Analog Integrated Circuit Design

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Lecture No - 7
Negative Feedback Amplifier with Ideal Delay- Large Delays

Hello and welcome to another lecture of analog integrated circuit design, just to summarise the previous structure, we have analysed the negative feedback amplifier with delay and we have determined the kind of solution that it is going to give noise.

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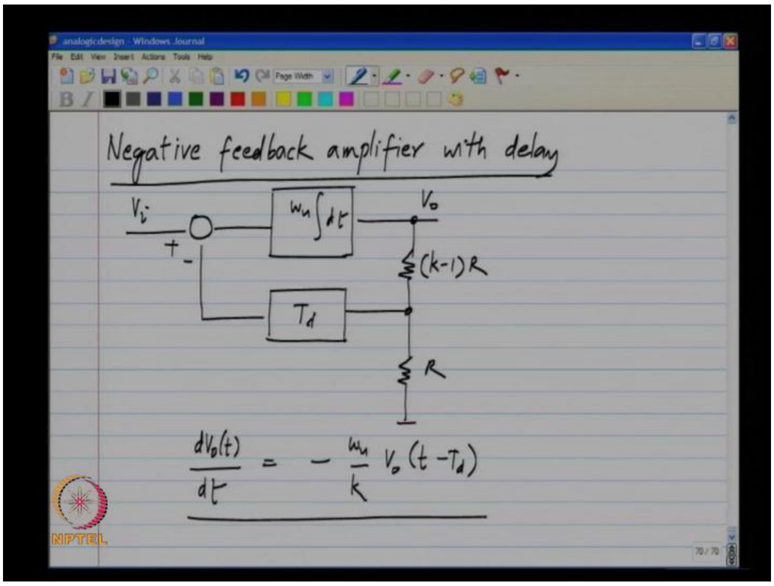

So, we have seen that if you have a delay in the negative feedback loop noise, it can be modally using a delay T d in the feedback path. This gives a differential equation which is that the derivative is the same as the delayed function. The derivative has the same shape as the delayed function and the particular solution for a given input can be obtained by substituting the right boundary conditions.

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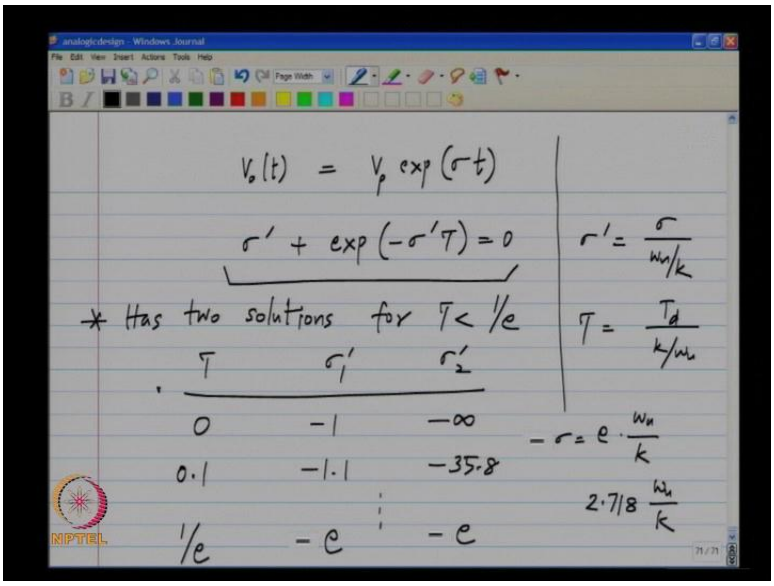

As we saw yesterday to obtain the solution to this, we assume that V naught of t is of the form of v p exponential sigma t. While solving this, we get the non linear equation which is sigma prime exponential of minus sigma prime tau equal 0. Here, sigma prime is the normalise sigma, sigma is normalised to omega u by k and tau is the normalised delay T d normalize to a time constant k by omega u. We find that this has two solutions for a tau less than 1 by e. So, we can list the solutions for different values of tau for tau equal to 0 that is no delay, we have just a single solution and the other solution can be thought of as being at minus infinity.

So, this minus 1 corresponds to a sigma value of minus omega u by k and we already know that without delay that is the solution you get solutions of the form exponential minus omega u by k times t. For t equals 0.1, we get something like minus 1.1 and minus 35.8 and so on, what happens is this sigma one goes on increasing that is it goes more and more negative to higher frequencies. That means, the exponential corresponding to sigma 1 becomes faster and faster and the exponential corresponding to sigma to becomes slower and slower and for tau equals 1 by e, sigma 1 will be e and sigma 2 will be e. Here, the sigma 1 and sigma 2 are the normalized quantities.

So, when tau equals 1 by e that is when the delay is approximately 0.36 of the time constant k by omega u, we get two identical roots which are at e times omega u by k or that is the value of sigma or 2.718 times omega u by k. As you saw yesterday, the interesting part was that with response delay, the response actually gets faster, there is no overshoot with this kind of delay. We have to see for larger values of delay what happens because this particular equation has solutions only for tau less than 1 by e.

