Communication Networks
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Module - 05
Queuing Theory
Lecture - 22
D T M C

Alright; so far we have been discussing some portion of Queuing Theory. Now we have come to the critical juncture of discussing the actual queuing process. So, we have already talked about two things, one discrete-time Markov chain and a continuous-time Markov chain. So, today we will start discussing about Discrete-Time Markov Chain.

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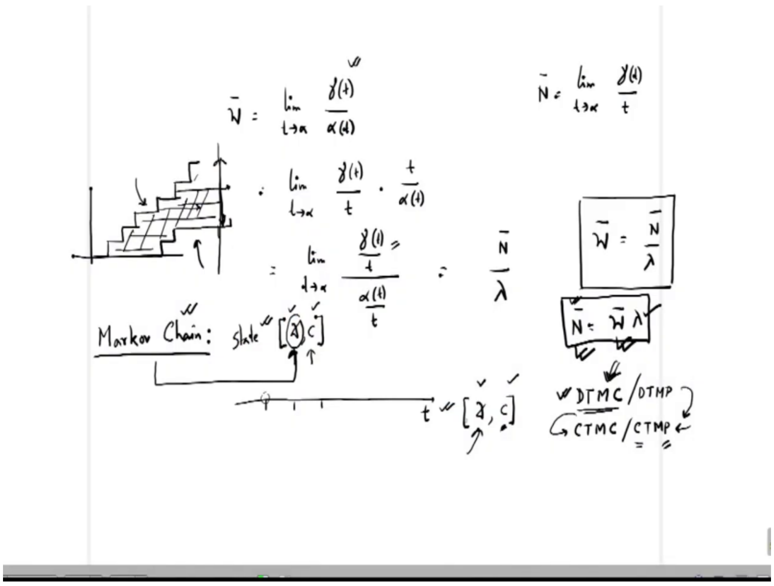

So, that was the discussion of the last one. So, we will start our discussion of this DTMC.

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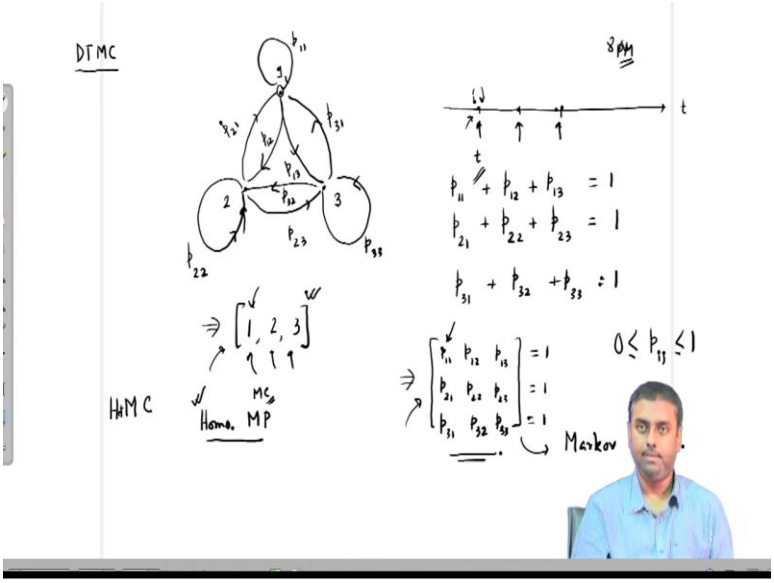

Let us try to understand some of the properties of these things. So, for that let me give one example, let us say just a system I am trying to create. Let us say there is a random traveler who actually goes from city to city ok? So, I am just taking one example, you can find this example also in the book of Kleinrock.

So, basically let us call that city 1, 2, or 3 we can we can take any city name. So, it might be Delhi, Calcutta, Bombay something like that ok? So, he is moving around from this city to these other cities ok. Now, how he does do that? So, there are some assumptions about this, of course, it is a hypothetical thing.

So, we will be taking some examples or some assumptions from that example. So, that is what we are saying that it is a hypothetical example and he might be in a particular sub every day, if I see the time the time is slotted because it is a DTMC. So, time is slotted.

So, let us say every day at, let us say 8 AM or 8 PM; let us say 8 PM he catches a bus to move to another city, but which bus he will be getting is random. So, beforehand he also does not know and beforehand even the observer also does not know.

