Networks and Systems
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Lecture-30
From Fourier Series to Fourier Transform

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In this lecture, we will introduce ourselves to the concept of the Fourier integral and the Fourier transform. Let us recall our motivation in discussing the Fourier series. To recapitulate, that the characteristic property of a linear time and variant system was, that if it is driven by sinusoidal excitation.

The responsible also be sinusoidal in character. This particular property is valid only for sinusoidal. No, other periodic function will have the such a characteristic, will provide such a characteristic. And once, this particular property is satisfied. We brought in additional concept, like frequency response function, impedance and impedance concepts, which will simplify the analysis of linear systems, driven with sinusoidal excitation functions.

Now, this naturally let us to enquire, how we can analyze the linear system when, it is driven by a periodic non sinusoidal excitation function. And the Fourier series came in very handy in the contest. In the sense that, very non sinusoidal periodic function can be thought of as composed of several sinusoidal of different frequencies.

And by virtue of linearity, principle, we can find out the response to each one of the excitation functions. And superpose the responses to get the total response, under steady state conditions. When, the linear system is driven by a periodic, but non sinusoidal excitation function. Naturally, you like to see, whether this analysis can be carried forward to a case.

When, the excitation function is no longer periodic. We will call such excitation functions, a periodic function. So, when a function is periodic, is it possible to for us to consider, that a periodic function has composed of different sinusoidal functions of different frequencies. Surprisingly, the answer is yes. And this constitutes the discussion of the subject matter of our discussion under the Fourier integral, Fourier transform methods.

I use the words surprisingly not without justification. Because, when Fourier proposed that any a periodic function can also be a thought of as composed of several sinusoids. You did not gain immediate acceptance from all the mathematicians. And actually, the fact that the periodic function can be composed of several trigonometric functions of different frequencies.

Appears have to be known, even before Fourier, even though, the earlier work is did not carried out this analysis to a logical limit logical extent. But, it is the singular and unique contribution of Fourier to extended this kind of analysis to a periodic functions as well. What he really said was, that if you have the periodic function, it can be thought of, as the sum of several sinusoidal with infinite number of frequencies.

Infinite number of such small components, each with vanishingly the small amplitude. And if you put all of them together, you will get the given the periodic function. So, let us see, how we go about splitting up a given a periodic f of t into various sinusoidal components of the description, which we gave just now.

The key to the whole idea is, that if a particular periodic function is given. We can assume that the period is infinity. We are extending from minus infinity to plus infinity. And what happens, beyond plus infinity on one side and what happens, beyond minus infinity, we will not bother about it.

So, if any periodic function can be a thought of as a periodic function with period extending from minus infinity to plus infinity it occurs just ones. So, that means, the period of this is going to infinity. What is the consequence? The consequence is, that the frequency of this is 0. And so, if the frequency of this 0 is 0, then how are we going to think of various harmonic components.

Because, in the Fourier series, various frequency components, that are presented integrally related, integral multiple of the fundamental frequency. Here, the fundamental frequency is 0. And in the Fourier series the only term, which has the fundamental frequency, whose frequency is 0 is the dc term and after all, the dc term can split, any periodic function. So, must look for a different approach all together.