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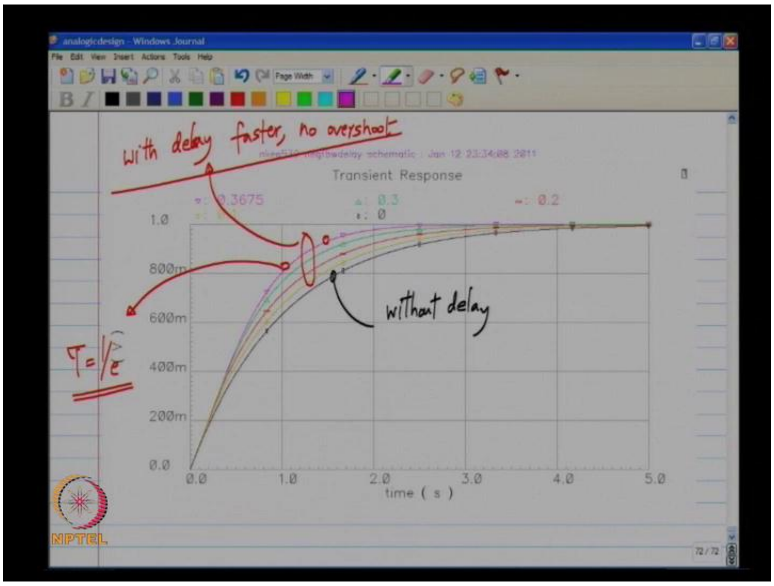

So, just to remind you of the solution again this is what we get without delay and these four are the solutions with successfully increasing delay, they are faster and there is no overshoot. In fact, it turns out that this one the uppermost curve which corresponds to tau equals 1 by e, this is the fastest response you can get without overshoot, so that is a good number to remember. So, to summarise a little bit of delay actually helps, but too much of delay can harm because it can easily see that there is a possibility of overshoot. The best thing to have is tau equals 1 by e, because that is the fastest response you can get without overshooting. In the step response over shoot refers to going beyond the target which is unity here and then coming down and settling down here slowly.

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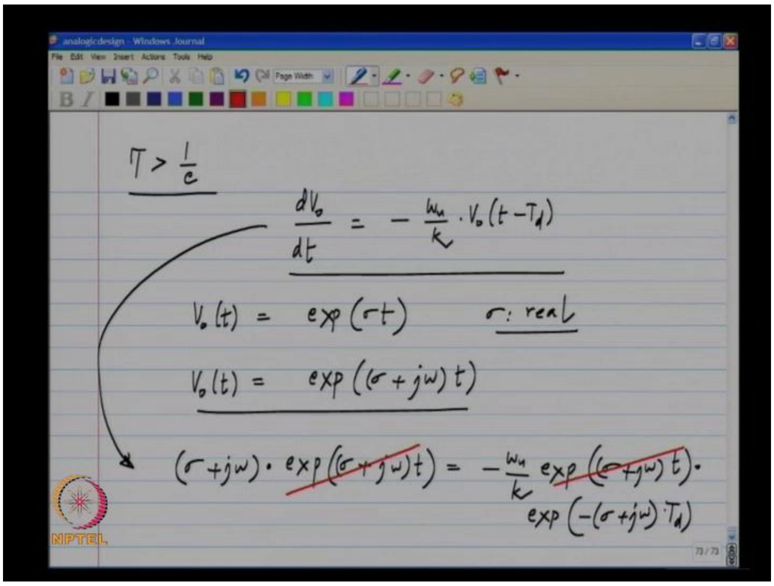

Now, we will examine what happens when tau is greater than 1 by e, the differential equation of course is still the same. Now, the only thing is earlier we assumed a solution of the form exponential of sigma t with a real coefficient in the exponent. That is what has to be relaxed, now we know that with the real coefficient, the exponent, we do not have solutions for tau greater than 1 by e.

So, we have to have a complex coefficient and with this we can substitute into the differential equation d V naught by d t will be sigma plus j omega exponential sigma plus j omega times t. That is equal to minus omega u by k and V naught of t minus T d gives rise to a product of two exponentials. One of this is simply exponential of sigma plus j omega times t and the other one is exponential of minus sigma plus j omega times T d.

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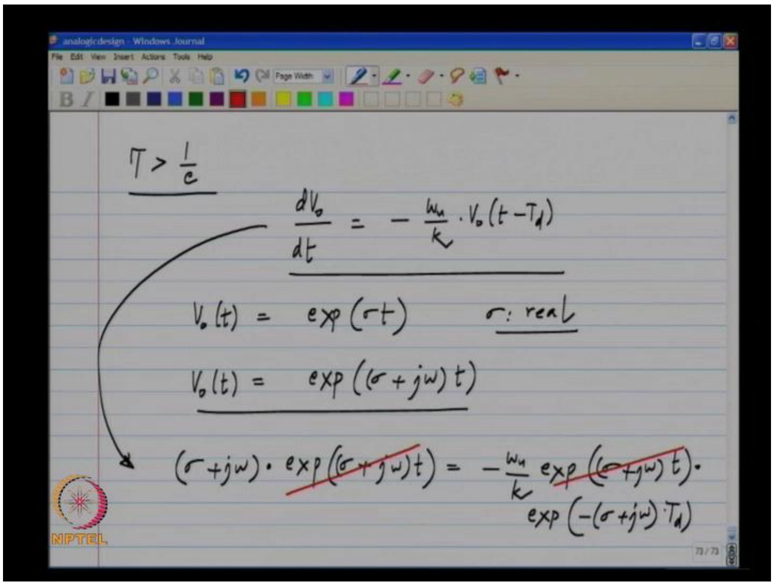

As before, noise this cancels out and we will be left with sigma plus j omega equals minus omega u by k exponential minus sigma plus j omega times T d, which can be written as minus omega u by k exponential minus sigma T d. This is corresponding to the real part of the exponent and exponential minus j omega T d which can be further expanded into its real and imaginary parts. Now, we have two variables sigma and omega and we need two equations to solve them and if you look at this equation, we have the real part and the imaginary part on the right and left hand sides.

We equate that two to get the two equations, so this is the real part and this has to be equal to this times cos and this is the imaginary part and that has to be equal to this times. So, the two equations are and omega equals omega u by k exponential minus sigma T d sin omega t d. So, from these two, we solve for sigma and omega, let me rewrite the equations here.

