Analog Integrated Circuit Design
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Lecture - 6
Negative Feedback Amplifier with Ideal Delay- Small Delays

Hello, and welcome to the sixth lecture of analog integrated circuit design. At the end of the previous class, we were looking at the effect of delay in the negative feedback system, we were looking at it qualitatively. In this lecture we will do so quantitatively, this one and the next few lectures.

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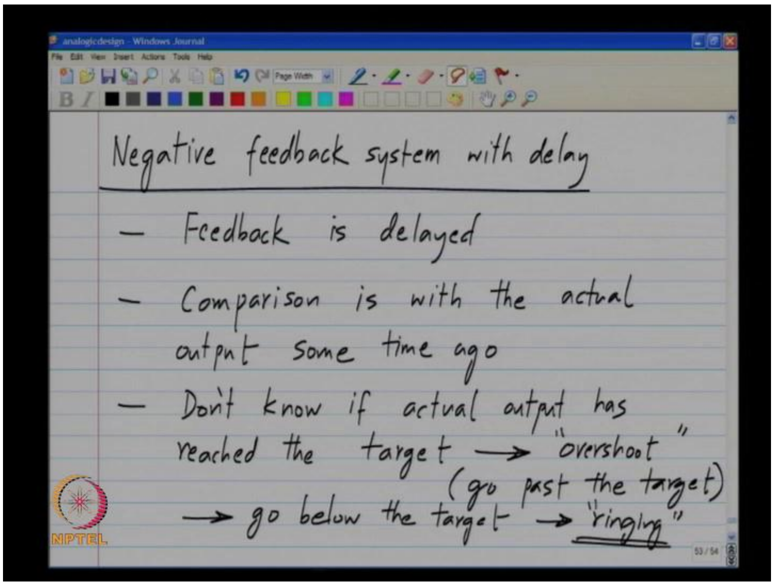

When we were looking at the negative feedback system with delay, basically the feedback is delayed. That means that, the comparison with the input is with the actual output not at this instant, but at some time ago. So, this the disadvantage of this, is that we do not know if the desired output, if the actual output has reached the desired target or if it is yet to reach, because of the delay, because of this, there is the possibility that the output also goes beyond the value.

So, that means, there is an overshoot. Sometimes this, the overshoot can persist for some time and results in a phenomenon known as ringing. That is you go past the desired target and you will realize it only later. Then you will start coming down, then you will go below the desired target and go back up and so on. This can happen repeatedly and this is undesirable. So, what we need to do now is to, analyze the negative feedback system with delay. And see the values of delays that are tolerable, because we can easily imagine that, if the delays are very small everything will be fine. We have to put a quantitative limit to the delay that is allowed in the negative feedback system. So, that we can design our negative feedback systems to behave properly.

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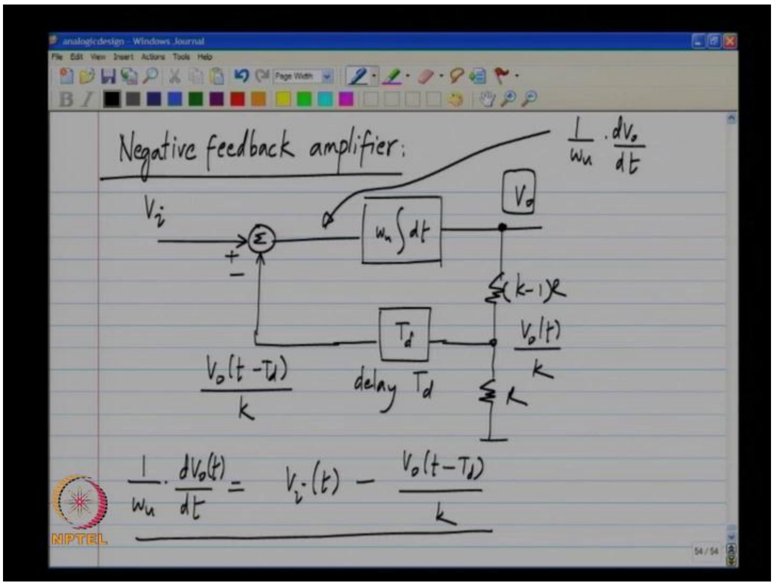

This is our negative feedback amplifier, the output voltage is divided by k and compared with the input V i, this is what we have analyzed so far. We have seen that, if the input is the constant and the output if it starts from any other value will reach k times the input value. Now a delay can appear in any part of the system, it can appear in the integrator due to the implementation of the integrator or it can appear in the feedback, it does not matter where it is.

The delay is the same. So, for now I am going to assume that the delay appears in the feedback path. That is, instead of the model, I am going to use a model with an additional delay. So, this additional delay is t d. So, now, we have to analyze the system just like before. We are going to write the differential equation for this system and then analyze the system. We see that here, we get V naught by k and, because we have delays I will make time-dependence explicit.

So, what appears at the feedback point is delayed V naught divided by k. Now, the things are very simple if the output is V naught, the input to the integrator has to be 1 over omega u, the time derivative of V naught. So, it is very easy to write the differential equation. The input of the integrator equals, the input minus the delayed output divided by k. This is the differential equation that we have to solve.