So, let us see, how we develop this. So, basically the key idea as I mentioned is, that we think of the period of this periodic phenomenon to be infinite extending from minus infinity to plus infinity. So, what we will do is, will start with the Fourier series of a periodic function. And then see, what happens, when the period becomes larger and larger, till it goes out to infinity. So, the limiting process, we will find out, how the various frequency components arise and what their amplitude will be and so on and so forth.
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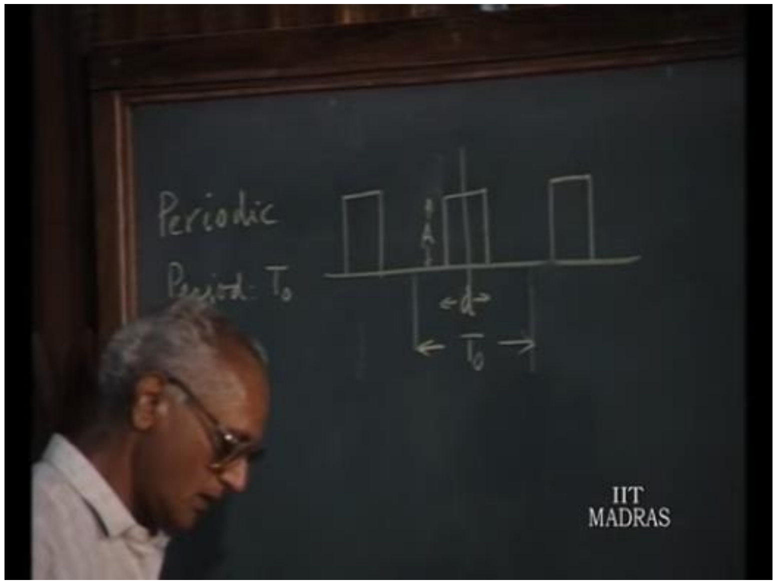

So, let me start with a function, which is periodic, period T 0. Just give a concrete example; I will take a specific wave form. But, all over discussion will be in completely general terms. So, we have periodic pulse strain of amplitude A and pulse duration d and the periodicity being the T 0.

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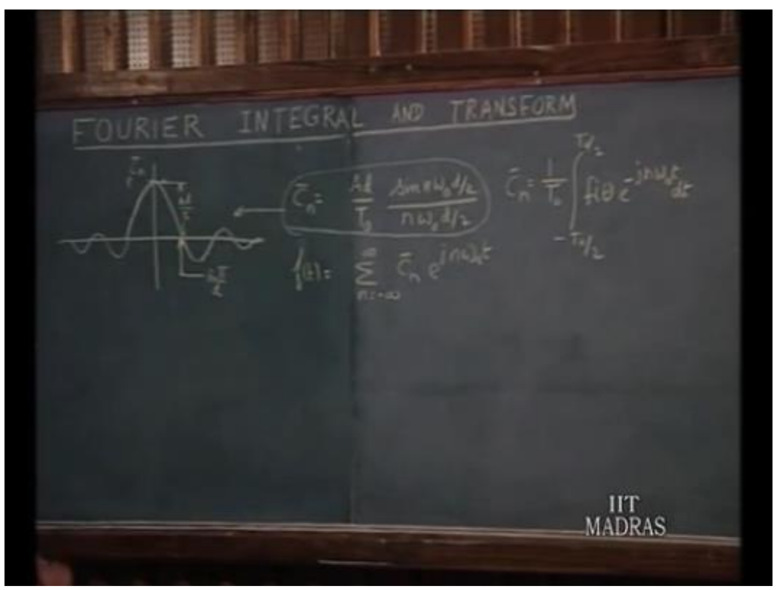

We know the spectrum for this, would be something like this. Where, this is C n and we know that, C of n is A d upon T 0 sin n omega 0 d upon 2 by n omega 0 d upon 2. This is something which we have talked about several times. So, let us look at the spectrum. And since, this is going to be a real number, because the function is even. So, it is going to be a real number.

I am not plotting the phase and the magnitude separately; I am plotting the entire C n itself directly. So, it is either way. So, it is always either positive or negative. The height of this is A d upon T 0 and the first 0 occurs, when n omega 0 d by 2, happens to be pi. Therefore, the frequency corresponding to this 2 pi upon d.

So, this is the nature of the spectrum, for a periodic pulse strain like this. And if we wanted to find out, f of t from this, we certainly write this as n from minus infinity to plus infinity of C n e to the power of j n omega 0 t. That is what gets. This is the general that we are getting. This particular C n; that we are getting apply to this specific wave form, this is the general formulation.

The general formulation for C n; in general case would be 1 over T 0 minus T 0 upon 2 to plus T 0 upon 2 of f of t, e to the power of minus j n omega 0 t, d t as we already know o, these are the two general expressions for C n and this is the f of t, which is the Fourier series for this wave form. So, these are two general expressions. In the particular case of a periodic pulse strain, this would be the particular value of C n. That would derive for our special case that we have considered.

