# Communication Networks
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## Module - 05
## Queuing Theory (Contd.)
## Lecture - 24
## DTMC to CTMC

Ok. So far we have discussed the Discrete Time Markov Chain or Markov process. So, this is something we have means we have a fair bit of idea right now. So, what we will try to do today means taking those queues from the discrete-time Markov chain and all the equations that we have derived for the discrete-time Markov Chain.

So, we will take that forward and we will actually go towards a Continuous Time Markov Chain, which is probably the one we are requiring for this trunk switch analysis. So, that is why we have started all these things. Of course, you will see later on in the course that discrete time Markov chain has a huge application.

So, when we will be doing some means packet switching protocol analysis for their delay throughput and all those things. We will see that DTMC will come in handy, but right now, we will be probably going toward the CTMC and this will cover the entire Markov process ok. So, as we are discussing what we have understood so far, is something like this, that.

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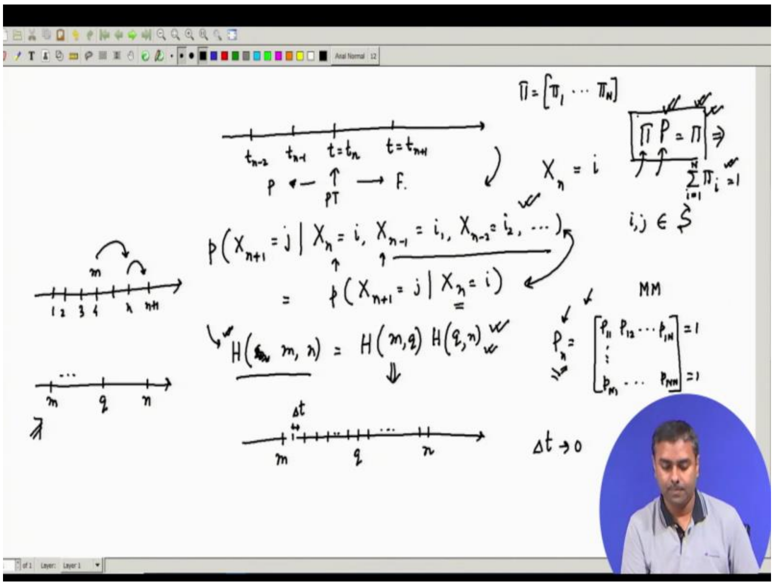

If in the time frame, we have a particular time instance, let us call that as or t 1 ok. And this we are calling as present time ok, and at that time we have some observation of the state ok. So, let us call that some X 1 ok. So, X 1 denotes that at t 1, this is happening ok.

So, this is our observation at the present time, we want to predict the future. So, somewhere in let us say t 2, I want to predict what will be happening right over here ok. So, that will be some X 2 and what we have is we have some knowledge of all the previous instances ok.

So, let us say at t; so, let us say t equal to t n minus 1 t n minus 2, I have all these observations. And observation means the state actually; that means, if I characterize the random variable at time instance let us say n this will be taking some i or j, where i j these are all i j these are all part of the state variables. So, that might be just count like how many customers are there in the system or it might be something like how many customers are there in the queue, the way you define your states ok.

So, now if I try to calculate the probability that my X n plus 1 is equal to let us say j, given that I have knowledge about all the previous things. So, let us say X n equals to some i, X n minus 1 equals some i 1, all these i 1 i 2 i 3 that will be describing they are all part of this state. So, that means it is if a count of customers or some count it might be 2 3 4 whatever that number, and this is the associated random this one random variable which takes value i 1 ok.

So, X n minus 2 that is i 2, and so on. So, what we have told, in a Markov process really does not depend on this past history, it only depends on the present. So, therefore this is equal to the probability. So, all other conditions I can take out and I can say just it depends on what is the current status.

So, these two probabilities are identical that is when the process is called Markovian. So, Markovian means it just depends on the previous state and it does not depend on any other state prior to that. So, this is one concept that we have already discussed. So, the consequence of Markov is the Markovian process, we have also started characterizing a probability transition matrix, if you remember. So, the probability transition matrix is the overall summary of state transition probabilities.

So, if there are means it is a matrix we have already talked about. So, which is having all values probability 1 1 1 2 dot dot dot up to if there are n state and so on; where p i j is actually in one step going from a particular state i to j ok. So, this associated probability, we have already given the example of the random traveler and we have seen that if cities are denoted by random observation of him being in one of the city i j or something.