So, he can catch any bus, but with some associated randomness ok. So, basically from there immediately he will be, and this transition from one city to another city, so that bus is almost like teleportation ok?

So, it transits him immediately ok? So, immediately he goes to another city according to whatever probability he takes some bus he catches, he might be in city 2 on a particular day t. So, he might be in city 2 and then at 8 PM he might catch any bus for city 1 with some probability any bus for city 3 with some other probability or he might remain over there ok.

So, that might happen. So, of course, something will be happening. So, therefore, his probability of transiting to City 1 City 3, or City 2 overall will always be 1 ok. So, that is something you have to keep in mind. So, whichever city he is there.

So, his probability to transit to other cities, that probability if you sum over all possibilities must be 1. Because that should be he must be doing something, he must be either going to some other city or he will remain in that same city ok he cannot do anything else.

So, therefore, overall some of those probabilities should be 1. Now, these probabilities what are those things? So, every time every 8 PM of every day, he actually inserts those probabilities. So, these are called actually the transition probability from one city to another city.

So, therefore, on day t we might have some probability, we might have some probability for suppose I have observed him in city 2 on day t, and then on that day, he is transiting. So basically, he might have some probability to transit to some other city and watch that associated probability.

So, those probabilities will be characterized by the transition probability given by the description. So, there might be some transition probability p 21, there might be some transition probability p 23 and there might be some transition probability to himself p 22, but whatever happens this p 21 plus p 22 plus p 23 must be equal to 1 this must be happening.

And this is true for every city, in this city also the same thing will be happening. So, there is a transition probability p 12, there is a transition probability p 13 and there is a transition probability to himself p 11. Again, the same thing will be happening so p 11, p 12, and p 13 must be equal to 1.

Similarly, for 3 also the same thing. So, he might transit over here, he might transit over here or he might remain over here. So, this will be p 31 p 32 and this is p 33. So, again p 31 plus p 32 plus p 33 must be equal to 1 ok.

And if you see I can represent this. So, basically, what you can see over here, these cities where he is that makes the state ok, that is actually making the state. So, the cities are the states where he will be where I observe him. So, any day I try to see I will be describing where that particular traveler is located.

He might be in a city because that is a random thing, he might be in City 1, City 2, or City 3. If I can describe that where he is that should be the description of, means description I wish to means make I wish to infer.

So, because of that, I describe that as to be state. So, he might be there in states 1, 2, or 3. So, that is the overall state space ok, because we have considered 3 cities. If we consider 5 cities then the state space will be 5 and he has a possibility of being in every city. Now, this is probably random because which city he will be that completely depends on where he has started and how he is transiting with all probabilities it is happening.

, therefore, which city he will be that is also probabilistic because it is a random process right, there are states, and associated between states there are transition probabilities, and if you see if there are 3 states I can have an associated transition probability matrix ok?

So, which is this one, p 11, p 12, p 13, p 21, p 22, p 23, p 31, p 32, p 33 of course, their summation, the rows summation should be all one. This typical matrix where the values are all probability therefore, they lie all these p ij's are always between 0 to 1. Their row sum is always 1 this typical matrix is actually a transition matrix for capturing the transition probabilities from one state to another state or all possible states to all possible states.

So, this is termed as Markov matrix ok. So, Markov matrix has this nice property, that they are generally probability value and their row sum is always 1 ok. So, that that property is there. There are additional properties that probably we will not be requiring ok, if we do a full-fledged queuing theory course probably that thing will be exploding.

But right now, probably this is good enough for us. Now, let us talk about this state and transition these two probability things we have got from this description ok? We have now understood what is a discrete-time process Of course, it is it is only happening at some time means boundary, it is not happening anytime it cannot happen anywhere ok?

So, the way we have described the system it can only happen at some particular boundary. So, ok; so it can only happen in some particular time boundary. So, therefore, it is a discrete-time process, I know the state transition is happening only on that boundary, on that means specialized boundaries only, it cannot take place at any time.

So, that is why it is a discrete-time process and it has an associated probability of transition from one state to another state. So, basically, it has a state description of which state it can take over here, and as we have told you we will be concerned about discrete states only. So, over here the example also we have taken as a discrete state.

In our queuing, it will be just the number of customers ok. So, it might be 1, 2, 3 something like that, or even 0. So, it will be just transiting from that. So, from 0, it will go to 1 like in our bookkeeping we have already seen. In queuing what might happen? It can go the queue length might increase from 0 it can go to 1, then 1 to 2, again it can come down 2 to 1 something like that.