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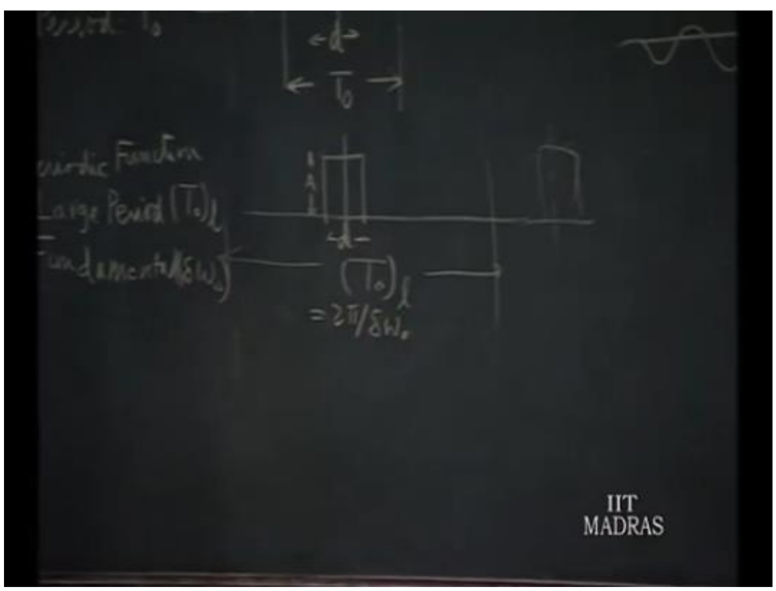

Now, let us see, what happens, when we have a periodic pulse strain, a periodic function again. But, the period become very large, large period. I will simply say T 0 of L will indicate as the large period. We have the same wave form, but the period becomes very large. So, I can put in this fashion.

I have the same pulse of amplitude A, duration d. But, the period is very large. That means, the next pulse occurs after long period, very large period. So, consequently, I will the angular frequency corresponding to this. Angular frequency here is fundamental period angular frequency is omega 0. So, we will say, fundamental of this is delta omega 0, I am saying. So, it include emphasize, there is the small angular frequency delta omega 0. So, T 0 of l is 2 pi over delta omega 0.

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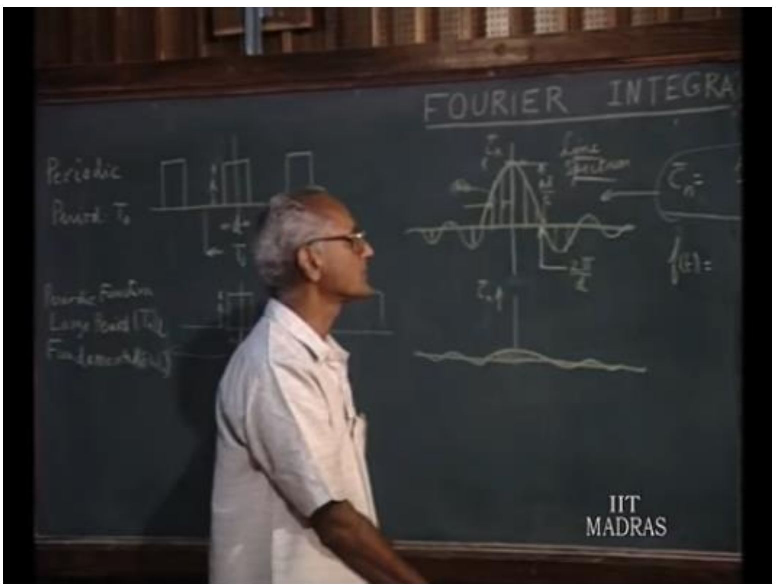

So, what would be the consequence as far the spectrum is concerned. Now, look at this, this is the formula for C n. Now, what happens, if T 0 becomes larger and larger, A d upon T 0 becomes smaller. That means, the initial height here, the amplitude will become smaller. And if you have very large T 0, this becomes almost negligible smaller and smaller.