So, p i j is the associated probability that one day he will take a bus or whatever transportation from city i to j, what is the probability associated probability of that? So, we have told that this is the Markov matrix we have also discussed that. So, the Markov matrix has a typical property that this row sum is 1. So, this is something we have already discussed.

So, this probability transition matrix might be dependent on the time instance you are talking about. So, it might be dependent on this n instance. We have also talked about a homogeneous Markov process, where this probability transition matrix is independent of this time instance. So, at any instance, this will remain the same, and the transition matrix will look similar.

So, this is something we have also talked about. We have also talked about a multi-step transition, some s or let us say m to n ok what does this mean? So, now earlier this transition matrix was at a particular time instance. So, if my times are here marked. So, 1 2 3 4 like this n n plus 1 something like that; so, P n actually means that from n to n plus 1 what will be the associated probability transition matrix, but this is a generalized multi-step transition.

So, it is actually starting from some value m and up to n. What is this multi-step transition? There might be a separation between m and n. So, we have also started characterizing and then putting this Markov property we could show that if we take some intermediate steps. So, therefore suppose in the timeline you start with m and then dot dot dot it goes intermediate q then goes to n.

So, if I have this then this probability transition matrix only for Markov gets separated. So, this probability transition matrix becomes m to q and matrix multiplication q to n ok. So, for any value of intermediate q, this is always true that is also something we have proven ok. So, this is called this is the Markovian property embedded in it.

If we have also shown that if it is not Markovian this will not be happening. This transition matrix will not be separated out; it will not be a matrix multiplication. But, fortunately for the Markovian process, this is what happens. So, that is a quick recapitulation of what we have, so far understood from our discussion.

And of course, we have also derived that in steady state, if it is a homogeneous stationary Markov process. So, therefore, what happens to this pi P equals pi; so, this equation we have also derived, where P is the transition matrix which is homogeneous and does not depend on time. Pi is the state probability vector and that is also because stationarity also does not depend on time.

If that is the case then this matrix equation gets satisfied and for pi, pi is a vector we have told being in state 1 up to state n, if there are n state. Summation of this pi i overall i from 1 to n this is 1. So, this gives me the n linear equation and then this gives me the normalization equation, from those n plus 1 equations we will be able to solve the corresponding state probability if I know I can characterize the transition matrix.

So, these are the things we have discussed so far. And that is actually the fundamental equation of DTMC. So, for any DTMC you have to solve you have to first see whether it is homogeneous and it is stationary. So, given it is homogeneous stationary and of course, ergodic also it has to be. So, if those things are satisfied then you just go about solving them by this equation.

So, you only have to do is you have to characterize the transition probability matrix, because it's homogeneous it will be always the same. So, just characterize the transition probability matrix from there you will be getting a set of linear equations you solve them you get your pi matrix. So, that is how will be generally solving any DTMC method ok.

So, now we have all this background of DTMC, can we now go and relax this condition that the time is slotted it can only choose some particular instances when we are talking about the event occurrence. So, like for that random traveler's case it was just let us say night 8 O’ o'clock he only pick the bus. So, at that time only there will be transition otherwise there is no transition.

The whole day it does not really give you any transition. So, from that can we come to a scenario where at any time transition can happen; our queuing requirement is in a way or trunk switch requirement is in a way that because, in the trunk switch the arrivals which are happening or the service which will be ended, those are actually marked as events in our means random process.

So, those events can occur at any time, we do not really have a particular time that ok only this time the arrival might happen, the customers are allowed to arrive or the packets are allowed to arrive to the switch only at this time, call can be initiated only at this time. So, that is something that restriction is in a realistic field that is not there.

So, there therefore our analysis must go towards that particular criteria that the arrival can happen at any instance ok. So, now if we have understood that part let us try to see if I relax try to relax this time instance, how do I actually relax it. So, this derivation means what we are trying to do will be something like this; we have this let us say I have a timeline right.

For this derivation I have told some time instance m it s a discrete instance, intermediate some time instance q and then there is some time instance n ok, for which this is true ok. So, this transition matrix going from m to q and transition matrix from q to n and the overall transition matrix from m to n they are interrelated like this in a Markovian process, this is something we have.

If you go to the continuous one it is still Markovian the Markovian properties; that means, this property still holds. Now, only the times are not discrete it will just be continuous. So, at any time instance all the future as many time you can take so, they will be all summarized into the present only. So, it will be only dependent on present instance. So, these things are still true. We want to just get to a equivalence of this one that is all we will have to do ok.