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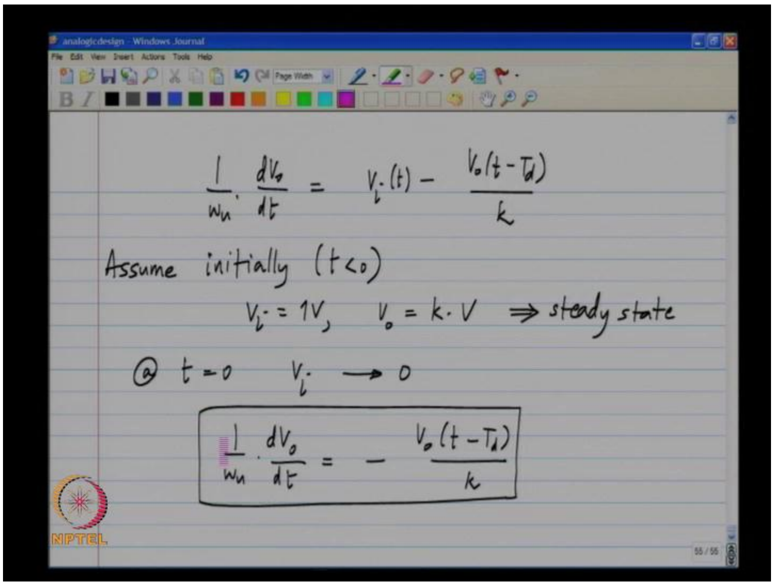

Now, let me put down the differential equation again. Previously when we solved the equation without delay, we assumed that the output initially at zero. And saw, what happens when an input is applied to it. The input is initially zero, the output is also zero and then the input steps to say one volt then, see what happens to the output.

Now, just for simplicity I will assume the opposite for now. I will assume that, the input is one volt and I will also assume that the output has reached the steady state of k volts. That is, the output corresponding to one volt. The feedback quantity here is one volt and here also it is one volt. So, the input of integrated is zero and the output has reached a steady state. I will assume that, that is what the case was before t equal to 0. And at t equal to zero, V i steps to 0.

So, the output is still at k volts and we want to see how it comes down. And the reason to do this is, the reason to do this is rather simple. That I would like to simply eliminate this term for the time period that I am solving. This is the steady case, which has been reached before t equal to 0. Now, at t equal to 0, V i, goes to 0. And we want to solve differential equations for t greater than 0. So, because V i steps to 0, the equation is simplified. And once things become simpler, it becomes easier to visualize the solution.

So, first before we go try to solve this, let us see, what kind of a solution we can expect? So, before we had solved an equation, without this, t d, that is 1 over omega u d V naught by t d equal minus V naught of d by k. The solution to that was an exponential. So, now, we can see what is the kind of the solution, this kind of an equation permits. So, here we will not be discussing, solving differential equations in general, but we would like to find out the solutions of this particular differential equation.

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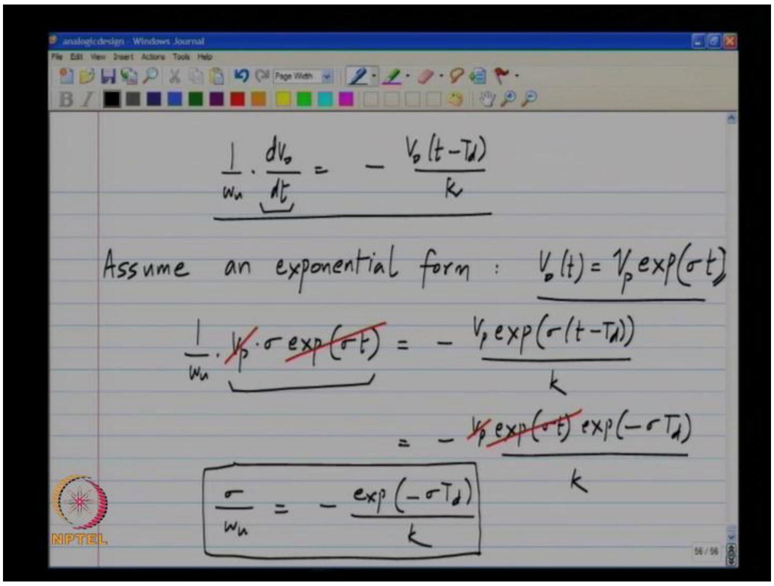

So, what is this equation saying? Now, without delay the equation would have been.

So, to solve this type of stuff, of course, you can use your knowledge of solutions of differential equations to solve it, but it also helps to have a picture of what the equation is saying, so that you can guess at the kind of solution. First we will do it for the case without delay. So, this I can manipulate it to be, now also you notice that this part omega you divided by k is a scaling factor. And the solution of the differential equation will remain the same whether you have the scaling factor or not, because it is linier differential equation.

So, what the, here I have rewritten the equation without the scaling factor and what the equation is really saying is that, the time derivative of the function is the same as the function. I have omitted the minus sign here, but that again can be absorbed into the scaling factor. So, what the equation is saying is that, the time derivative is the same as the function itself. Now, even if I dint know, how to solve the differential equations, you could have done this. Like for instance after high school, where you learn about differential coefficients, when you learn about differentiation. You have a table of functions and its derivatives.

So, you look at the table and you see which of the entries of the table, as the same column, same on the left column and the right column. Which of the entries of the table has the same function on the left column and the right column and the answer is obvious, it is of the form of the exponential. Then you have to add some scaling factors, to make the complete equation be satisfied. Now you can look at our case, that is the differential equation including a delay in a similar way and try to find the solution. So, although this looks different from before, it is more or less the same. I will rearrange the terms, as follows and I will rewrite it, without the scaling factor. So, what this equation is saying, is that, the derivative of the function V naught is the same as the delayed function.

So, again with a little bit of knowledge of functions and the derivative, you will be able to easily see that an exponential satisfies this relationship, because an exponential that is delayed, is still an exponential with the scaling factor. So, an exponential will again satisfy this. We will have to find the right parameters of the exponential. So, that this equation is satisfied. So, that is what I am going to do now. So, to quickly summarize what we did, we have written down the differential equation for the negative feedback amplifier with a delay in the feedback path, then we have not yet solved it, but we have try to find out the nature of the solution.