So, all those transitions associated with probability we are now talking about over here. And of course, remember it is a discrete time process earlier whatever bookkeeping we were doing was all for continuous time, but right now we are concerned about the discrete time process ok.

So, that is that is our whole description of the system. Now, let us try to see what can we say about these two things, state, and transition; first, we will start with the transition. Now, in the transition of what might happen, there is no restriction. So, every day there might be different probabilities that he takes.

So that means, this transition probability itself evolves over time, this might happen that on day 1 probably he had some other probability it depends on his interest, it depends on the associated bus availability their arrival randomness, and all those things. If that randomness changes over time then probably on different days there will be different probability of him going to another city from that city to another city; that means, the transition probability will vary over time.

If that happens then it is called a heterogeneous Markov chain. Of course, I should put heterogeneous Markov chain ok if the transition probability will be more of our interest, that generally the transition probabilities do not the system description does not really change over time, the kind of system will be taking ok.

If that is the case then that particular kind of Markov-associated Markov chain or associated Markov process is called homogeneous Markov process or associated homogeneous Markov chain. We still have not talked about what is Markov chain, but we will come to that in due course ok.

So, we will be mostly interested in this homogeneous Markov process. So, therefore, this transition probability will not be really varying over time, if the transition probability is not varying over time can you now say the associated state probabilities? Will that vary over time? Let us try to enquire if that is ok. So, that is exactly what will be now next doing, let us try to this part ok?

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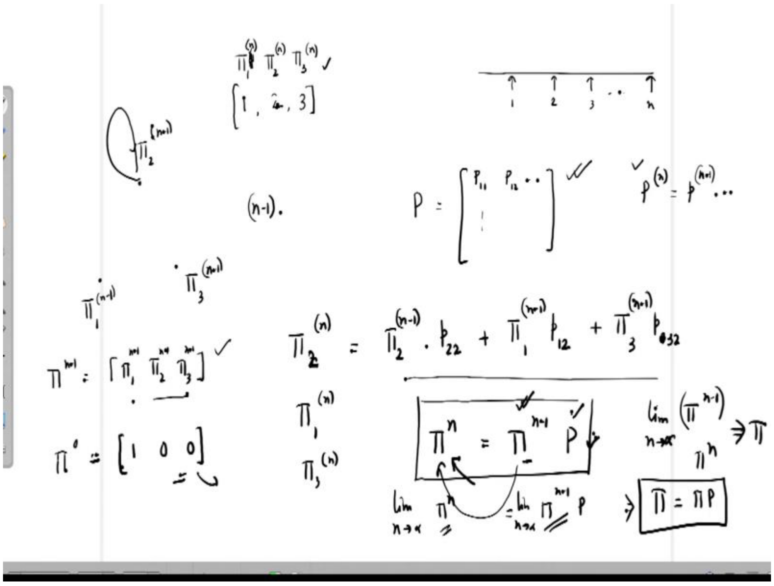

So, what we have been told? That lets us have that particular random traveler example. So, we have state 1 2 3, let us say associated probabilities we put with pi. So, pi 1 pi 2 pi 3 initially we will think that they vary with time.

So, basically, we can say pi 1 time is now discrete. So, I will give pi 1 n pi 2 n pi 3 n. So, in the nth time instance because it is a discrete time. So, time instance 1 2 like this it goes 3 up to n. So, in some arbitrary nth time instance these are the state probability ok.

Let us try to see how I characterize this state. So, right now what we have taken is a homogeneous Markov chain; that means, my transition probability matrix which is the P matrix which is consists of P 11, P 12, and so on. So, all, these things do not vary over time if it was heterogeneous then I just had to write P n; that means, in time instance n what is the associated transition probability?

If it is homogeneous then P n's are all for all n they are equal. So, P n is equal to p n minus 1 is equal to, so everybody is equal because the transition probability matrix is not varying. Now, how do I get a relationship between the transition probability and this one? Can we now try to characterize that?

So, let us try to understand these things from the probability perspective, the basic probability perspective. So, let us say on day n minus 1 he has an associated probability pi n minus 1 that he is in city 1, pi 2 n minus 1 that he is in city 2, and pi 3 n minus 1 that he is in city 3 this is the associated probability.