So, let us try to see how do we do that? That is that will be our first target. So, the target is very simple I have slotted time right and between slots let us say they are uniform slots, is a assumption fairly I can take ok. So, we can always take uniform slot that will still be a Markovian process of course. So, if they are uniformly spaced; that means, in time their separation is always some delta t ok.

Now, what will happen if I start putting this delta t tends to 0 this limit? Immediately all these discrete events will come very close by and then almost it will become like when I make by delta t infinitesimally small, then I will be accounting for the entire real time axis every point of it.

So, therefore my time will automatically become continuous, that is the trick will be trying to put, but before putting that trick let us try to derive something which can be meaningfully taken into the continuous time. So, I will be just deriving it from here only ok?

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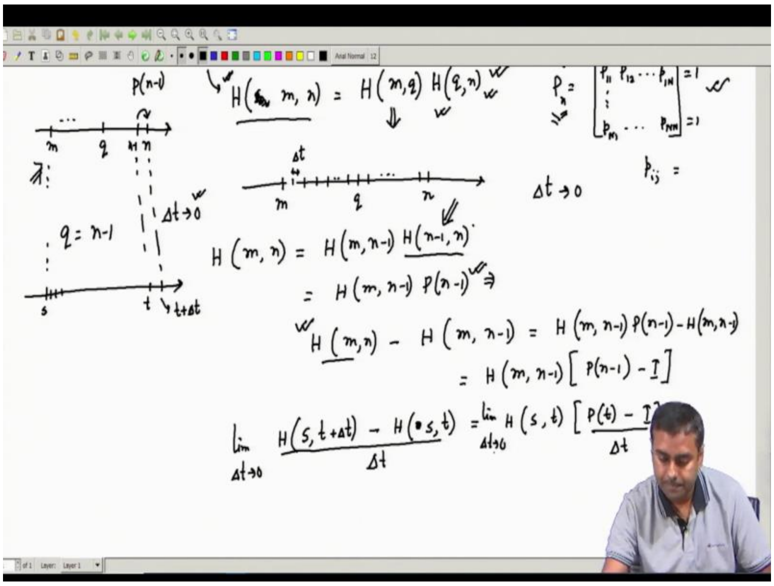

So, let us try to see this H m n I want to put, any q I can put. So, therefore, I can always put q equal to n minus 1, this is something I can always put that it is valid for any q. So, therefore, for q equal to n minus 1 also it is valid. So, then I can put this to be H m q is n minus 1 into H n minus 1 n right.

So, I can write it this way; what is this H n minus 1 n? So, if I try to see it carefully this is nothing but going from n minus 1 to n. So, this is the transition one step transition matrix, because I am just going from previous step to next step.

So, this must be our P by definition at n minus 1. So, I can write this as p n minus 1. So, this is H m n minus 1 P n minus 1 again I can write this ok. So, this is true still we are in discrete domain. So, we still have not put that limit t tends to 0. So, that we have not done.

Whenever it goes to continuous this one; so, I my target will be like the previous one when we have started deriving the Poisson process, so, my target will be formulating a differential equation. So, for formulating a differential equation what will be targeting, we will first formulate a difference ok. So, that we take the difference and then put the differential over that.

So, put divided by delta t and limit delta t tends to 0. So, that will formulate the differential equation. So, let us try to put a difference over here, how do I put a difference? So, my difference will be now this H m n minus H m n minus 1 ok. So, this difference I am trying to calculate evaluate, what will be this, now H m n I can actually replace by this one.

So, I can just write that. So, it is just algebraic manipulation H m n minus 1 P n minus 1 minus H m n minus 1 ok. I can take H m n minus 1 common, so, this gives me P n minus 1 minus it is a matrix. So, matrix if I take the identity matrix will remain over here ok. All the matrixes are n cross n like this ok.

So, this is if it is having n state; if it is having infinite state then it will be infinite cross infinite matrix ok. So, it will be infinite dimensional matrix. If the states are finite, then it will be finite like our random travelers experience, there the state where; states were finite because there were only 3 cities.

So, it is n equal to 3, but if I talk about our queuing with any number of customers who can be there inside the system then it goes up to infinity. It is countable infinity, but it goes to infinity. So, therefore, all the matrices will be associative matrixes will be of infinite dimension, but we are right now not bothered about that. So, this is what is happening; now I have got this ok.

