even call A as A transpose of A prime, and all these theories, you can work out on this side.

Null space of A transpose will be same as null space of A A transpose, and rank of A transpose is same as rank of AA transpose. In matrix theory, very elementary matrix, you have seen that rank of A and rank of A transpose are same, which means rank of A transpose A and rank of AA transpose are also same; you can see that in another form, another angle here also. So, you do not have to remember that elementary matrix theory that rank of A and rank of AA transpose. The fact that the rank of A transpose A rank of AA transpose are same, that will come even from another angle of this thing; another logical argument, which will come. For the time being we keep here.

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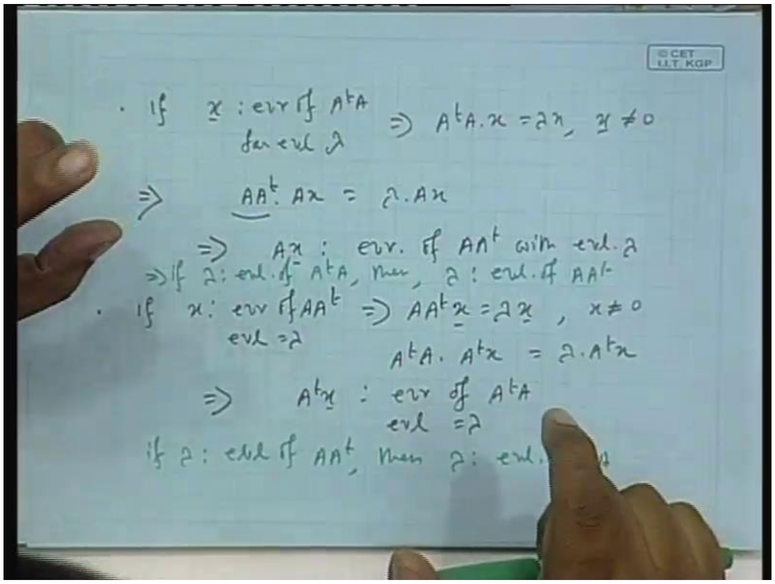

Now, suppose I prove some basic results, that is, if x evr is Eigen vector of A transpose A for Eigen value lambda, meaning A transpose A x lambda x and x is not equal to 0. Then, AA transpose, if you consider this vector Ax, then A transpose A x is lambda x; if you put that back is again lambda AX. So, that means in that case Ax is what, is an Eigen vector, evr of this matrix AA transpose with evl lambda. Similarly, if x is an evr of A transpose, meaning A transpose, not A transpose, AA transpose, meaning AA transpose x is with evl is equal to lambda, meaning AA transpose x is lambda x and x not equal to 0. Then, A transpose A, A transpose x, if you carry out what will you get? You can take A A transpose. and A transpose x is lambda x. You will get lambda A transpose x, meaning A transpose x is an evr of this matrix, A transpose A with evl equal to lambda. This means, if lambda is an Eigen value, here it implied; if lambda is evl of A transpose A, then lambda also, evl of AA transpose and this implies if lambda is evl of AA transpose, then lambda also, Eigen value of A transpose A.

This only means that A transpose A and AA transpose have the same Eigen values. If lambda is an Eigen value of A transpose A, it is the Eigen value of AA transpose, and if it is an Eigen value of A transpose, it is the Eigen value of A transpose A.

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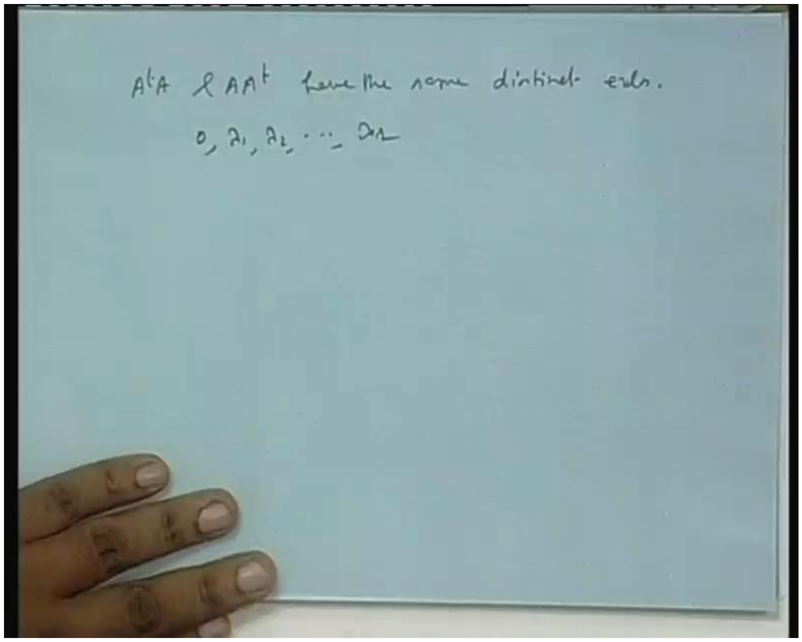