The equation then reduced to the their essentials, simply says that, derivative is the same as the delayed function and from that, we can guess that the solution is of the form of exponential. So, now, will plug the exponential into this equation and see the parameters of the exponential that are required. So, that the equation is satisfied.

So, we will assume that V naught is an exponential, there will be some scaling factor. Which let me just call it V p exponential and I will call this sigma t. Assuming this I can rewrite the differential equation and I get V p sigma exponential of sigma t is d V naught by d t, it is the same as minus V p exponential of sigma t minus T d divided by k. And which in turn can be written as, minus V p exponential of sigma t, exponential of minus sigma T d divided by k. So, you very easily see that, V p exponential of sigma t cancels out and what you are left with is sigma by omega u equals minus exponential minus sigma T d divided by k.

Now, I am going to show the solution to this. It cannot be solved in an explicit form, I cannot write sigma equal to something, but we can look at the function and try to solve for sigma numerically, but before I do that, I would encourage you to pause the lecture at this point and try to solve the equation yourself, because it is only when you solve all the steps in the problem yourself, that you will be able to fully understand the solution.

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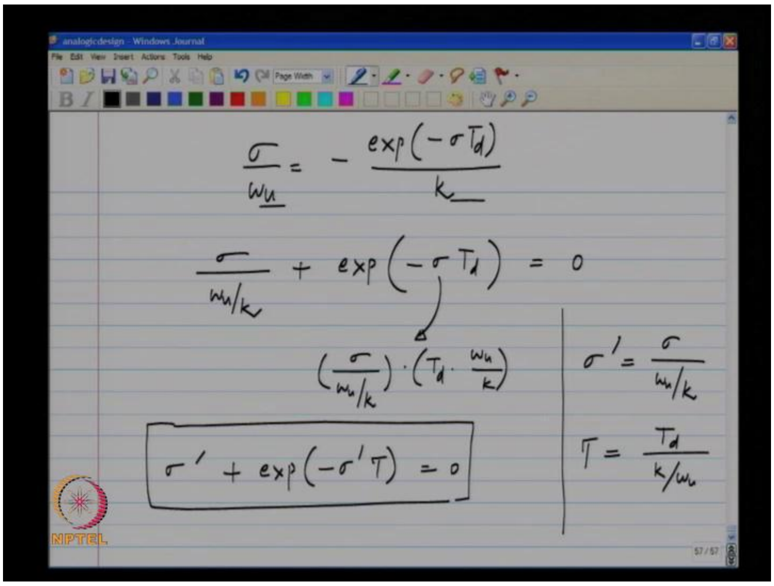

I will reconstruct it as, now I will redefine the variables. You know that, because you have sigma t in the exponential, sigma has dimensions of frequency and in this form it is normalized to omega u by k. So, let me rewrite, this, I will normalize sigma to omega u by k plus I get exponential minus sigma T d, but I am going to rewrite this as, sigma by omega u by k times T d times omega u divided by k. It is exactly the same, it is equal to sigma T d, I have just normalized everything to omega u by k. Now, the reason for this will become clearer later, but you already know that k divided by omega u is, the time constant of the system and omega u divided by k is the bandwidth of the system, when there are no delays.

So, essentially we are normalizing everything. We are normalizing the frequency to omega u by k and time to k by omega u, which is the time constant of the system. This leaves all the variables dimensions less and it becomes less messy to handle. So, I will define sigma prime to be sigma by omega u by k and I will define a normalized delay tau which is T d divided by k by omega u. Rewriting in terms of the new variables, I have sigma prime plus exponential of minus sigma prime tau to be 0.

If we know how much delay T d there is, we already know omega u and k. So, we know tau in this equation and we have to solve for sigma. Again I would encourage you to pause at this point and plot this function yourself. And try to find the zeros, of the function. Plot this function, sigma prime plus exponential minus sigma prime tau verses sigma and you will able to get an idea of what solutions there are to this system.

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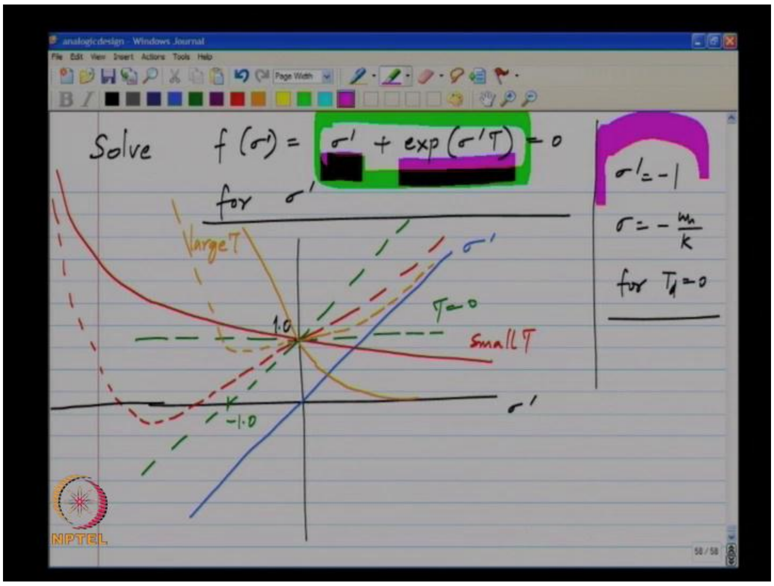

To visualize this, it will be useful to plot this function versus sigma prime, before we go there, we can substitute tau equal to 0, which is the case without delay and see what happens, if tau equal to 0. We have sigma prime plus 1 is 0, that is sigma prime to be minus one or sigma has to be minus omega u by k. This we already know to be the right answer, because when solved the system without delay, we had got exponential of minus omega u by k times t.