Now, I will start putting by discrete to continuous time this one. So, m is now I will put delta t tends to 0 ok. So, this spacing will be very small now ok. m let us denote as s in the continuous domain of course, they are same time ok, I am just denoting it you know different this one.

My q is n minus 1 so, that is corresponding to this one again and n is this one. Let us call this n minus1 to be t, then what will be this one this should be t plus delta t. And of course, I have this limit ok? So, now replace the new notations and reformulate this equation, let us try to see what happens.

So, I can now write H in place of m I will be putting s, n I will be putting t plus delta t minus H m is replaced by s and n minus 1 replaced by t which is nothing but H s t ok. So, this is something we can talk about as P t minus I ok. Now, we have got the difference as you can see t plus delta t minus t. So, divide by delta t on both sides and take limit ok. What is this? This is the differential with respect to t of this parameter H s t.

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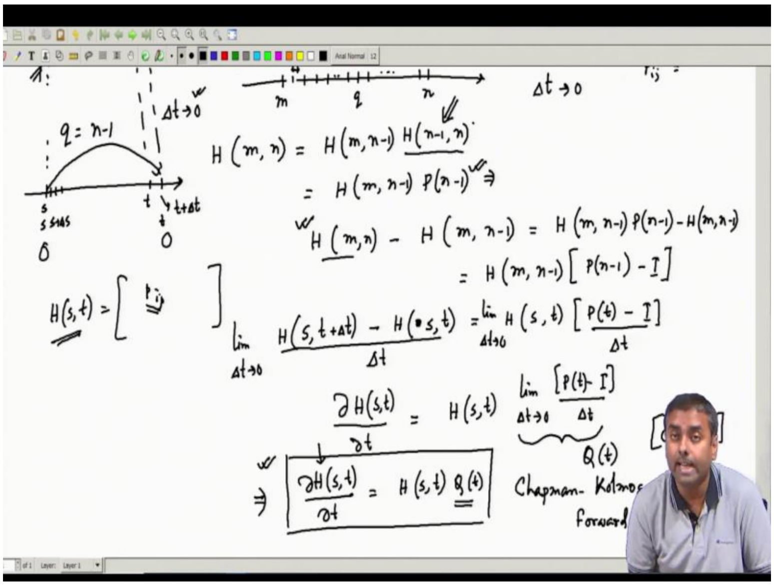

So, this is del we know that H s t does not depend on this delta t. So, that can come out and we are left with something of this nature, this entire thing now we give a new term it is dependent on t. So, therefore we put it as a new matrix. Of course, it is also a matrix because P is a matrix, I is a matrix. So, of course, it will be a matrix. So, it will have the same dimension as P and I which is n cross n. So, that is from there we get a new equation, to describe the corresponding continuous time Markov process.

This is famously known as the Chapman Kolmogorov Forward or simply CK Forward equation. There is a corresponding backward equation also that is very easy to see, this is with respect to t the other one I can do, over here I can actually start means describing it with s and s plus delta s something like that ok. And this last one becomes t and then actually segregate it into because over here q I can put anything.

So, in place of n minus 1 I can put m plus 1 and then do the whole thing I will get another equation which will be differential with respect to s ok, and that will be termed as Chapman Kolmogorov backward equation. Because, it is in the backward part it is in the forward part ok, but we do not have to bother about that our this only single equation will suffice for our description ok.

So, what we have done eventually, from the description for the multi stage transition of a simple discrete time Markov chain, we could now put that delta t tends to 0 and we could get the corresponding differential equation that describes the state transition. So, this is the state transition now, H s t says that it was in s means in s it was in state i.

Now, in t it is going to state j. So, that is the corresponding means H s t also will have corresponding matrix with i j index. So, this any i j let us call that p i j. So, that will be the probability that it was in state i at time s, now a it is a continuous variable time s and it is going to state j in time t ok.

So, that it tells and the corresponding guideline or corresponding guiding equation is this one, the differential equation associated with it; where I have got a strange matrix called q t this is new, this was not there. Earlier I had a P matrix that was the transition matrix ok, but now, I have got a new matrix called q matrix later on you will see this is called the rate matrix, or rate of transition matrix.