That means, A transpose A and AA transpose have the same distinct evl, not Eigen vectors, only evls. Eigen vectors are different; one of length n another length n, are totally different, but that distinct Eigen values could be like this; one could be 0, if I am picking up the permission of 0 Eigen value and then one could be lambda 1, lambda 2, dot, dot, dot, lambda r. We have also seen, I have written the argument, that A transpose A and AA transpose; they are symmetric matrices and non negative definite matrices and therefore, Eigen subspaces corresponding to w lambda 1, w lambda 2, then lambda 1, lambda 2, they are mutually orthogonal, because we have already seen that Eigen vectors of Hermitian matrices, belonging to distinct Eigen value are mutually orthogonal. Within w lambda 1, we can construct a basis, which is also orthogonal by Grand- Schmith procedure, so thereby, I can construct an orthogonal basis of the entire vector space rn.

I mean, here actually, we had made a statement earlier; we will not prove, that A transpose A and AA transpose, they are diagonalizable. That is, I mean, if you consider the distinct Eigen values, corresponding Eigen subspaces, if you find out orthogonal basis of all the distinct Eigen subspaces, append them; you get a basis of the original subspace. If you are considering them as the Eigen values for A transpose A, then by this process, I mean, you will get the w lambda 1, w lambda 2, all belonging to rm, and you find out their orthogonal basis, append them; you get orthogonal basis of rm, similar logic for rn. But that is a separate point. Now, you show in fact, these values are called singular. This is very important; singular values of A, and they are real and greater than or equal to 0. That means lambda 1 dot, dot, dot, lambda r are strictly greater than 0, because there, they can be either 0 or greater than 0. So, 0 has separated out; lambda 1 to lambda, they are strictly positive and there is one 0. They are called the singular values of A, and this will take us to the singular value decomposition. You only have to show, that if suppose W lambda 1, W lambda I; it belongs to Rm, and is the Eigen subspace of what; the operator A transpose A which works on Rm, gives you Rm, takes you to Rm; that operator. That is the Eigen value lambda i; the corresponding Eigen subspace; that is with respect to Eigen value lambda i. Similarly, for the same Eigen value, Wi prime, this is actually, same Eigen value lambda I and because, they have the same Eigen value; if lambda is here, lambda is there; lambda corresponds to Eigen subspace W lambda Rm and Wi lambda i prime in Rn.

If dimension of W lambda i is say ni; dimension of W lambda i prime is ni prime; there is multiplicity of lambda i in A transpose A is ni; multiplicity of lambda i AA transpose matrix is ni prime. Then, we prove that ni and ni prime are same, that two multiplications are same, we proved; is not difficult.

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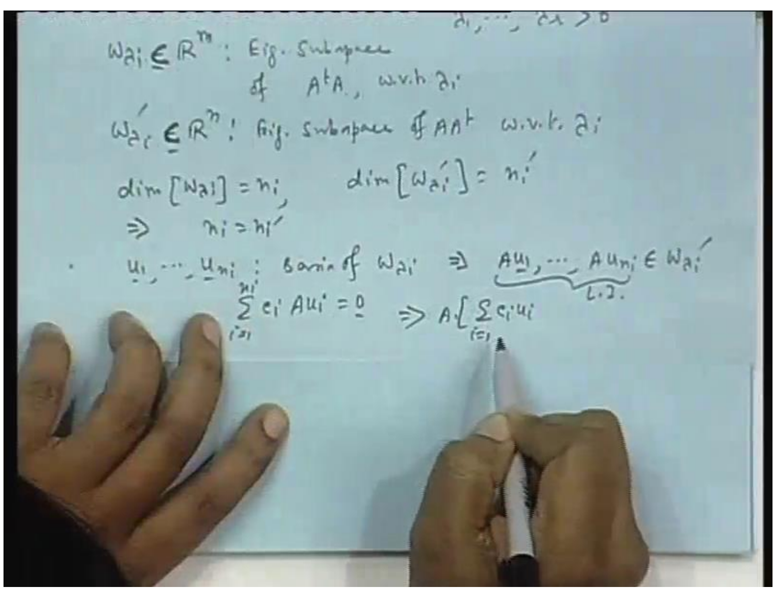

