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NPTEL ONLINE CERTIFICATION COURSE

Course
on
Analog Communication

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Lecture 06: Fourier Series (Contd.)

Okay, so what we have done so far we have try to discuss about Fourier series right, and especially the origin of Fourier's it means how Fourier's has come into picture. So I will just quickly recapitulate what we have achieved so far, so given a vector analogy we have said that may vector if you know the dimension or whatever vector space you have if you know the dimension and if you know means as many orthogonal vector can exist in that particular dimension.

Suppose you have a two dimension thing and you have two orthogonal vector, so what we have said that as long as you know this two orthogonal vector you can represent any vector in that space in that two dimension space with respect to these two vectors especially the linear combination of these two vector and these two orthogonal vector are actually termed as the basis vector, right. We have taken that same analogy into signal so our target there was of course we have started targeting a periodic signal.

So what we have said given any periodic signal that can have any nature can we really find out some known signal a linear combination of them can give me back that same periodic signal, so any signal possible in the world which are periodic with some period let us say T_0 okay.

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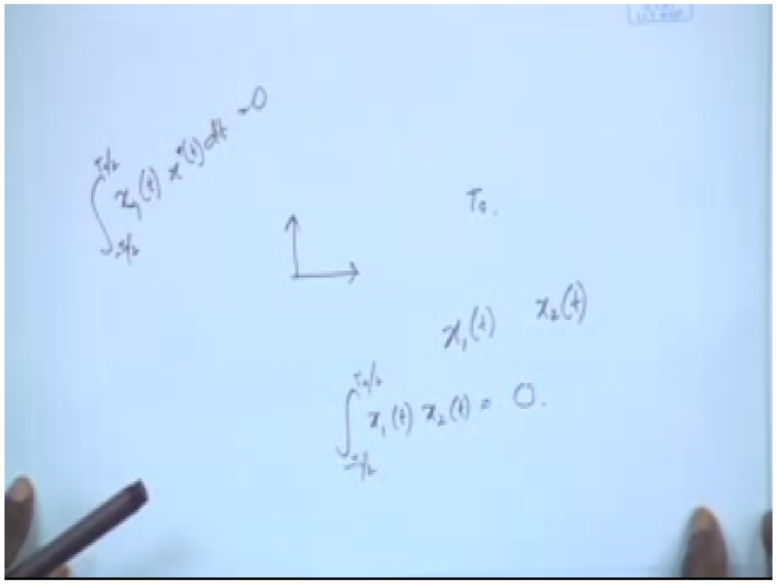

So can we get set of basis signal okay, or set of signals which are already known like this orthogonal signal so already known and can be represent and can we guarantee that any signal in that particular signal space can be represented as linear combination of all those known signals. So what in that aspect we have defined few things I am not going it to the details of those things so I have defined what we mean by orthogonality condition in signal space, okay.

So if two signals $x_1(t)$ and $x_2(t)$ if these two signals are there both are periodic signal with of period T_0 let us say and then how do we actually prove that these two signals or how do we test that these signals are orthogonal, so this is something we have already evaluated, okay. We have evaluated two conditions one is if both signals are real then we have evaluated one condition and then we have said if both signals are complex then also we have evaluated one condition.

So the ultimate condition which we have also proven is that $x_1(t)$, $x_2(t)$ over that period let us say $-T_0/2$ to $+T_0/2$ this must be 0 if both the signals are real, and at the same time if signals are complex then you have to take $x_1(t)$ and a complex conjugate of the other signal $x_2(t)$ and you have to do the same integration over that same period and this must give me 0, if $x_1(t)$ and $x_2(t)$ are orthogonal to each other.

So what we have to now do we have after that we have said we should have a quest of this known signal all the orthogonal signal in that signal space we must be able to find out as many are there, okay. So in that process we have defined two things one is trigonometric Fourier series that means all the signals must be decomposed into means series of sinusoidal or cosinusoidal so that is one another one was exponential Fourier series again all the signals must be decomposed as linear combination of exponential functions, right.

Remember if the signal targeted signal has a period T_0 all those exponential signal or it is a complex exponential signal all the exponential signal also must have period of T_0. So in that process trigonometric one we have already done exponential we have said that.