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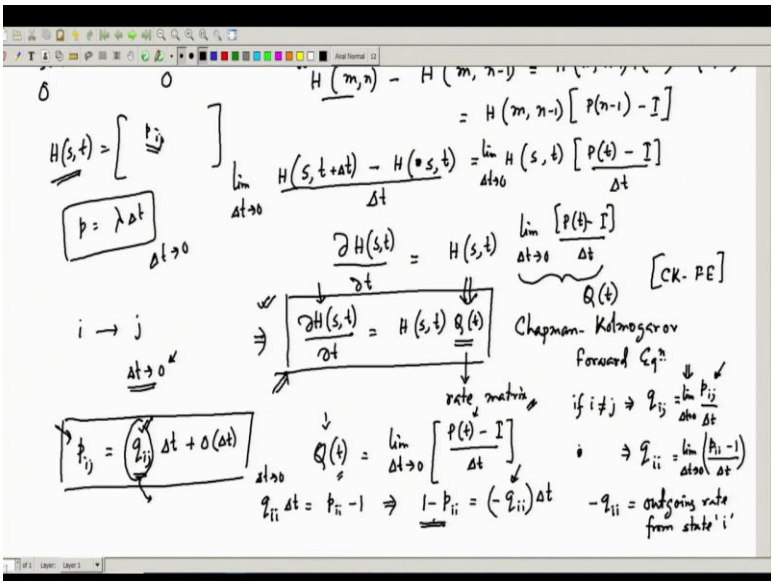


We will see how that. This is typical to CTMC it only happens in CTMC. So, what will now try to do, before we go any further let us try to analyze this equation completely; that means, specifically H s t we already know. This unknown parameter that we have to understand, what is this rate matrix this has to be characterized properly, then only we can go forward towards developing the CTMC theory ok.

So, let us try to see what is q t. So, q t by definition is this so, let us put q t limit delta t tends to 0 ok this is q t. So, q t will have corresponding let us say q i j I am just dropping this t ok. So, if I have a homogeneous process then I can drop this t because it will mean that p t will not change with time. So, therefore, q t also will not change with time any time instance it will have the same value.

So, I will just for the time being I will drop this t, I will assume that it is a homogeneous process; that means, from time to time the transition probability state transition probabilities do not change. Correspondingly, we will also see that because if this does not change from time to time for any t if it has the same value, therefore, any t q t also will have to have the same value. We only want to see the relationship between these two ok? So, the corresponding q i j what will be the value of that?

So, q i j will be something like this, corresponding p i j will come minus if i is not so, if i is not equal to j that means, it is not the diagonal part of the matrix or diagonal elements of the matrix. So, i is not equal to j, in that case, it will be i will not have any value it will be 0. So, it will be p i j divided by delta t and of course, this limit will be the limit delta t tends to 0.

And if I equals to j that means, q i i what that will be, that will be limit delta t tends to 0 P i i minus 1 divided by delta t, because then identity matrix because its diagonal 1 identity matrix will have term 1. So, these are the two definitions that we can see over here ok, of our description of q t. Now, let us try to see what is happening over here.

What we know is the associated probability of this p i j ok. So, p i j I know is the associated probability state transition probability from state i to state j ok. So, p i j I know if I just try to see what exactly it is. So, if I for the time being I just ignore this delta it tends to be 0 of course, we have to put that. So, I can see p i j is nothing but, actually q i j into delta t with the limiting condition that delta t tends to 0 ok.

So, again basically what happens is this probability becomes this two multiplication, when delta t tends to 0. If delta t is not tending to 0, there should be some error term that will be vanishing as delta t goes to 0. So, I can again take the same definition of O delta t over here and add that. So, I can account for this delta t tends to 0, because when delta t is not tending to 0, it will be something else.

This O delta t will account for that, but when delta t tends to 0, I also know this O delta t that has a definition that O delta t by delta t when delta t tends to 0 that goes to 0. So, therefore, definitely, O delta t will be going to 0. So, I can talk about this relationship p i j will become equal to q i j into delta t.

Can you identify this relationship from our previous discussion when we have done the Poisson derivation? What was happening over there, if you remember care means carefully if you see this, earlier I also had a probability and then this was actually rate, the lambda; earlier when Poisson we were characterizing lambda into delta t plus O delta t was coming.

So, therefore, this must be associated with a rate, whenever I have this kind of equation that probability is equal to something into delta t ok, where delta t tends to 0. So, then this happens to be the rate that was our understanding, we have derived this right. We have talked about this, one event occurring and then counted that and then tried to equate that we could get that.

So, we got that probability was some rate into delta t. So, that was something we have got when delta t tends to 0. So, this is something this relationship we have got. So, therefore, from that particular inference, we can also derive inference over here and there I can say that q i j is nothing but now the rate of transition from state i to j, because this is associated probability of transition from i to j. So, that must be the rate. So, q i j happens to be the rate of transition from state i to j.