This is another useful practice, because as you go along, you will solve more and more complicated problems and when you setup the equation for a more complicated problem it will always pay to reduce it to the simpler case by setting the sum of variables to the simpler case. Make sure that the complicated equation that you have set up in details passing the sanity check, at least it satisfies the simpler case, then you go on solve the more complicated case.

So, now, will go ahead and plot this particular function versus sigma prime. First I plot the two parts of the function separately. Sigma prime versus sigma prime is nothing, but a slight line passing through the origin. So, this is the first part of the function. Now for the second part of the function, exponential of sigma prime tau, this depends on what tau is. Sigma prime is 0, this is always unity and if tau is very small, it will be a shallow curve like this and if tau is very large, it will be a steep curve like that.

A special case that we can also look at is tau equal to 0. So, for tau equal to 0, what do you see? Simply, we see the sum of this straight line, the blue one and the green dash one and we will get the sum to be something like that and it crosses 0 at minus one. We already know that, the solution for t d equal to 0, is sigma prime of minus one or sigma of omega u by k.

So, with reference to this, we can see what happens to the other cases. For a small value of tau, what happens is that, for positive value of sigma prime, for a large value of sigma prime, the exponential goes to 0 and this curve, the sum of the two approaches the straight line. So, we will get a curve that is something like, it keeps approaching the blues straight line and then for negative values of sigma prime, it does something like that. It goes to 0, but for very large negative values of sigma prime, the exponential takes over. The exponential for negative values will be like that. So, this comes back up and follows the exponential.

So, as you can see, there are two things to observe, one is that the roots will be both more negative than minus one. So, you will have a root here and you have a root here. There will be two of them, because for positive values of sigma prime the straight line is dominant. Finally, the some approaches the straight line and for negative values of sigma prime, the exponential is dominate and some approaches the exponential and in the middle somewhere, it reaches a minimum and the minimum can go below 0. For very small values of tau, it does go below 0. There will be two solutions, both of which are more negative than minus one.

Now for a very large value of tau what happens is that, if you look at this orange case then on the positive side it quickly approaches the straight line. Not much interesting there, but on the negative side what happens is, it does comes below, but then the exponential is rising so fast that, it goes a very quickly without ever reaching 0. So, this is what we can tell by a little bit of qualitative examination of this particular function for tau equal to 0. We get a solution of sigma equal to minus one, there is only one solution, because this straight line goes off to larger and larger negative values.

For small values of tau, there are two solutions, both of which are more negative than minus one, that can be got from this red dashed line. And for large values of tau, there will be no solution at all, that is if we assume a function of this type, the exponential and of sigma t for our solution with respect to time, there will be no solution at all. We will see what to do about that case later. We will also see, what is the value of tau, for which there will be no solution, but when there is a solution, there will be two of them, both of which are more negative than minus one.

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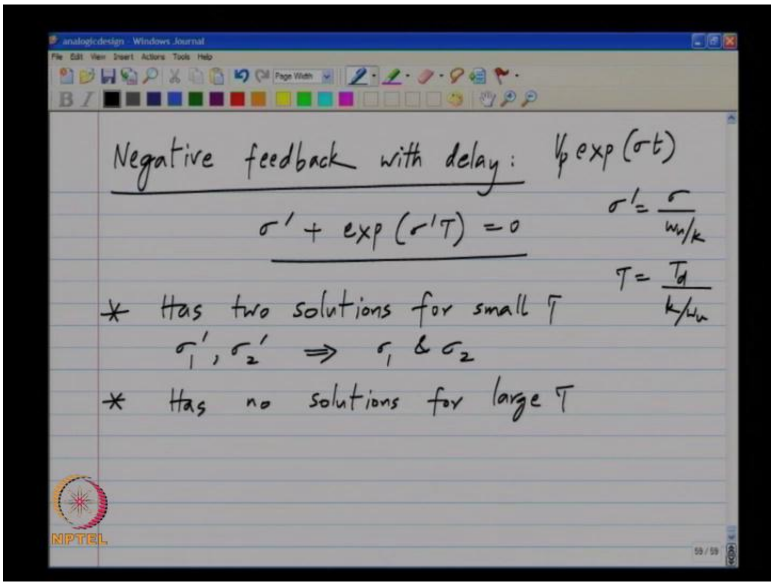

So, for the case of negative feedback with delay, what happens is that, we get an equation of this form. We assume a solution, which is exponential sigma t and that results in equation of this form. Where sigma prime is the normalized sigma and tau is the normalized delay. And this particular equation has two solutions, for small values of tau. Again tau is the delay, in the system normalized to the time constant. So, if the delay is much smaller than the time constant, then there are solutions.

Let me call this, sigma 1 prime and sigma 2 prime, these give to sigma 1 and sigma 2, when you want to do the normalization and it has no solutions for large tau. We can also calculate the largest value of tau, for which there are solutions. In other words, the value of tau, beyond which there are no solutions to this equation. Let us say, what we can do, if before we go there, let us see what we do with this sigma 1 and sigma 2. We have two solutions, sigma 1 and sigma 2 which means that.