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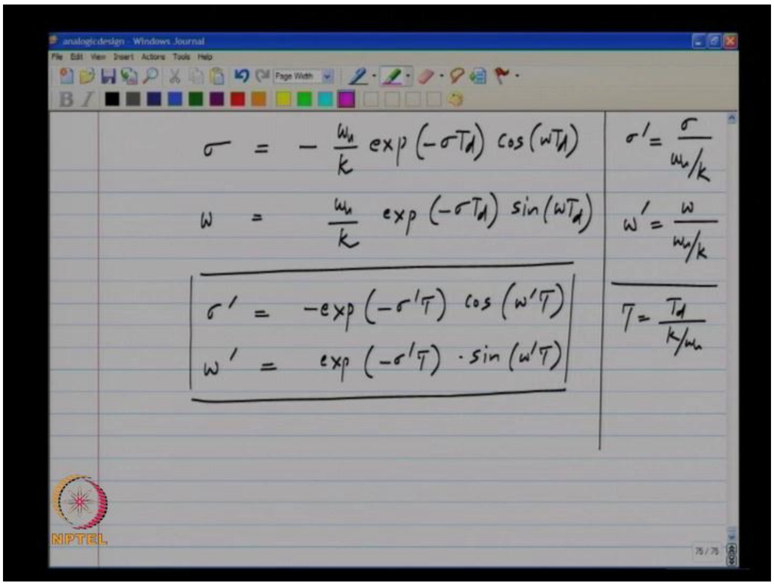

As usual, we do our sanity check for T d equal to 0, the only satisfactory solution is to have sigma equals minus omega u by k and omega equal to 0. That corresponds to the case without delay, now again I will normalise the terms. So, we land up with a dimension less quantities, I will define sigma prime to be sigma normalise to omega u by k. Similarly, omega prime is omega normalises to omega u by k and tau is T d normalise to k by omega u with this normalisation. We can rewrite this as sigma prime is minus exponential minus sigma prime tau cos omega prime tau and omega prime is exponential minus sigma prime tau sin omega prime tau.

So, now we have a set of non linear equations before we had a single non linear equation and even that could not be solved explicitly for sigma prime. Now, we have two non linear equations simultaneous equations, which have to be solved sigma prime and omega prime. This is not possible, explicitly we cannot write sigma prime equals to this and omega prime equal to that, but we can try to get a feel for the nature of the solution. Also, one of the most important things is when you are qualitatively discussing the effect of delay, we said it is possible that we overshoot and then undershoot and then we overshoot and then undershoot and never actually settle.

Now, that will happen when sigma is positive because the solution is of the form exponential sigma plus j omega times t and if sigma is positive, then this keeps growing and if sigma is negative, it keeps falling. So, we have to find out at what value of delay the sigma becomes positive, but we know that for a very large value of delay, it is going to happen because for a very large value of delay. It means that there is no feedback for a long time and we keep going past the target by a large amount. So, we have to find out exactly what value of delay will cause this to happen and that is why we need to get an idea of how the solutions to this equation look like.

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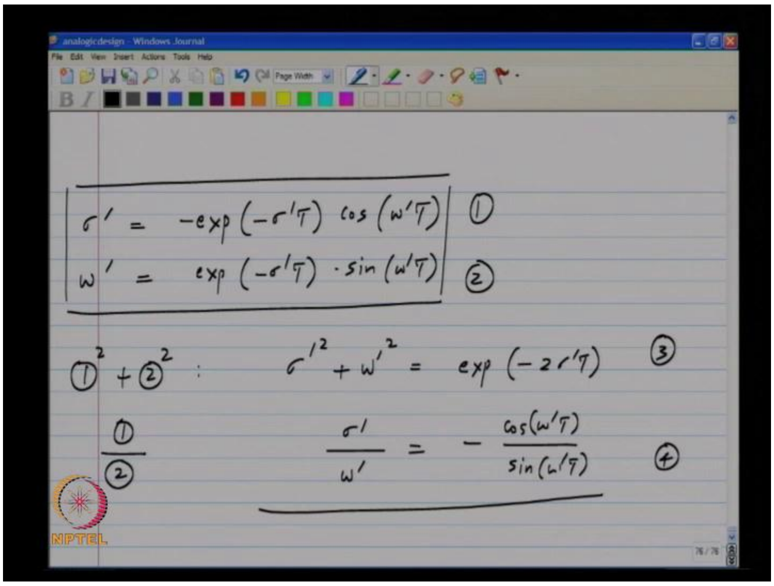

Now, to get a feel for the solutions, we can transform it into another form, essentially what we have to do is plot both of these equations on the same set of axis sigma versus omega or omega versus sigma. Then, we find the solution, but these particular equations are rather more difficult to visualise. So, instead of these, we can transform them into two other equations, first what I will do is I will take the square of the first and the second equation. This gives me because cos square plus sin square becomes 1, I add sigma prime square plus omega prime square is exponential of minus 2 sigma prime tau.

The next equation is obtained by taking the ratio of the two, which gives me sigma prime by omega prime as minus cos omega prime tau by sin omega prime tau 3 and 4 are easier to visualise. From this, we will try to visualise the solution and especially we are interested in the value of tau at which the system becomes unstable. The sigma becomes positive, now in general a set of equations like this for a particular value of tau will be solved numerically and you can substitute the numerical value and find the solution after applying the boundary conditions.