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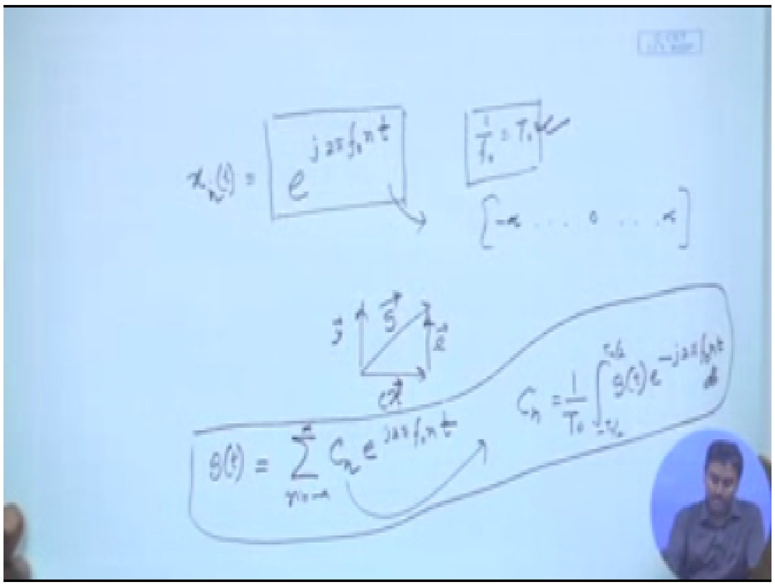

$e^{j2\pi f0nt}$ okay, so that is the signal we are targeting and we are calling this as all those $x_i(t)$ or $x_n(t)$ okay, so basically it is a as you can see it is exponential and you can put Euler's theorem and then you can get 1cos+jsin and then the period will definitely be 1/f0 right, so f0 is the frequency so for any value of n there will be a period which is defined by 1/f0 which is called as T_0, okay so this period will always be there.

So all the signals are periodic signal effectively with period this and remember these are all complex signal so we need to orthogonality condition we need to actually prove it from that complex perspective or complex signal perspective that is the first thing, okay and with this we have actually shown that these signals are orthogonal with respect to different values of n, right so we take different values of n you can immediately show that it is orthogonal so whenever suppose we take two values m and n and then you put that orthogonality condition that we have just discussed you put that you will be seeing that if m is not equal to n it will be always 0, okay.

So these signals are all for different values of n up n starting from I mean 0 to ∞ and -∞ so for all this values of n they are all orthogonal to each other. There is another aspect of it in vector also we have told that eventually what we were targeting any vector we were targeting to represent by another known vector right, and in that process we are targeting the error of this representation so we have said that this is that error so if suppose this is my g and this is my x and if g has to be represented by x where given some linear coefficients C_x and then this was the error and our target was to minimize the error same thing we have also done with respect to the signals and then we have evaluated all the values of that is required for different signals right, so that is something we have already evaluated.

So it is the error that we are targeting to minimize right, and eventually we also said that in the vector space suppose in this one we are just trying to represent g which is a two dimension vector by one vector so that is why there will be always a error we can minimize that but there will always be a error, but if we have another vector which is orthogonal to this x let us say y we can always completely represent g with respect to linear combination of x and y we know that already in a two dimensional space.

We have two orthogonal vector we can always represent that, so that means if you have a completeness of the basis definition whichever space you are defining if all the orthogonal vectors in that or all the orthogonal basis vector are already defined then you can always have a linear combination and you can make your error 0 same thing happens to the signal if I have a complete definition of basis set then I can always represent a signal with respect to linear combination of them and we also know how do we calculate the individual coefficients.
So there is just a complete recapitulation of what we have achieved so far, okay so with that aspect we have talked about completeness and we are just saying without prove that for all values of m let us say all values of n, n going from -∞ touching 0 to +∞ okay, every integer value it takes if we take all this then this makes a complete basis set this particular $e^{j2\pi}$. So we are just stating it without prove and then we can say any particular signal g(t) which has a period T_0 it might have inertia that can be always represented as a linear combination so with coefficient $e^{j2\pi f0nt}$ n going for -∞ to +∞, right.