Now, what is the probability of finding him the next day in City 2? Let us say I want to calculate pi n sorry pi 2 n, let us try to see how do I evaluate this. So, the evaluation is very easy in the previous day where he was and the associated transition you multiply because it is where he was that is independent of what transition probability he will be taking.

So, these 2 events are independent so I can multiply them ok. So, therefore, suppose he was in city 2, pi 2 n minus 1, and then if on the next day, he has to be in city 2 it has to take a transition p 22 from the same city 2 to himself he has to take a transition.

So, therefore, this is the probability that if he was in city 2, if I multiply with p 22 then he will remain in the next day city 2 that is the said probability. But is that all? No, he might also be in City 2 where on the previous day he was in other cities and those events are mutually exclusive ok.

So, therefore, that probability will be added. So, pi 1 n minus 1. So, he was in city 1 and then he did a transition 1 to 2 plus pi 3 n minus 1 and then he did a transition sorry 3 to 2 ok that is the associated thing, if you carefully see I can like that I can also calculate pi 1 n and I can also calculate pi 3 n.

If you carefully see this is just a matrix multiplication of associated vector, if I just construct associated vector which is actually pi 1 pi 2 pi 3 with time index let us say n minus 1 if I put, then it is an overall pi n minus 1 ok. So, that vector if I put pi n then this will be index will be n. So, if you carefully see it is just associated with this vector multiplied by this transition probability matrix. So, pi n minus 1 vector multiplied by this P vector must give me my pi n this relationship in a very simple way is always true.

Now, of course, if the state variable has more value then definitely the transition matrix also will have means matching values, and this multiplication of this vector to vector and matrix multiplication will always match the dimension. And I will accordingly get, how many equations I get? As many number of variables are there that many equations I will be getting.

So, this gives me the number of equations if I know the transition matrix, if I know the previous time values then I can always calculate from there the next time will be the value. So, the state probability evolution I will be able to capture from the previous one.

So, if I know at 0 where is that? This means, that if I say ok I drop him in city 1, then the associated pi matrix pi 0 will be. So, I have dropped him in city 1, the others will be 0 and from there with the transition probability matrix I will be able to capture what will be the associated probability on the second day. And then from the second day again I can capture what will be the associated probability the next day and so on, this I will be able to capture.

Now, even if the transition probability matrix is P ok which is homogeneous, I have no guarantee that over time this probability will not keep on varying ok. If I have an additional condition that P means such a matrix that this to this transition they get a stationary value; what does that mean?

That means if I go up to time infinity then I will see that actually wherever in the initial condition or whatever initial condition I put or wherever I drop that random traveler at

0th day enough time if I go, then the probability associated state probabilities will become convergent.

That means, it will take a similar kind of value, if this happens then that particular random associated process or the Markov chain is termed a stationary Markov chain. Not only stationary in time also if it is getting similar things over time then we also call that it is probably an ergodic Markov chain. Will not go into the details of that description because that requires a full again full-fledged queuing theory course.

But right now, we can just understand this property that if this happens; that means, not only the transition probability but the state probabilities also are not varying, they are being constant over time.

So, what does that mean? If I put limit n tends to infinity then this pi n minus 1 or even pi n these are all going towards a stationary value of pi ok. So, immediately what I can do? In this equation, I can take that limit. So, if I take limit n tends to infinity on the left-hand side and if I take limit n tends to infinity on the right-hand side.

So, basically, both of them will go to pi and P does not depend on n. So, therefore, I will get an equation which is called pi equal to pi P, which is a famous stationary discrete-time Markov chain queuing equation. You know what has happened? Now, I have got see the state probability.

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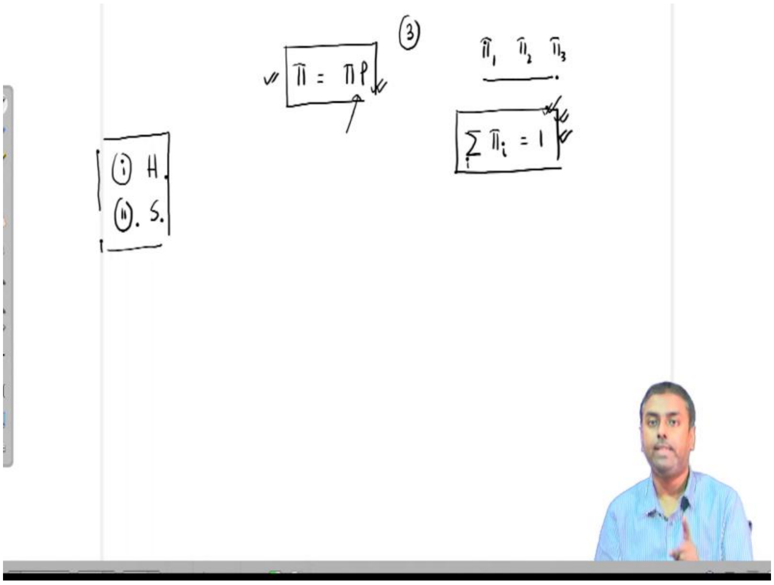