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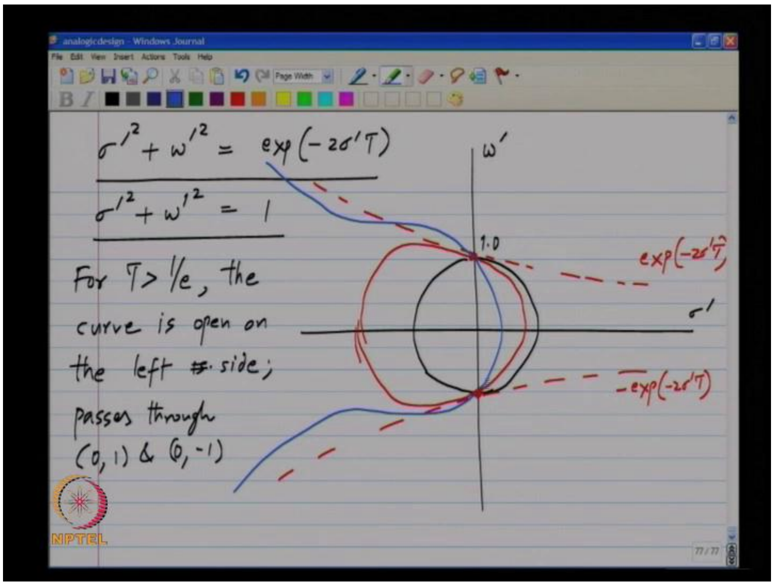

So, the first equation is sigma prime plus omega prime square equals exponential of minus 2 sigma prime tau, how does this equation look like, when we plot it on noise a omega versus sigma plane to see this. We can first consider the very familiar equation sigma prime square plus omega prime square is one which is the equation of a circle. A circle is something whose x coordinates and y coordinate and squared and added up add up to unity noise. Now, in this case, actually equation right hand side is exponential of minus two sigma prime tau and not one. So, what it means is the solution to this is some kind of a distorted circle whose radius goes on changing with sigma prime.

To get a feel for this noise, I will plot exponential of minus 2 sigma prime tau for some arbitrary value of tau. So, what the first equation is saying is that sum of sigma prime square and omega prime square is equal to this value that is equal to this curve, which is becoming larger for negative values of sigma prime. This is becoming smaller for positive values of sigma prime. So, compared to the circle the new solution will look like this on the omega prime axis, it passes through unity plus and minus 1.

Then, on the right side, you can think of it as constantly reducing radius and on the left side the radius is increasing. So, this is what the curve will look like, this is for a particular value of tau and if you increase the value of tau, you will find that it may not even form a close loop. It may go off to infinity because the exponential for negative values of sigma prime becomes very large. So, In fact, it may look something like this and it turns out that for tau greater than 1 by e. This is exactly what happens, it will not form a close loop it is closed on the right side passes through plus 1 and minus 1 and then diverts this to infinity on the left side.

We do not have to worry about the exact shape of this this much detail is enough to get an idea of what the solutions are and when the system becomes completely unstable and it passes through 0,1 and 0, minus 1, this is true for all the curves. So, this gives us an idea of what one of the equations look like what we need to do is to plot the other equation and superimpose it on this graph and see what the simultaneous solution looks like, let us now consider the second equation noise.

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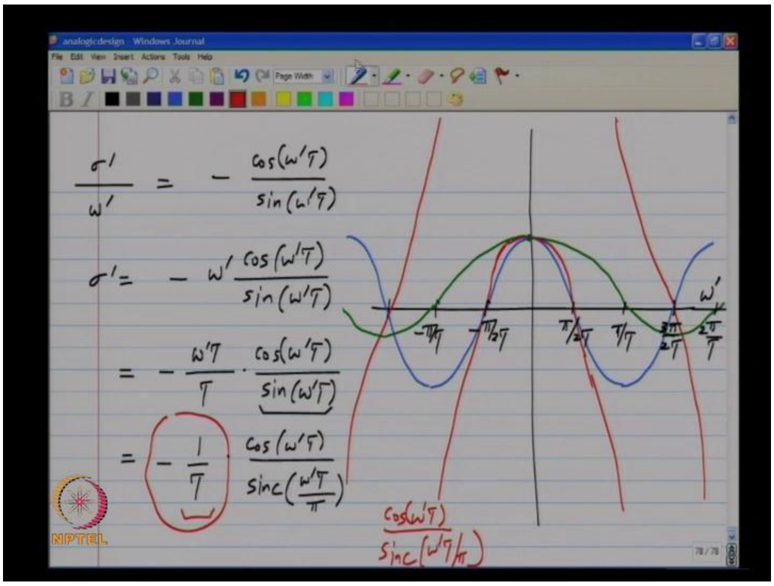

This says that sigma prime by omega prime is minus cos omega prime tau by sin omega prime tau and previously, we plotted sigma prime on the x axis and omega prime on the y axis. Now, because here sigma prime appears only in one place, it is easier to plot omega prime on the x axis, we can then turn the graph around and superimpose it on the other plot noise. I will rearrange it like this sigma prime as a function of omega and this can be further rewritten as noise, I have taken a factor of tau out of the equation. If you look at this, we have sin omega prime tau divided by omega prime tau which is a sin c function noise. So, that is what it is, it makes it easier to find out the value at omega prime equal to 0 etcetera.

So, I am going to plot this on omega prime as the as the x axis and sigma prime as the y axis. So, first of all for omega prime equals 0, the cosine is 1 and the sin c is 1, so the result will be just 1 and tau, now before I plot the whole curve, I will plot cos and the sin c. We know that both the cos and sin c are symmetric about the y axis noise and so on. So, first I will plot the cos, it starts at unity goes to 0 at plus minus pi by 2 tau becomes minus 1 at plus minus pi tau and does something like this. I will plot the sin c separately starts from a unity and this becomes 0 at plus minus pi tau.

Then, it goes to a negative value, but not quite the same magnitude at 3 pi by tau and then becomes 0 again at 2 pi by tau. This should be 2 pi by tau here and the function is the ratio of these two, if i plot the ratio of this excluding the factor of a 1 by tau, I will get something more than the cos and it goes to 0 over here and here it goes to minus infinity and again at pi by tau. It goes off to minus infinity and if you look at the next segment between pi by tau and 3 pi by 2 tau or minus pi by tau and minus 3 pi by 2 tau, the result is again positive.