So this is something we can always talk about where this is are evaluated just by that error minimization process which was evaluated as $1/T_0$ integration whatever that g(t) is there $e^{-j2\pi f0nt}$ dt okay, and it should be integrated over that period let us say $-T_0/2$ to $+T_0$ so that is the Fourier series and corresponding coefficient we already know about, right this is something we have already know. Now let us try to see if this is the representation in this representation what do we get, we have already seen that in trigonometry representation we actually get different sinusoidal right, so at different frequency this is also exponential representation so it must be a sinusoidal it is a complex sinusoidal, so let us try to see what do we get

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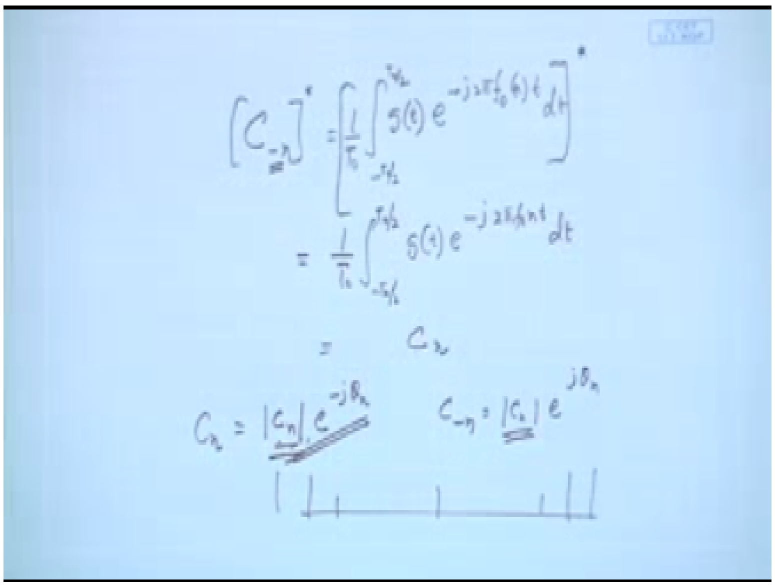

So first let us try to evaluated what do we mean by this okay, so see that coefficient evaluated at –n and I am trying to take the complex conjugate of that, so let us first put what is C_{-n} that should be whatever my signal is let us say my g(t) $e^{-j2\pi f0}$ now here that coefficient should come at that coefficient it has to be evaluated so that particular coefficient –nt dt it must be integrated from $-T_0/2$ to $+T_0/2$ and there is a complex conjugate of this so the whole thing has to be conjugated right, so it should be –n I forgotten that right.

Because we are evaluating the coefficient at –n so corresponding to the coefficient should be –n and if this is –n what happen this -,- it becomes + right and then we are taking complex conjugate so integration has nothing to do with it so it will come inside the integration g(t) is a real signal we have already talked about that, that we are evaluating the Fourier series of a real periodic signal g(t) okay.

So g(t) is real so a complex conjugate of that will be just g(t) and the complex conjugate of this one so what that will be it is already -,- + and you take the conjugate of that, that should be again – so I should eventually get $1/T_0$ $–T_0/2$ to $+T_0/2$ g(t) e to the power again get back -, so there are two negation -,- becomes + and then you take complex conjugate so it again becomes -.

You can identity this, this is exactly right, so what a very nice thing we have observed out now, what happens whenever we evaluate this coefficients C_{-n} is actually a complex conjugate of okay, so always C_{-n} for any value of n they are always a complex conjugate of what does that means. Let us say my these are complex number so it must have a amplitude and a phase so I can represent that as and then there should be a phase which is let us say $e^{-j\theta n}$ okay, I can represent it this way.

Because is a complex number so it must have a amplitude and must have a phase right, so that is all we are doing we are just putting this representation as amplitude and phase, okay. So this is the case what must be the complex conjugate of that the C_{-n} must have amplitude remains the same because it is complex conjugate so it must be and this will be just conjugated so $e^{-j\theta n}$ complex conjugate will be $+\theta_n$

So what is happening if because now every coefficient have two parameter if I can see it has a amplitude it has a phase, so I can have eventually two plots one is with respect to this n I can plot the amplitude so what happens to the amplitude if at n positive I get some value at n negative the amplitude should remain the same right, because for C_n also it is $|C_n|$ for C_{-n} also it is $|C_n|$ right, so the amplitude should remain the same therefore what do we get for different values of n the values might be different.