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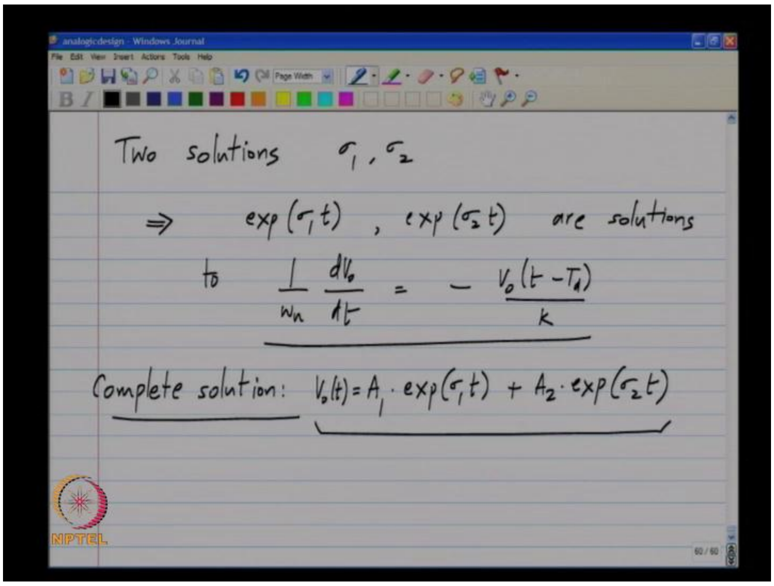

This means that, both exponential sigma 1 t and exponential sigma 2 t are solutions to our differential equation. Now, when you have two possible solutions, to a linear differential equation the actual solution could be any arbitral linear combination of the two. Complete solution will be a linear combination of these two. The particular constant A 1 and A 2 are determined from initial conditions and let us see how to do that.

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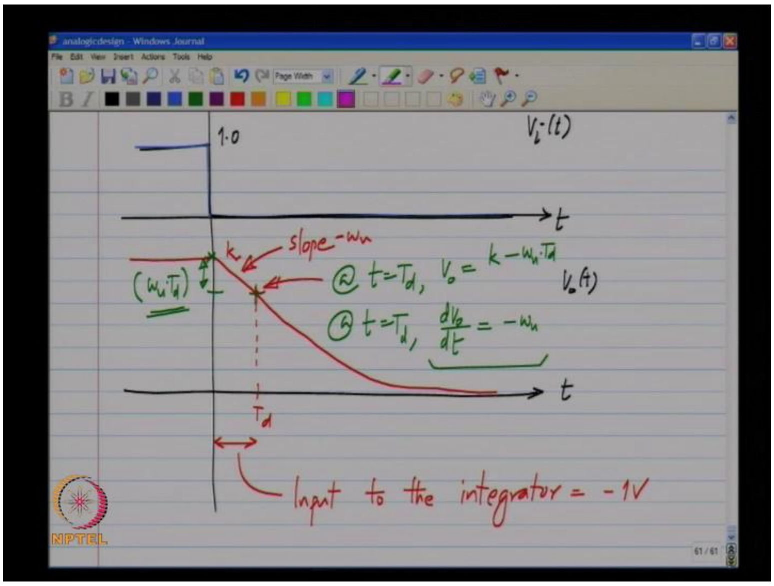

In our particular case, we have assumed that, V i, falls from one to 0. V i falls on one to 0 and V naught of t up to t equal to 0, it has the value of k. Now, if you notice this system initially V i is 1 and then it falls to 0. The input falls to 0 and the output is k volts and this is at V naught divided by k or one volt. Now, because there is a delay in the system, the input to the integrator changes, it is initially 0 volts, when V i. The feedback volts are both one volt and then V i becomes 0.

So, the feedback voltage is at one volt, but the feedback voltage cannot change for a period of t d. So, for a period of t d, any change that happening here will not appear here. So, the output starts changing when V i goes down and V naught divided by k also starts changing, but the actual feedback voltage does not change until a period of t d, until a time t d. So, what happens is until a time t d, this feedback voltage will be held constant at one volt. The input to the integrator will held constant at minus one volt.

So, I will show a period t d, here. What happens is, during this period, the input to the integrator is minus one volt. Which means that the output of the integrator goes down under the slope omega u, until a time t equals t d is reached, and after this, this is V naught of t and V naught of t minus t d will simply delayed version of this. This start changing the feedback voltage and what happens is, the slope starts reducing and finally, it can go to 0. I am only showing this qualitatively, I do not know exactly what it does, whether it is going beyond 0 or below 0 etcetera. But it is a some of exponential, it is probably going down like that.

So, now we have two initial conditions with which, we can solve the particular solution. That is, we can get the constants A 1 and A 2. So, at t equals t d the value is, because the initial value is k and you have gone down at a slope omega u for t equals t d. V naught will be k minus omega u times t d. The amount of fall during this time is omega u times t d. Also add t equals t d, the slope is continuous, a slope as being omega all along and t equal to t d, also the slope remains omega u, that is minus omega u, the slope here these minus omega you.

So, the value of the slope is minus omega u. So, we have two questions and two unknowns A 1 and A 2. From these two, we can calculate the particular solution. I will show a case where the input falls from one to 0. It could equally well go from 0 to 1. Exactly the same procedure holds, the picture will be inverted. You can still equate the value of the output voltage and the slope of the output voltage at t equal to t d, and find the particular solution.

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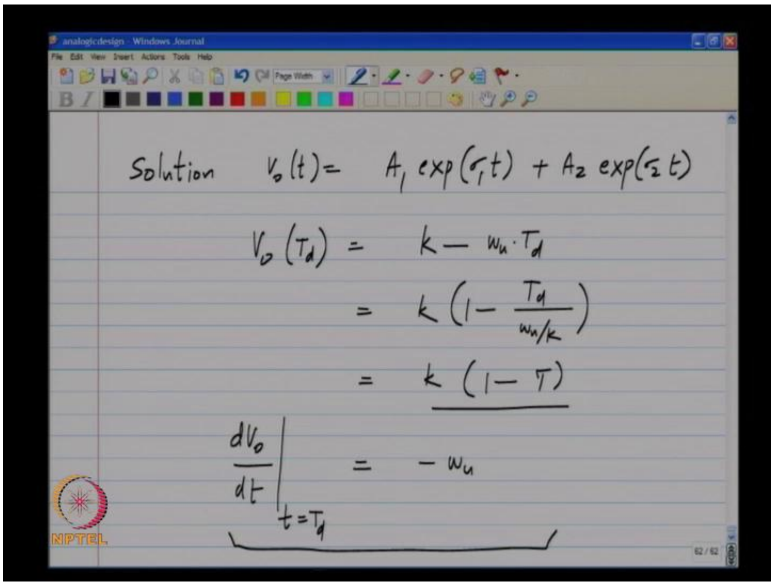