So, there is a discontinuity here and then it also goes off to infinity at 2 pi by tau in general, it goes off to infinity at every integral multiple of pi by tau, it will be plus infinity or minus infinity and this is what it looks like. Now, the red curve is the plot of cos omega prime tau divided by sin c omega prime tau by pi as I have mentioned before. It is not sufficient to just watch me do this it is much better. If you try to solve this by yourself and plot it either using a numerical package for by hand by looking at the nature of the functions.

So, then we will get a much better grasp of what the function behaves like and I cannot write here, because of lack of space this keeps repeating in every segment from 2 pi by tau to 5 pi by 2 tau and the next one, the function jumps to plus infinity. Then, it goes through 0 and minus infinity and so on. So, this keeps repeating, so now this function has to be multiplied by minus 1 by tau to get what we want, to get the value of sigma prime. So, let me not put sigma prime here because that is not what I have plotted, I will do that in a new plot.

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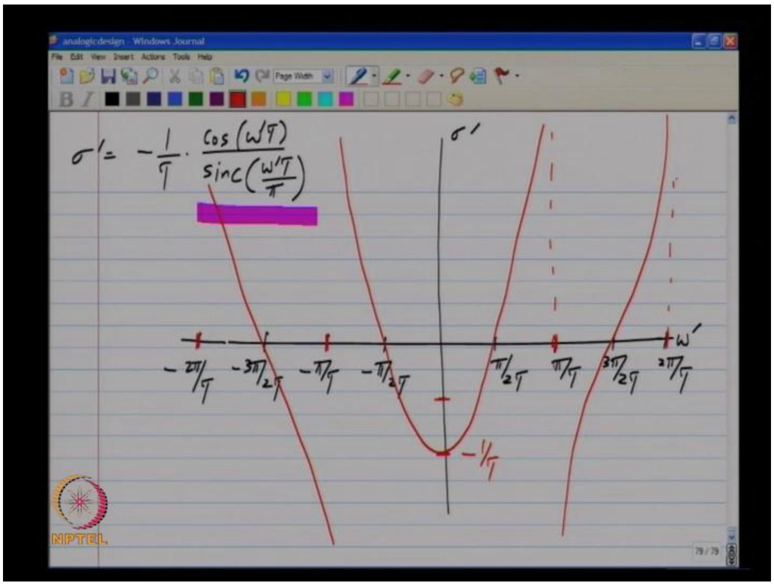

If I plot it on a sigma prime versus omega prime axis, the value of this factor at omega prime equal to 0 is unity. So, our function starts from minus 1 by tau and it goes through 0 at these values pi by 2 tau and then it jumps plus jumps from plus infinity to minus infinity and has something like this and so on. So, there are discontinuities at minus pi by tau minus 2 pi by tau plus pi by tau plus 2 pi by tau. In general, every integer multiple of pi by tau, so this is the nature of the solution sigma prime plus sigma prime versus omega prime looks like.

A cup in the middle and a lot of other segments with discontinuities between them. Now, what we have to do is to overlap this with the other plot which is the blue curve here. We have to rotate the axis because this is plotted omega prime versus sigma prime and the other one is plotted sigma prime versus omega prime. So, we have to turn the plot around and then plot it, so let us do that and see what the solutions do look like.

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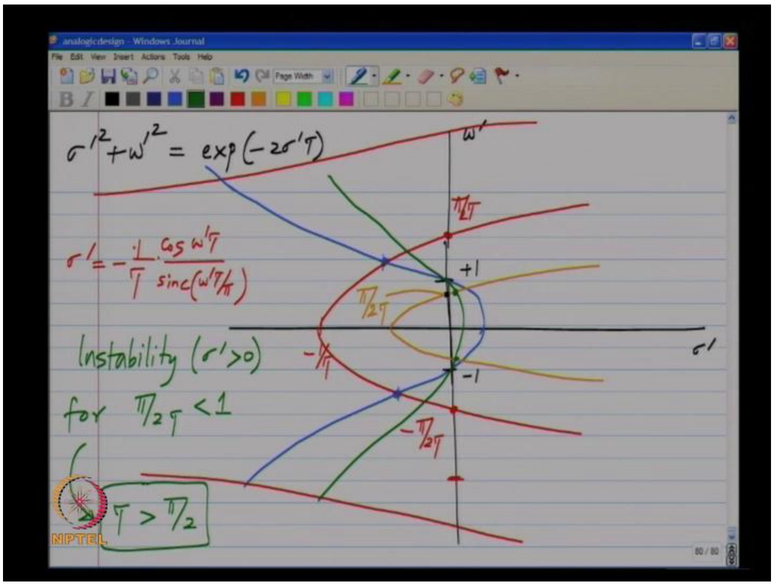

So, first equation is sigma prime plus omega prime sigma prime square plus omega prime square is exponential minus 2 sigma prime tau and I have already plotted this omega prime versus sigma prime and the solution looks something like this. Let me mark unity here plus 1 and minus 1 and as I said earlier, the solution will be an open curve something like that. That other equation sigma prime by the other equation sigma prime is minus one by tau cos omega prime tau by sin c omega prime tau by pi. This has a value of a minus 1 by tau for omega prime equal to 0 and then it goes off like that and it intersects the y axis at pi by 2 tau and minus pi by 2 tau.

Then, there are a number of other segments because of the discontinuity something like this noise there are a number of other segments like that. So, there are an infinite number of segments and therefore an infinite number of intersection points. So, this means that in this particular case we have an infinite number of values of sigma prime and omega prime, but except for the first pair here and here all the other values are at higher frequencies. The value of sigma prime is higher and the value of omega prime is higher and because the value of sigma prime is higher in magnitude. Those exponentials die of quickly and the solution is dominated by this one.