But whatever it is at –n the value should be same right, so we get a even symmetry in terms of the plot right, whatever we get at positive half we get same thing at the negative half, okay so that is why it is call even symmetry and for phase just the opposite whatever we get at positive we will getting just the negative of that right, because you can see if it is $–\theta_n$ it should be $+\theta_n$ if it is $+\theta$ it should be $–\theta_n$ so the angle should be same but it is just negated right, so we odd symmetry for the phase plot.

So basically what we are getting now we have a signal g(t) right, it is continuous in time what we can see now that g(t) as long as it is periodic it can be decomposed into multiple complex sinusoidal this is something we can immediately observe and those sinusoidal for different values of n it is actually representing different frequency component if n=1 it is f0, n=2 it is 2f0 if n=-1 it is –f0, n=-2 it is 2f0 so it is actually representing the front frequency component or you should say different sinusoidal at different frequency value, okay.

So all we are doing is now because we have got if you see this we have got two equivalent representation of the same signal g(t) in time I can plot I can also plot g(t) or I can represent g(t) as a ∑ in infinite sum of course as a ∑ of sinusoidal at different because this is just a complex sinusoidal which is known okay, so it is just that complex sinusoidal all we need to know is at every frequency component what kind of amplitude it has and what kind of frequency it has.

So the information about the signal is completely carried over here because this is a known thing okay, all the signals are known they are just representing different complex frequencies nothing else, we just have to know the corresponding coefficient because as the coefficient changes it will represent different, different signal that is all we are targeting, okay. So therefore times to prove Fourier what we get is a separate representation of a signal. Now we do not see the signal in time domain whenever we have a signal we know that it has a equivalent Fourier series representation as long as it is periodic and immediately we can plot.

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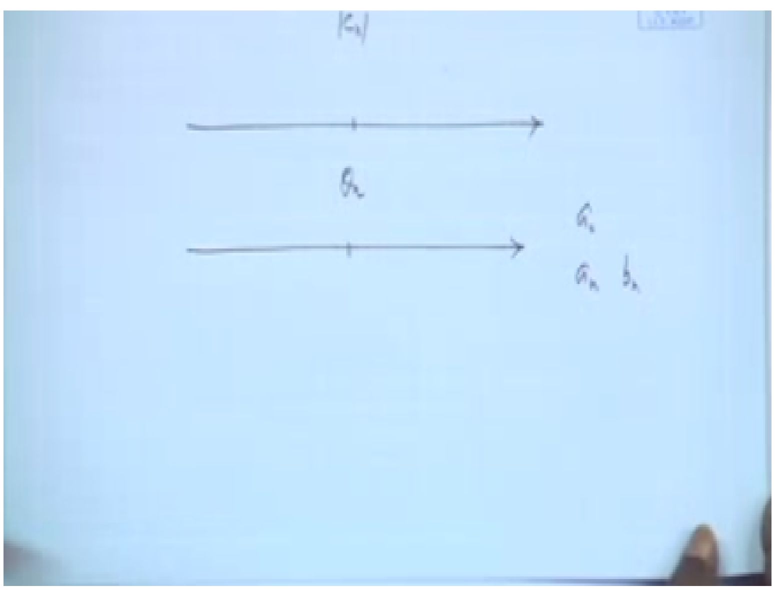

And effectively we have to give two plots one for amplitude, one for phase and we also know that the amplitude plot should be symmetric or even symmetric, so therefore positive or whatever it is it should be newer image at a negative half and the phase plot should be odd symmetric that means whatever we get in the positive half that should be just negative of that in the negative half, right so that should be the case so we should plot $|C_n|$ and θ_n, so all we have to do is this and we know that for which point in this independent axis we have to plot this are just those n value or more precisely nxf0 values.

So those frequency those complex frequency component how much of that component is present in phase as well as in amplitude, so how much of those components if I mix them all together my desired signal will be generated. So this representation is called a frequency domain representation because we are actually decomposing our whole signal into different frequency components that is all, all that we are doing is this only.

There are two strikingly different representation when we do it in trigonometric series and when we do it in exponential series, but the function must have unique representation in frequency. So let us now try to appreciate what exactly is the relationship between these two representation, when we started representing it trigonometric series what we were getting we are also getting two plots actually because we had that sinusoidal as well as cosinusoidal at every frequency we were having both the things, right barring 0, 0 was only having a DC values so it does not have, it does not recognize whether it is cos or sin because it is not having any frequency component, right it is Dc.