This is the general solution and we have two conditions, V naught at the time t d is k minus omega u, t d which can also be written as, k times 1 minus t d divided by omega u by k or k times 1 minus tau. Where, tau is the normalized delay. We also have, d V naught by d t, at t equals t d to be minus omega u and this is for the particular case, where the input is following from 1 to 0. If the input is going 0 to 1, it will be the opposite, but for any case you can find out the boundary conditions, apply these two conditions and find out A 1 and A 2. So, that is the way to solve any differential equation. That is the method, we have used here as well. Now, you can try to look at what kind of solutions we get, for sigma as well as for the final differential equation.

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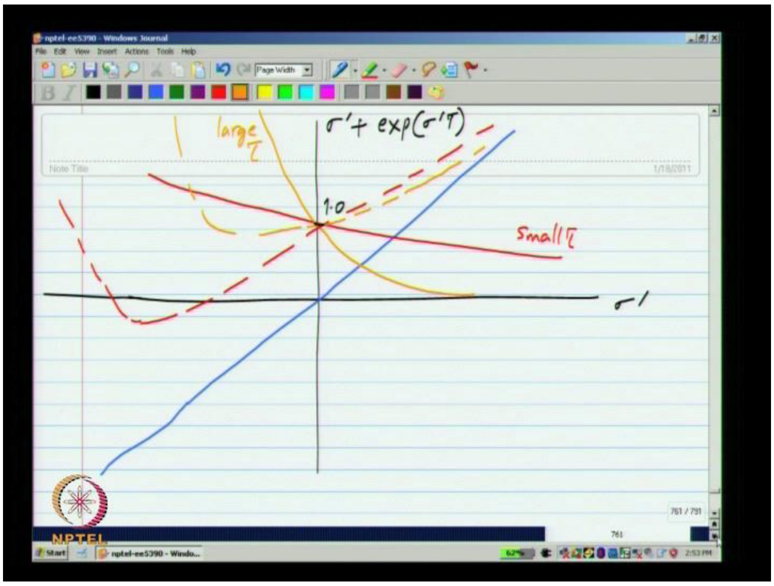

So, let me redraw the function sigma prime plus exponential sigma prime tau versus sigma prime. So, as we saw earlier, the first part of the function is a straight line passing through the origin and the second part of the function is an exponential, whose rate depends on the delay tau.

We also saw that, if the exponential is very shallow, we will get two solutions to this equation. If the exponential is very steep, you would not get any solution at all. Now, I can show your neater picture, obtained from numerical simulator. This function cannot be solved explicitly, I cannot write sigma prime equal to some analytical expression. I have to solve it numerically. You can use any of your numerical tools, to be able to solve this problem. If you want to apply initial conditions to your numerical solver, you can use the fact that, the solutions will lie beyond sigma prime equal to minus one. It will be more negative than minus one.

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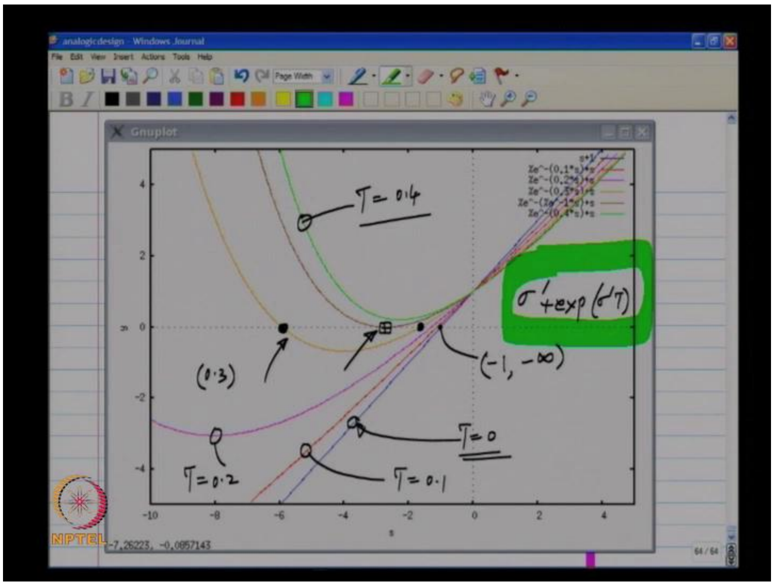

We can look at a solution here, obtained from numerical solver. Now, there are many curves here, for different values of tau. The lowest curve, the blue curve it is for tau equal to 0. That is no delay at all. You see that, its solution is minus one and there is only one solution. In fact, you can think of it is having two solutions, one is at minus one and another one is at minus infinity.

You can see as the value of tau increases, the red curve here, the red curve here is for tau equals point one and the pink one is for tau equals point two and this is for point three and so on. So, as you can see, as tau increases the red curve was almost close to the straight-line and then it will come back up, for much more negative values of sigma prime.

The pink one is already going back towards positive values and the orange one has gone back to positive values. You can clearly also see the two solutions. So, one of the solutions is here and one of the solutions is here. This solution is close to two and this solution is close to minus six, this solution is close to minus two and this solution is close to minus six. Also shown here, a particular case for which, the minimum of the function is just touching the x axis. So, for this, there are again two solutions, but both the solutions are identical. This green curve, the last one, is for tau equals point four and for this there are no solutions. Is this clear?