I am sorry, what we should say is, C n is 0 continuous thing. C n will be, this is the envelope, what we have drawn is the envelope. The actual C n will be at discrete point on this. Because, the spacing between two lines being delta omega 0. The fundamental frequency, this is what we had in the first place. What I have to drawn here is, the envelope of C n.

So, the actual C n will be presented only at discrete points. And this is the line spectrum and the spacing between two adjacent components is omega 0. Now, when we make the period larger and larger, keeping the pulse with same d; what you have is, A d upon T 0 is smaller, very, very small. So, the result is, that after all 2 pi by d remains invariant. Because, the 0 occurs here, when n omega 0 t by 2 is pi and that corresponds to 2 pi upon d.

And therefore, what you are having here therefore is, that you will have, if you plot C n, the envelope will be like this. And what happens to the lines, previously the separation between two lines is omega 0. Now, since the fundamental frequency is delta omega 0, which is smaller, the lines comes closer and closer. The lines become closer and closer, the whole amplitude comes down.

Now, what we really want to know is, what happens when T 0 close to infinity because, we want to treat a periodic function as function of infinite period. So, if T 0 becomes infinitely large. Then, obviously, the whole thing squashes down to the horizontal line, because the amplitude becomes 0.

So, that will 0 give us any useful information. So, must see some other way of handling this information. So, what we should like to see is, you have some kind of new quantity, which we should plot, which will not become 0, which will be meaningful. And which will give us an idea of the relative changes of the various wave harmonic components. So, to do this, what we will think of is, instead of C n, after all the whole problem has come because, C n has 1 over T 0 upon this.

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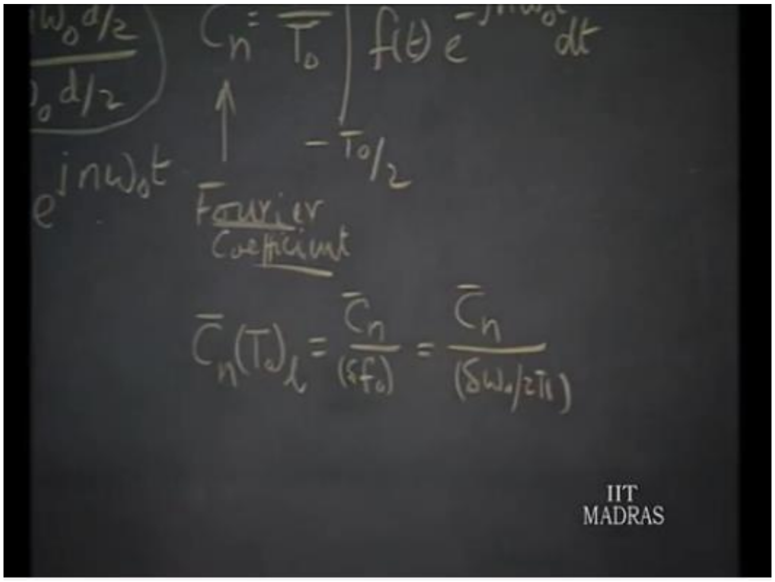

So, C n is the Fourier coefficient. It represents to us the amplitude in a sort of fashion of e to the power of j n omega 0 t term. Instead of that, if I plot C n time T 0, so in our case, C n stands T 0 of l. This I can write it as, C n times delta f 0. Because, the delta omega 0 is the fundamental angular frequency, delta f 0 is the corresponding frequency. So, I can write this as C n upon delta omega 0 upon 2 pi.

So, if you plot this quantity, we are going to multiply this C n by T 0 of l. That means, as far this particular wave form is concerned. Here, multiplying C n with T 0, then you are going to plot A d sin omega 0 t by 2 upon n omega 0 d upon 2. Therefore, the spectrum will now be restored to it is original side.