Suppose, you consider U 1 dot, dot, dot, Uni as a basis of W lambda i, that is the basis of W lambda i. That means, each is an Eigen vector, corresponding to lambda i. This means, we have already seen, if u 1 is an Eigen vector of A transpose A, then A u 1 also is an Eigen vector of AA transpose for the same Eigen value. That means, Au 1 dot, dot, dot, Au ni; they belong to W lambda i prime from our theory, which we have shown earlier, that A transpose A has Eigen value lambda and Eigen vector x, then AA transpose will havee same Eigen value, lambda, but Eigen vector Ax. If u 1 here, A 1 here; if Uni here, AUni; so they are all Eigen vectors of AA transpose for the same Eigen value, lambda i. So, they all belong to W lambda i prime. I will show that they are linearly independent, that is, this set linearly independent, you can see. That is, if you form ci Aui, i equal to say 1 to ni and equate to 0, ci should be 0; that follows very easily.

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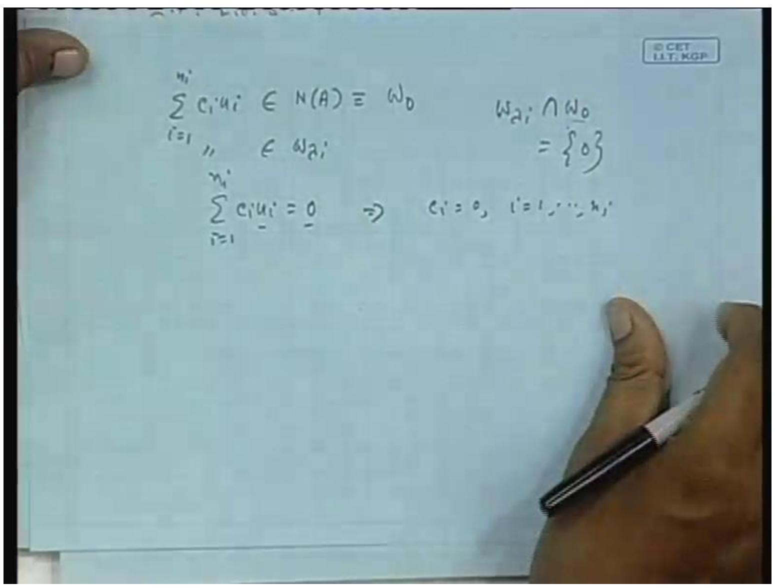

This, you can write as A on, that means, ciui, i equal to 1 to ni; should belong to what? That null space of A. What is the null space of A? Null space of A means, W 0; that is the Eigen subspace corresponding to Eigen value 0. But where from ui is taken? ui is taken from W lambda i. So, W lambda i and W 0, but this also belongs to W lambda i, and W lambda i intersection with W 0 is nothing but only the 0 vector. Because, this is corresponding to Eigen value 0; this is corresponding to Eigen value lambda i; they are distinct and therefore, the corresponding Eigen vectors, any vector from here, and any vector from here; they are linearly independent. So, only place where these two subspaces intersect is 0; that means, summation c i u i is 0, but u 1 to u n i, they are the basis; they are linearly independent.

That means, c i equal to 0; i equal to 1 dot, dot, dot, n i. This only means that these elements are linearly independent; that means, the dimension of these elements are linearly independent, right.

That is, a u 1 dot, dot, dot, a u n i is linearly independent; that means, dimension of w lambda i prime, which is actually, n i prime; it is greater than or equal to n i. Total number of elements here, n i. They are linearly independent contained in this. So, dimension of this cannot be less than n i. So, it can be greater than or equal to n i, but we saw that it is actually equal to. If not, now, suppose we take any vector. It is like this, that suppose, n i prime greater than n i. Then, you take a vector; take v from w lambda i prime, but outside, just span of a u 1 to a u n i; that is, these vectors, which from a linearly independent set within w lambda i prime, this minus space, go outside that, still you remain in this and take v from there.

That v is an Eigen vector of AA transpose, corresponding to Eigen value lambda i. That means, A transpose v is an Eigen vector, belonging to w lambda i, is the same Eigen value. Because, I said that, if AA transpose has an Eigen value lambda i and Eigen vector v, then A transpose v is an Eigen vector for A transpose A, for the same Eigen value lambda i. So, A transpose v is here and if A transpose v is here, then if this is an Eigen vector, then A working on that. That means, A working on A transpose v, will take you inside this; this is an Eigen vector. But AA transpose v will be what? Actually, this needs little extra. So, in the next class, because the time is short, I will pick it up in the next class. I will just work out these proofs here. We will start from here in the next class.

Thank you very much.