So at that point it was just a0 but rest of the case it was always having a pair a_n and b_n corresponding a_n is due to the basis function cos2πf0nt and b_n is due to sin okay, so these two frequency component represent. If you see that representation there was nothing called negative frequency okay, which we are getting over here in the exponential representation, so exponential representation have the spectrum starting from $-\infty$ to $+\infty$ so it has some component at the negative frequency as well as positive frequency.

Whereas when we are representing it in trigonometric that also has and this also have two plots one is amplitude one is phase. Whereas when we are representing in trigonometric form it was always having two plot corresponding to sinusoidal and cosinusoidal but it does not contain any negative frequency, right so this is something we have observe so we are always getting two plots and in this two plot what was coming it was just those $|C_n|$ at every value of n $|C_n|$ and θ_n

that is what we are getting, okay and then we are getting this for the other case we are getting a_n and B_n. Now let us try to see if we can correlate these two things that is first task.

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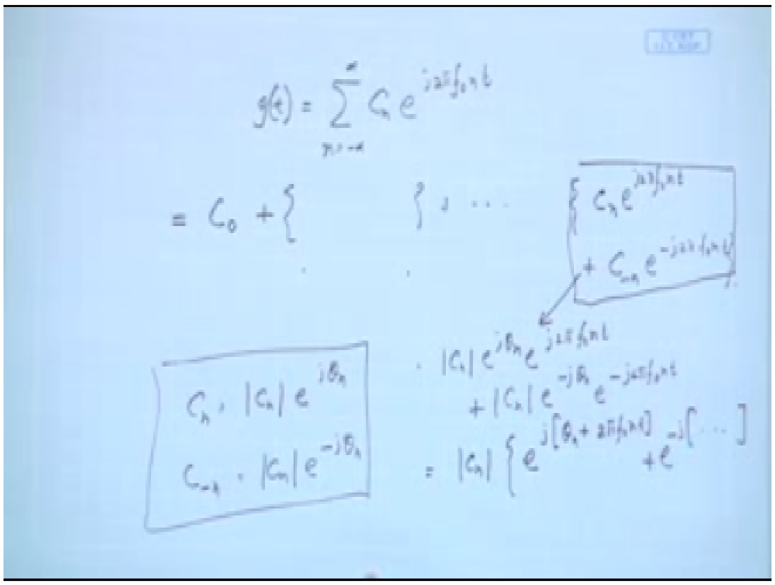

And then we will come back and try to explain what do you mean by this particular thing call negative frequency okay, so first let us be concerned about this relationship so what happens we have seen that there is a already we have explored that there is a relationship between C_{+n} and C_{-n} so what we will try to do is we try to pair up these two things, so we have a representation of g(t) as ∑ $e^{j2\pi font}$ n going for -∞ to +∞ okay.

So let us separate out the c0 term okay, so at C0 n is 0 so this becomes e^1 right, so that is alright and then we start pairing each of those n and –n right, so here we will do a pair of C1 and C-1 so on let us try to do it for $C_n$ej2πf0nt+C_{-n} e to the power, definitely it will be –n so –j2πf0nt okay, let us try to see what this term to greater gives me, see if I can evaluate all this terms I am almost coming to a conclusion that it is almost like this C0 is nothing but A0 and this is actually almost like a_n cos and $+b_n$ sin if I can finally represent them in that format, then I can give a relationship between this C_n and a_n, b_n right.

So let us try to see, now all we know is C_n is a complex conjugate of C_{-n} therefore I can immediately put our representation so I can say that suppose C_n is represented as $|C_n|e^{j\theta n}$ then C_{-n} must be represented as $|C_n|e^{-j\theta n}$ right, this is something I can always do as long as I know this θn and $|C_n|$ and that I should be knowing because if I know the C_n I can immediately calculate it is modulus and its phase right, so this is something I can always do.

Now let us try to see if I put this replace this over here what do I get so for that n^{th} term I can get $|C_n|e^{j\theta n}$ $e^{j2\pi font}$+$|C_n|e^{-j\theta n}$ $e^{-j2\pi f0nt}$ okay, so this is the n^{th} term I am getting which is nothing but $|C_n|$ gets common and I get $e^{j\theta n+2\pi f0nt+}$ the negative of that right, e^{-j} same thing okay.