So, now we have got a good understanding of this one. So, that should be the rate. So, that is why this was called rate matrix, but what happens to this q ii. Let us now try to see what is q ii; again we will do the same thing q ii into delta t what is that, that is actually p ii minus 1 ok? Again there is an associated probability, but probably still not fully probability if i just tilt it a little bit and put a minus sign over here.

Of course, this is only happening when delta t tends to 0. So, put a minus sign over here. So, this will become the right-hand side will become one minus p ii this is nothing but, minus q ii into delta t. What is 1 minus p ii? p ii is the probability that it will remain in i 1 minus p ii is all outgoing probability. So, it is actually leaving a state i.

So, this is the associated probability that itself it is actually probability that at time t or whatever time instance. It is leaving state I, it was in I and it is leaving state I it is the associated probability. So, therefore minus q ii must be the associated rate according to our new understanding. So, minus q ii says that it is the rate, it is the outgoing rate from state i, it is the outgoing rate from state i ok. So, now we have very nicely characterized the whole matrix if you now ask.

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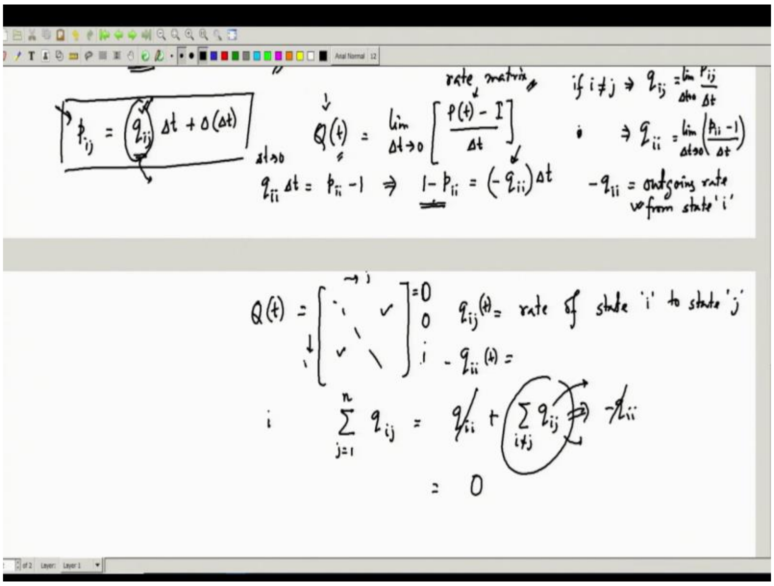

So, if you now ask, we have characterized the whole q matrix I can bring back the t term I can also drop that t term. So, if t is there then p i i q i i j also will have t. So, this matrix has some diagonal terms and some off-diagonal terms. So, off-diagonal terms q I j is talking about the rate of state or rate of transition from state i to state j ok at time t, if I have time also.

And this q i i t minus of that is actually the rate of going out. So, therefore I have got the whole transition matrix prepared for me. Now, from this transition matrix, all I will have to do is I have to see, if this has a typical characteristic like the Markov matrix. The transition probability matrix was there, now I have a rate matrix for the continuous time case.

Now, do I have any Markov matrix that has a criterion that this row sum was 1, over here do I have some criteria like that ok? So, that is something I will have to check. So, let us do the row sum, what is for a particular let us say we do sum over. So, this side it is i this side it is j. So, for a particular I let us try to do the sum ok. So, let us try to do q I j summation over all j 1 to let us say it has state n. This is the sum I want to do, what is this?

Now, only ith term has some other meaning. So, I will separate out q I i plus the rest of the things I will keep q i j where i is not equal to j. So, overall i, I am summing ok. What is this? This is each individual one's rate of going from i to j, if i sum over all other j this is the overall outgoing probability. What is the total outgoing probability that is minus q ii.

So, therefore, this can be replaced by minus q ii. So, what is the answer then,, this gets canceled so, I get 0. So, therefore, q t has another typical criterion that all the row sums are 0, that is a typical q matrix ok. So, this is how we get a q matrix over here. So, with this q matrix, now I can characterize this as the understanding of the whole q matrix.

So, once I have this q matrix now I can go back to the Chapman Kolmogorov forward equation. So, I have a full understanding of this equation. So, all I will have to do is I will have to solve this equation right. So, that will be our next target. In the next class, we will try to see how we solve this equation and how we apply this to our queuing analysis.

Thank you.