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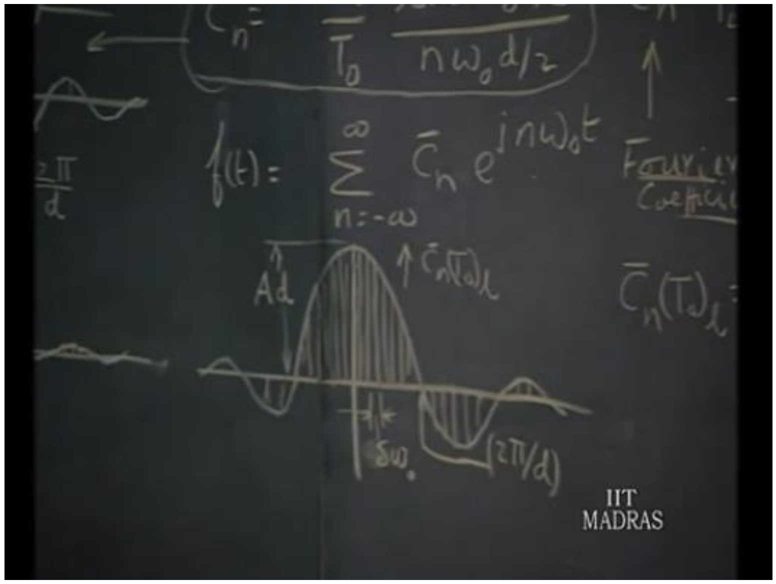

So, what you are plotting now will be having this envelop, which is exactly the same as this. But, the lines will become closer and closer. And what you are plotted here is, C n time T 0 of l large T 0. And the amplitude will be, since A d is multiplied by T 0. This will be A d. And the spacing between two particular lines becomes delta omega 0.

And this frequency corresponds to 2 pi upon d, because that is not changing, d is the same. So, we now see, that if we do not want to deal with C n, but C n times T 0 of l. Then, we can think of a meaningful way of representation the various harmonic components and what is the relative size.

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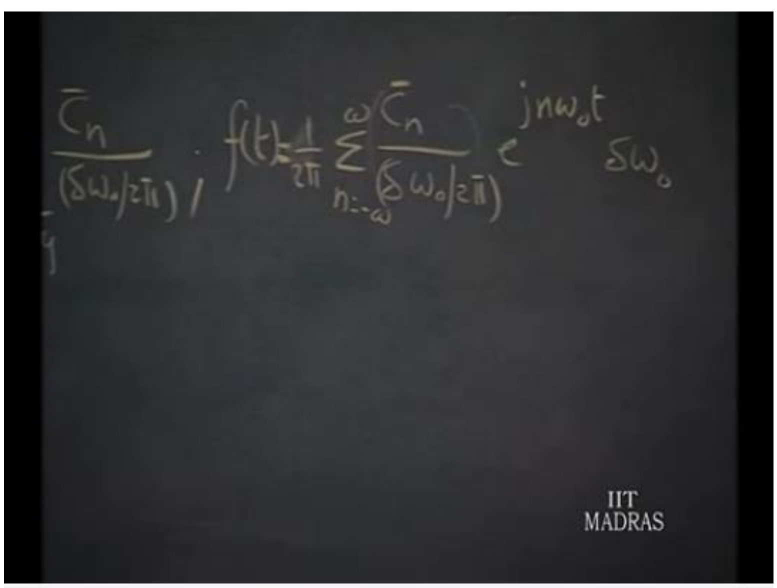

Since, this is C n. The coefficient per unit frequency, this is called coefficient density. Whereas, this is Fourier coefficient, this is called Fourier coefficient density. This is called Fourier coefficient density and once, you have the concept of coefficient density. Then, f of t can be written as further. I will write here, f of t can be written as, you recall f of t is the Fourier series expansion is this, n minus infinity to plus infinity C n, e to the power of j n omega 0 t.

But, instead of C n, I would like to talk in terms of the coefficient density. So, I will write this as, C n upon delta omega 0 over 2 pi; e to the power of j n omega 0 t, sum from n minus infinity to plus infinity. And since, I have divided by delta omega 0 by 2 pi. I have to multiply by delta omega 0 and then, I have to write here 1 over 2 pi. So, that would be the form of the Fourier series now, in terms of coefficient density rather than the coefficients.