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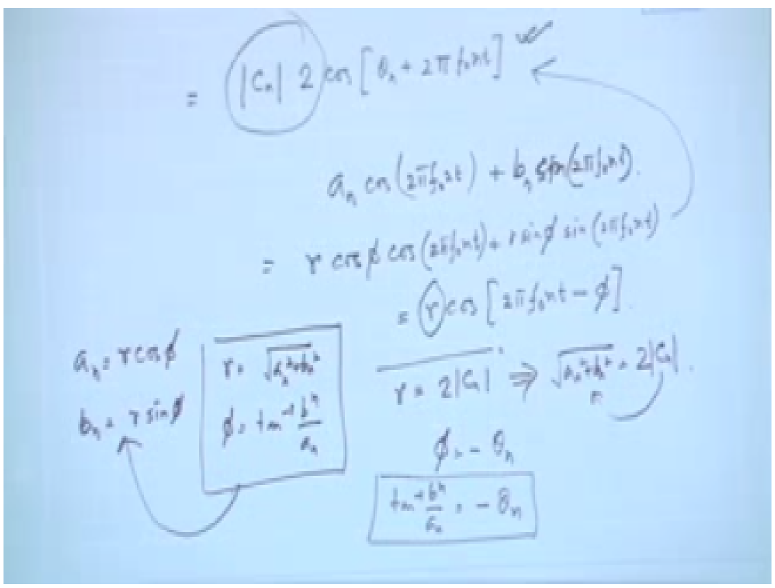

So if I just say it is e^{j} some θ or let us ϕ, $+e^{-j\phi}$ put all as theorem what do we get, 2xcos of that cos of inside whatever ϕ is there so I can write this easily as $2\cos[\theta_n+2\pi font]$ right, what I have eventually got is a single cosinusoidal term right, I have got a single cosinusoidal term nothing else this is something I am getting, okay. Similarly, for the trigonometric series for every n^{th} value I will be getting $1a_n$ cos2πf0nt+$B_n$cos2πf0nt sorry sin, right I will be getting this.

Now this one what I can do a_n I can write as sum rcosϕ and bn I can write as rsinϕ this is something I can always write, right because immediately I can see I can calculate r, r should be what $\sqrt{a_n^2+b_n^2}$ and ϕ should be if I just divide these two $\tan^{-1} b_n/a_n$ such long as I know this relationship I will be always able to put this representation put this over here so I get rcosϕ cos(2πf0nt)+rsinϕ sin(2πf0nt) why I am doing this I just want this representation again, right.

Can I see a single cos representation it is cosa cosb+sina sinb so that must be cos a-b right, so I can write as cos[2πf0nt-ϕ] fine. So now I can see I have got two series one was exponential series another one was trigonometric series from both the series I could actually get I was trying to see the n^{th} term and I could see that by unifying this cosiusoidal and sinusoidal and I unifying the C_n and C_{-n} I could get similar representation, only thing is that those coefficient has to be now matched right, because I will be getting exactly similar term for every n.

So therefore the coefficient because it is a unique signal so it must have unique representation unique addition of sinusoidal okay, so therefore coefficient must match so therefore r must be 2|Cn| right, and this ϕ must be minus of this θn right, so this is what we are getting. Now ϕ is $\tan^{-1} b_n/a_n$ that must be $–\theta_n$ so this is one relationship I get and r is basically $\sqrt{a_n^2+b_n^2}$ that must be $2|C_n|$.

So basically what is happening I have got a trigonometric series I have got a exponential series, exponential series whatever amplitudes plot I get I immediately can get a relationship between the corresponding trigonometric series, okay. So if these two are equal the representation should be equal and what has happen fundamentally as you can see that because of this complex conjugate you see finally when I added those two terms C_n and corresponding to C_{-n} I have actually got a real signal.

So that means my signal was real people might be thinking that when I do full exponential Fourier series I am represented them in terms of complex signals, so what is happening to my real signal complex signals are now creating their signal that is not the case the C_n's are arranged in such a way that if you take two n^{th} terms they will those two complex signal will actually give you back the real signal, okay. So eventually it is nothing but both the representation tells you that it is a particular representation where the, for a particular frequency there is an amplitude and there is a phase nothing else both the representation gives you similar value, okay.

So with this I will end this class the next class what we will try to do we will try to evaluate what do we mean by this negative frequency.