If you are you do not know about them, so if I just; so if I just write this equation that newly derived equation pi equals to pi P, what is this equation actually? So, if you see it is actually a set of linear equations of pi, pi is my unknown state probabilities I do not know I want to solve them. So basically, I get a state probability-associated equation. How many equations do we get?

We actually get if there are in the state, suppose in our random travelers experiment we had three things. So, I will be getting 3 equations involving 3 variables, pi 1 pi 2, and pi 3. So, I will get 3 equations from here now from these 3 equations there are 3 variables, will not be able to completely solve.

Because we do not have any constant things. So, we will get a nontrivial of these things at max what we can get we can get an interrelationship pi 1 to pi 2 pi 1 2 pi 3. So, in this relationship, we need another equation to solve the whole thing. So, that extra equation will be we also know that the P must be found somewhere. So, this summation pi i is equal to 1 overall i.

So, this last equation will get interrelationships from this particular matrix equation will be getting interrelationships between each of pi 1 to pi 2 and pi 3, and then this normalization because all these pi summations should be 1 because of its probability. So, their summation should be 1, I must be able to find that particular random traveler somewhere.

So, that summation will be 1. So, just by doing this, we will be able to solve the whole Markov process. So, basically, a discrete-time Markov chain which now we understand is it is means as long as we take two assumptions one is the homogeneous assumption and the second one is the stationarity assumption.

So, basically, the transition probability matrix does not change over time, and then the last thing that we have discussed is the stationarity or the stabilizability. So, that also is taken to be true; that means, if I go observe the system for time tends to infinity then the state probability also does not change over time, it has some probability and it stabilizes to that ok.

So, it is stationary. So, if we take the homogeneity and stationarity assumption that will be taking, there are some conditions we will right now probably we would discuss later on we can discuss them. So, with what condition they happen, we will discuss that later on. But if these two things happen and in our system, it's guaranteed that these two things will be happening.

So, then we can describe the whole system we are interested in solving the state probability for a queuing. How many times do I find the queue or what is the probability associated probability that I found the system without any customers? So; that means, in 0 state with 1 customer that is in one state, it 2 customer that is in two states. So, all these associated probabilities I need to find out ok.

So, these probabilities if I wish to find out I get a setup from this particular equation set, I get a set of linear equations. So, all I have to do is along with that I will have this normalization equation. So, I solve this linear set of equations and I solve the state probabilities. So, it is as simple as that.

So, therefore, means I can guarantee that these two conditions are satisfied homogeneity and stationarity or ergodicity also for the time being. If these 2 are satisfied we will be able to characterize the system by a set of linear equations as many states are there many sets of linear equations will be formed. We just solve the solution of linear equations is very easy and we can we can just solve that.

So, therefore, the only thing you will have to do is over here the thing that has to be supplied is the transition probability matrix. So, understanding the system how do you construct this transition probability matrix that will be your task ok. So, that is the only thing that you will have to do. Once you have done that you are almost there you can solve the associated things, but remember over here we still have not described anything that is Markovian ok?

So, this without going into the description means Markovian was taken in the underlying process we have not actually discussed that, we have very carefully avoided that description, but actually that must be taken into account, this cannot happen if the system is not Markovian.

Because then the whole system the whole history has to be taken over here what we are doing the next state where it will go. So, where the Markovian assumption was taken?

We have written that particular matrix equation that pi n will be only dependent on the previous state and the transition probability, this is the Markovian assumption.

That means it only depends on the previous state and it does not depend on anything beyond that. So, that was the associated Markovian assumption that we have already inherently taken over here ok? So, what we will try to do? We will try to see the implication of that one more time and then from there we will try to build up the continuous-time Markov chain in the next class.

Thank you.