The others do matter, they matter less than the first set of solutions and also you see that the way I have drawn the plot, all the intersection happens on the left half side of this plane that is for sigma prime values, which are negative. So, this means that all the exponentials decay out and the system will be stable, but this is not unconditionally true, we can plot it for different value of tau and see what happens. So, you observe that for a smaller value of tau this point minus 1 by tau will be further to the left and this point pi by 2 tau will be further up on the y axis and further down on the y axis here.

So, that means that the points of interaction will be further negative conversely for a large value of tau this end of this cup moves inwards, it moves to the right and also these points of inter section move in they move closer to the origin. So, for a larger value of tau, I will plot it here, noise the point of intersection may be in the right half plane for a larger value of tau the blue curve also changes to this green curve, but, it always passes through plus 1 and minus 1. So, there is a certain value of tau beyond which the intersection happens on the right side of the curve for sigma prime greater than 0.

We know that sigma prime greater than 0, the solution does not decay to 0, but it blows up exponentially. So, this means that the system is unstable and it does not matter, if you have a number of intersection points with sigma prime less than 0, what matters is even if you have single pair of points with sigma prime greater than 0. The solution will blow up and the amplifier is not usable what is meant by blowing up is that when you do not have an input because of some initial condition. The output simply diverges; we expect that the amplifier will respond only to its input, it will give u k times the input voltage after reaching steady state, but if the amplifier is unstable.

It is not going to do that even without an input, it is going to give its own output and the output is going to be unrelated to the input and soon become very large. So, the amplifier becomes un unusable if the poles are in the right half plane know, we have to find out the value of tau for which the solution moves to the right. This is rather simple because the solution to the first equation the plot of the first equation always passes through plus 1 minus 1 on the omega prime axis. The plot of the second equation passes through pi by 2 tau and minus pi by 2 tau for the intersection to happen in the right half plane.

This cup has to be within plus 1 and minus 1 that is to say this point of intersection pi by 2 tau has to be less than one. So, we will get instability which means some value of sigma prime more than 0 for pi by 2 tau less than 1, in other words tau greater than pi by 2. So, that is what we have learned by plotting these two non linear equations on a single pair of axis and finding out the points of intersection.

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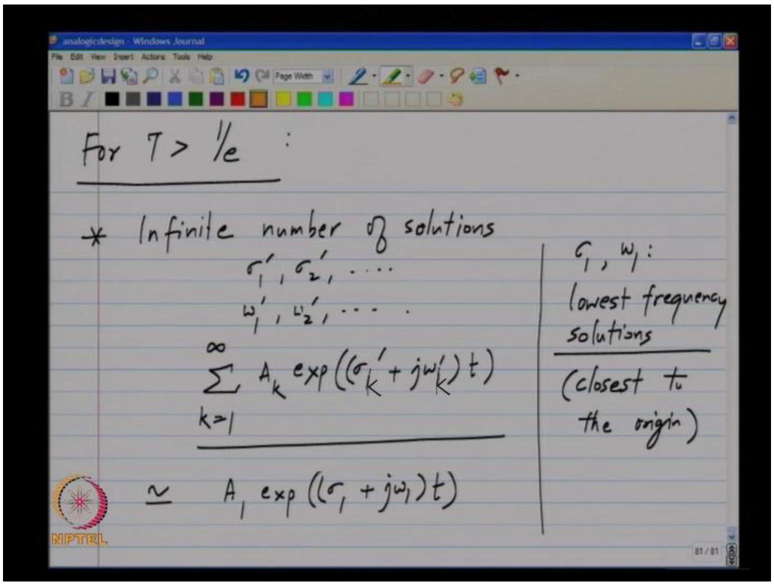

These summarise for tau greater than 1 e for which we could not find the points of intersection from the earlier method. We had to admit a complex solution sigma prime plus j omega prime, first of all noise there are an infinite number of solutions, you have sigma one prime sigma to prime. So, the list never ends and similarly omega 1 prime omega 2 prime and so on and the list never ends. So, if you have an infinite number of solutions what is the final form of V naught as a function of time, it will be sum of all possible noise. I have to run the summation from noise, one sum of all possible exponential solutions.

So, this is what we have to do and from the boundary conditions, we can try to determine a k, of course it is not possible to take all the infinite terms, but what happens is that. There is a pair of solutions here and here which are closer to the origin and all the others are much further out. So, you can take only this and get a reasonable approximation to the solution or maybe you can take this and the next one and get a reasonable approximation.

So, you do not have to take all the infinite number of terms, if you want to write down the solution, you approximate this by a 1 exponential sigma 1 prime, sorry I had to make this sigma 1 not sigma 1 prime sigma 1 plus j omega 1 times to k. Here, sigma 1 and omega 1 are the lowest frequency solutions that is the once that are closest to the origin. So, this is what happens, you get an infinite number of possibilities for sigma and omega, but you take the lowest frequency, once and you will get a reasonable approximation of the final solution.