So, again we see that, there is straight line, which is the solution for tau equal to 0. There is a straight line, which is a curve for tau equal to 0 and solution to that is minus one. If you want to think of it two roots, minus one and minus infinity and all the other curves start bending back upwards. That is most clearly seen in the orange curve. There is a solution here and the solution there. One way to think about it is, the solution minus one for tau equal to 0 is increasing in frequency, that these it is becoming more negative. The solution and minus infinity is decreasing in frequency it is becoming less negative.

So, that is what happens. So, now, for this particular value of sigma, you can calculate the particular solution and this value the value of tau, for which the function just touches the x axis, can be found out. The way to do that is, put down the function sigma prime plus exponential sigma prime tau, this is your function, differentiate this with respect to sigma prime and equate that 0.

That will give you the value of sigma prime for which the function is the minimum and you said that, the minimum itself to be equal to 0. So, that means, the curve is just touching the x-axis, and you can solve for the value of tau. Again, I encourage you to solve the problem by yourself, before looking at my solution. Here I have showed the same graph over a much wider range.

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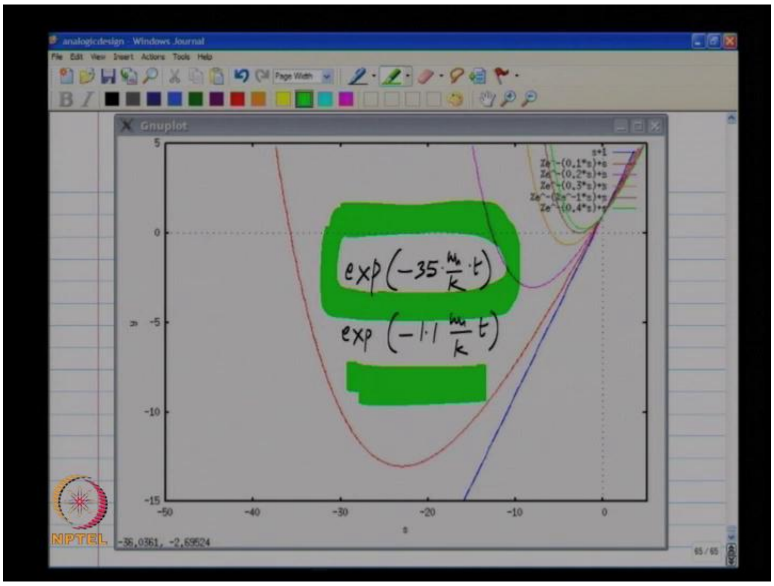

So, the orange one was here and the pink one holds backup and the red one also holds backup. So, for the red one, one of the solutions is very close to minus one, that is the solution without any delay. The other solution is something like minus 35 or something of that sort. As you can intuitively imagine, for very small values of delay the behavior will be the same, as that without the delay. So, you can see the solutions to the red curve is very close to the blue one, that is one of the roots is very close to minus one. The other root is close to minus 35. So, the two solutions, due to the two values of sigma will be of the form, exponential minus 35 omega u by k times t.

Due to the other one, exponential of minus, it is very close to one, let us say one point one or something like that. One point one omega u by k times t, you can easily see that this function dies out very quickly. It becomes 0, very quickly and the solution is dominated by this function. That is to say, the final solution is very close to the case without any delay. As the value of delay increases, the two solutions will become close to each other. They also become quite different from minus one. So, then something interesting happens, what was that, will see shortly.

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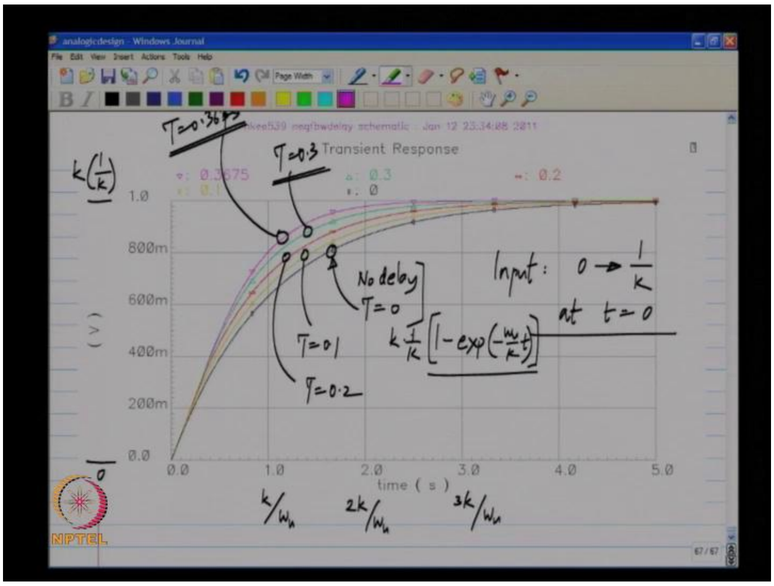

Here, I am showing an example of the solution that is obtained to the differential equation. This is obtained from simulator, essentially what this is, is the solution A 1 exponential of sigma 1 t plus A 2 exponential of sigma 2 t with the particular conditions that are applied. In this particular case, I have used in input which goes from 0 to one by k, at t equal to 0, this is why the output is going from 0 to 1. Which is k times the input voltage one by k.

So, you can say that, it is going from 0 to one. Also the x-axis is graduated in normalized time constants one corresponds to k by omega u, two u corresponds to k by omega u, three k by omega u and so on. Lowest curve here, this corresponds to no delay. So, this is simply nothing but, this curve as one minus exponential minus omega u by k times t times k. Times k times the input voltage. So, one minus exponential, this is the traditional solution to the first order equation. The next curve here, this is for tau equals point 1 and this is for tau equals point 2 and tau equals point 3. Actually, what you say is rather interesting, it will looks like first of all, there is no overshoot, which we thought was a possibility.

We were not sure, but at least for these values of delay, tau equals point 1, point 1, and point 3, there is no overshoot. Overshoot means that, the output is supposed to go from 0 to 1 and it goes beyond 1.You can see that, nowhere is it going to 1. If I plot it up, to infinity it will not go beyond 1. It will asymptotically reach 1.

So, and the second thing is that, the response with tau equals to point 1 is actually faster than with tau equals 0. That is little bit of delay, is helping the response of the system. You are going faster and faster and this red color was faster than the yellow one and the green one is faster than the red one and so on. The very last curve here, is for tau equals point 3 6 7 5 and this is the largest value of tau, for which there is a solution. That is, this corresponds to the function being at tangent to the x-axis. Earlier we said that, for very large values of tau the function remains completely positive. There is no solution and the tau equals point 3 6 7 5 is the highest value of tau for which there is a solution.

So, the message to be taken home from here is that, first of all, for small values of tau there will be two possible sigmas. Which gives you two possible exponential of solutions and the actual solution in any case will be a linear combination of the two exponentials or more significantly for small values of tau, like point 1 and point 2 or point 3. The response A does not have an overshoot and two, it is actually faster than the response without delay. So, you can see that yellow, red and green are faster than the black one, which has no delay.

So, it seems actually, it is desirable to have a certain amount of delay in the system, so that you can reach the steady state faster. We will examine in this detail. The highest value of tau for which there is no overshoot happens to be tau equals point 3 6 7 5. So, will now see what this particular value is, how this value comes about.

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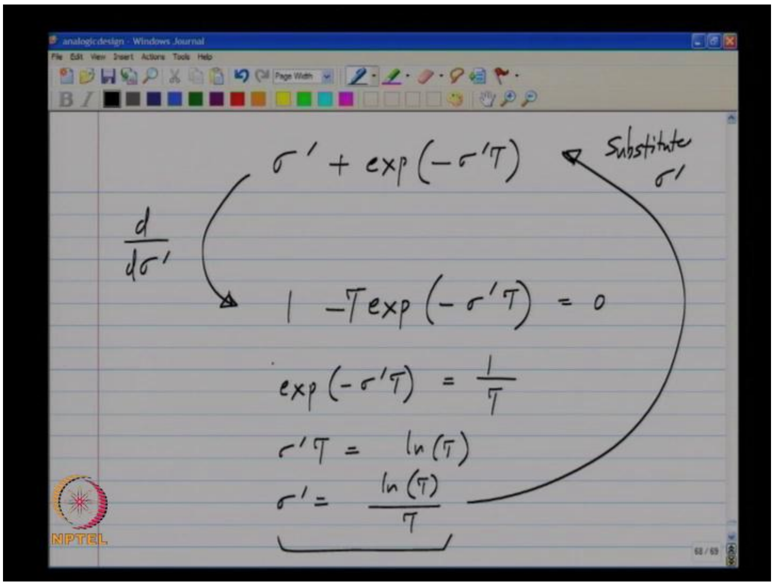

So, we were looking at this function sigma prime plus exponential sigma prime tau equals 0 and this is what we were plotting. Now, there will be no solutions to this, if the function remains entirely positive. This happens for large values of tau and we said that, the way to find it is, find the value of tau for which the minimum of the function is exactly 0. So, the bottom of the curve touches the x-axis and then goes back up.

We have to find the minimum value of this function sigma prime plus exponential of minus sigma prime tau. To do this, we do the usual process of the differentiation. Differentiating this, we get 1 minus tau exponential of minus sigma prime tau to be 0 and from this, we get exponential of minus sigma prime tau to be 1 by tau or sigma prime tau to be ln tau or sigma prime to be the natural logarithm of tau divided by tau itself. This is the value of sigma prime, for which we get the minimum value of the function. And the minimum value obtained by substituting the value of sigma prime into this one and that will be…

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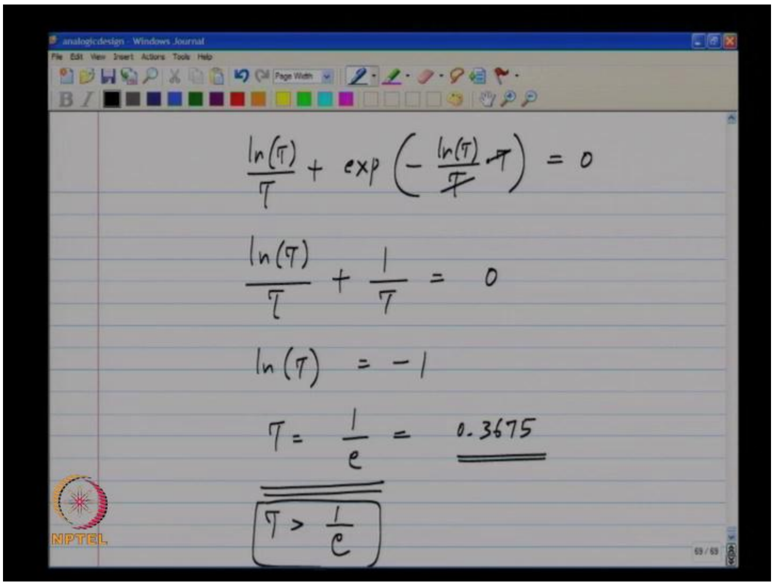

We want to find the value of tau for which, this is 0. This gives us logarithm less explanation of minus natural logarithm tau which is 0 or the natural logarithm tau is minus 1 or tau is 1 by e. And 1 by e is point 3 6 7 5, when the natural logarithm tau is 1 or tau is 1 by e the natural exponent, we will have a case were the function sigma prime plus exponential of minus sigma prime tau, just touches the x-axis and we have two identical roots and for tau greater than 1 by e. We have no real solutions and we have to resort to complex solutions, to this particular function, that we will see in the next lecture.