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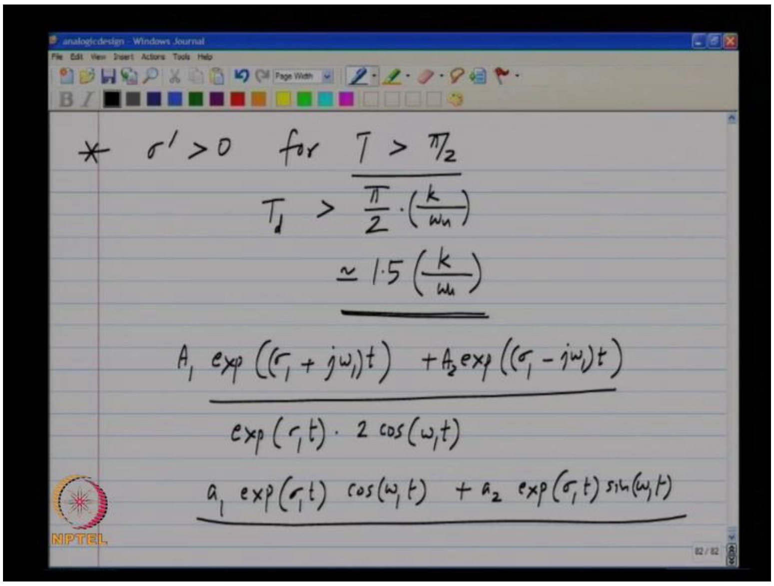

So, another important thing that we noticed is that there is at least one value of sigma prime which is more than 0, which means one value of sigma which is more than 0 for tau greater than pi by 2. This means that the delay T d is greater than pi by 2 times k by omega u and pi by 2, we know is approximately 1.5 noise k by omega u, we have all along been saying that if you have a delay that is very large, the system will become unstable. Now, we know exactly how large that is if you have one and half times the time constant. The system will become unstable if you have a delay that is one and a half times the constant the system will be unstable.

So, you have to keep the delay much smaller than 1.5 k by omega u and what do the solutions actually look like because earlier for a tau less than 1 by e. So, the output voltage V naught started from 0 and then reached the output without overshooting. Now, in this case, we have said that we have admitted exponential solutions, in this case we have said that we have admitted complex coefficients inside the exponent, what it means is solutions of the firm exponential sigma 1 plus j omega 1 t noise.

Accompanying this, you will always have exponential sigma 1 minus j omega 1 t because you see that you will always have a symmetrical pair. If you have a point of intersection here above on its mirror image above the x axis, you will also have a solution and this you know can be written as exponential sigma 1 t and 2 cosine omega 1 t. Now, if have different coefficients here and there let us say a 1 and a 2 you will be able to write it as a one exponential sigma one t cos omega 1 t and some other a 2 exponential sigma 1 t sin omega 1 t. So, the solutions will be in terms of sinusoids cosine of sin which is modulated by an exponential.

If the value of sigma is negative the exponential dies out and the sinusoid also dies out and if the value of sigma 1 is positive exponential blows up, but we will not be very interested in looking at positive values of the exponential. That simply says that the amplifier is not usable what we want to find out is the delay for which that happens and make sure that we never get there. We have to find reasonable values of delay for which you have sinusoids that are decaying and the amplifier is still usable.

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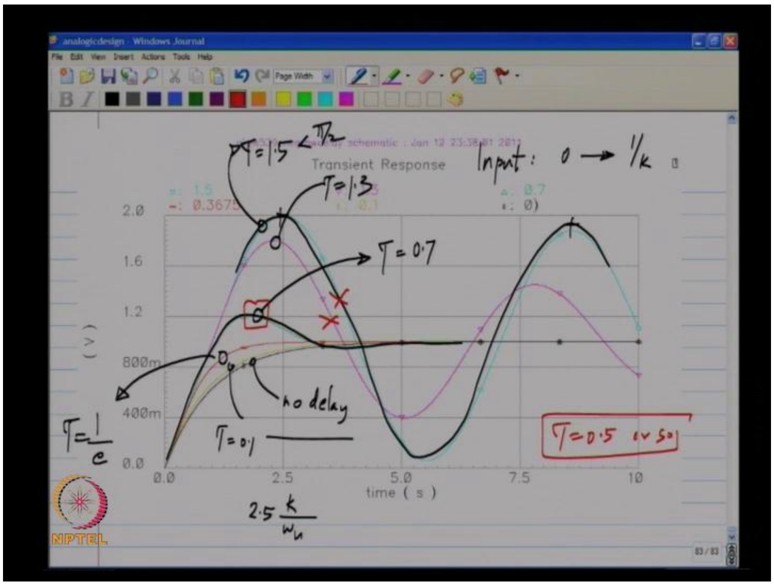

Let us look at some example solutions, so here is what happens again for reference I have shown the case without delay and this plot is obtained in a way similar to the last one that the input goes from 0 to 1 by k and the x axis is normalised to k by omega u 2.5. So, no delay case, of course is the exponential and I have shown it also for a small value of delay that is tau equals 0.1 and you see that is quite close to the case with no delay, this is what we knew all along and then this is 1 for which tau is 1 by e. We said this is the fastest response we could get without overshooting at any tau that is larger than this results can overshoot as exemplified by the green curve.

So, this is for tau equals points even the output goes out and then comes down below 1 and then settles this is not going back and forth too many times. So, this may be still acceptable, but if you look at the next case this one for which tau is 1.3. Then, you can see that it is going up and then coming down and going up and coming down and at 10 time constants the output is nowhere mere settle it will go up and down many times. Finally, we have the pale blue one which corresponds to tau equals 0.5, 1.5, this is just smaller than pi by 2 for pi by 2, and we know that it becomes unstable. This is stable because you see that the amplitude of the ringing is decreasing here.

It is 2 above 2 and here it is smaller and it will become smaller, but, it will really take a long time to settle. So, you do not want any of these things you would not want to use this much delay. So, this is about the maximum that you can tolerate or in practice even smaller about tau equals point five or. So, for that there will be just a little bit of ringing and not much overshoot and the response dies out in a reasonable time and you will get the steady state solution in a reasonable time.

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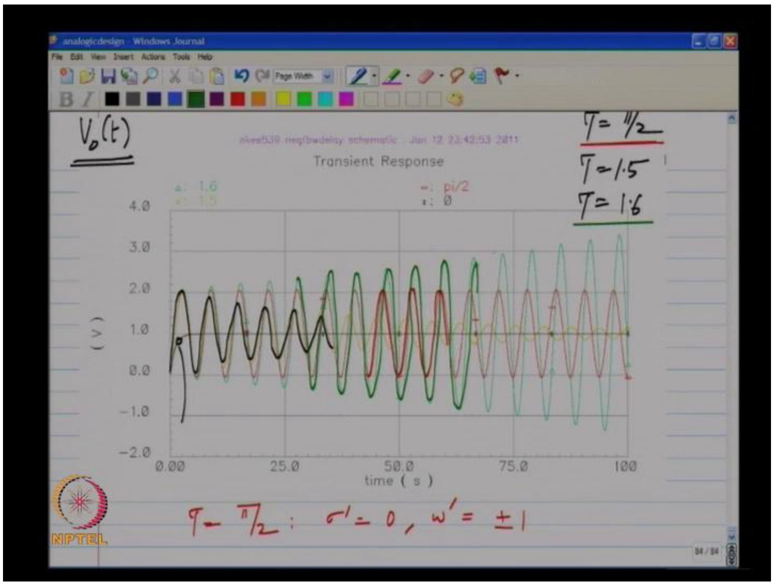

Now, let us look at another plot to appreciate this instability better, so in this case what I have done is to calculate the output V naught of t by numerically solving all the equations that I had put up earlier. For three cases I have done it for tau equals pi by 2 and tau equals 1.5 which is a slightly smaller delay than pi by 2 and tau equals 1.56 which is a slightly higher delay then pi by 2. The pale yellow curve whose amplitude is decreasing is the solution for tau equals 1.5, we can see that it is stable that is the sinusoidal ringing thus die out after t equal to 100, 100 time constants it is been still not completely died out, but if you wait for 1,000 time constants, it will die out.

Now, this is a stable system technically, but this is not a usable amplifier though, because it takes a really long time to settle because you can compare it to the case without delay. That is settling within a few time constants where as for toils 1.5, it has not settled even after 100 time constants, but technically it is stable because if you do leave it for long enough, it will settle to the steady state. Now, the red one the red curve here is the case for which tau is by 2 and you can see that it produces a sinusoid of constant amplitude.

This makes sense because for tau equals pi by 2 the points of interaction section will be exactly on the y axis. So, that means, that sigma prime will be 0, the exponential will simply become unity and the omega prime will be corresponding to these two points plus or minus 1. So, you get a sinusoid with constant amplitude without any damping for tau equals pi by 2, you get sigma prime to be 0 and omega prime to be plus or minus 1 and that corresponds to the constant amplitude sinusoid. The polls are being exactly on the imaginary axis or the y axis and if you now go to the other case, where tau is slightly greater than what is required, we have the polls in the right of plane, so you will see that sigma is positive.

So, the amplitude of this sinusoid goes on increasing, so in this case sigma prime is positive. So, the solution blows up and if you wait for long enough it will blow up to infinity. So, this is the general nature of the solution you always want to be sufficiently below tau equals 1.5 probably close to tau is tau being 1 by e, where you get the fastest response without overshoot or approximately tau of half which gives you very little overshoot.

This tau equals 1.5 or tau equals pi by 2 puts a absolute limit, after this the amplifier becomes unusable because you will get an exponentially increasing sinusoid. Soon, because of nonlinearities in the system, the system becomes an oscillator with a constant amplitude this is what you want to do if you want an oscillator, not if you want an amplifier.

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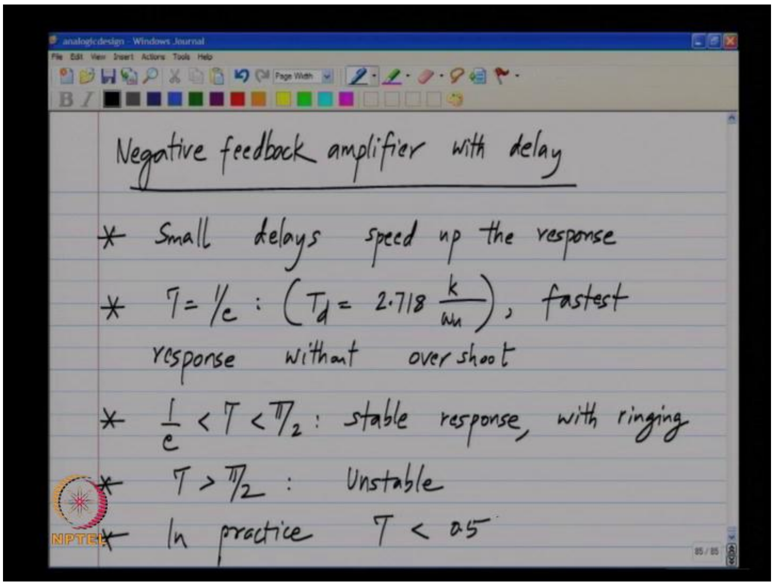

Finally, a quick summary of negative feedback amplifier with delay a small delays actually help you the speed up the response and this is because with a small amount of delay you are always accelerating a little more. Then, you should for tau equals 1 by e that is T d is about 2 points, even times the time constant k by omega u, you get the noise fastest response without overshoot and for tau between 1 by e and pi by 2. You get a stable response, but with ringing and for tau greater than pi by two the system is unstable and in practice, you would limit tau to about 0.5 or less.

So, even if there is a ringing, it is very small in magnitude and it dies out very quickly. So, that is a summary of what happens to a negative feedback amplifier in presence of delay. So, that should give you an idea of what kind of delays are tolerable and how this affects you in practice is that when you try to implement the amplifier the technology with which you implement imposes a certain lower limit on the delay. So, this means that your time constants and speed of the amplifier will be limited, see you in the next class.