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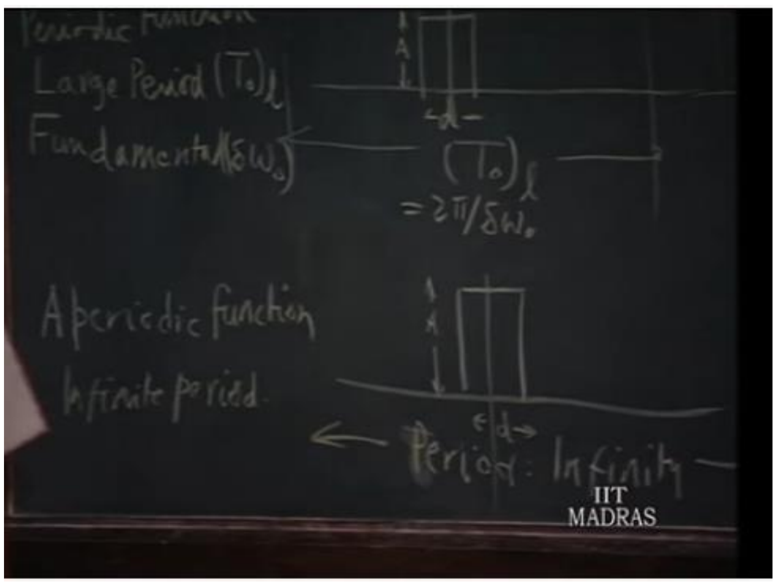

Now, let us see what happens, when you have a periodic function. In the periodic function, we still take this same function of same pulse of amplitude A, duration d. But, there is no other pulse involved, the period is infinity. So, infinite period, so imagine that we have this situation here, where now delta omega 0 tends to 0.

So, what this consequence of this T 0 goes to infinity, delta omega 0 become smaller and smaller. So, what happens the spectrum now, A d is independent of your T 0. So, that remains the same. This 2 pi by d remains the same. All that happens now is, as you increase the period, delta omega 0 become smaller and smaller.

These lines come closer and closer will be more lines. And the limit, you will find a particular line at every point on the x axis. So, that in other words, you will get a frequency component at every point on the x axis rather than the discrete point along the x axis. Because, delta omega 0 become smaller and smaller. So, the line gets crowding almost next each other at every point on the curve.

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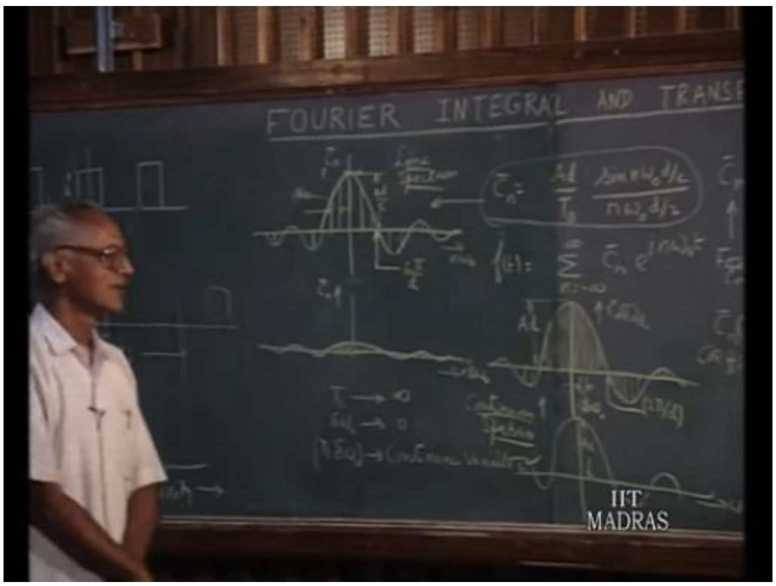

And therefore, we can say n delta omega 0, which n delta omega 0 what is significant n delta omega n delta omega is the frequency at which a line exists in this spectrum now a line theoretically exists at every point on the spectrum.

So, n delta omega can be replaced by continuous running variable omega. So, that is how the limiting process occurs as the fundamental frequency become smaller and smaller the harmonic exists at every point along the omega axis. So, in the limit you will have a frequency component at every point along the omega axis.

So, the spectrum now would be just like this. But, a continuous spectrum, because you have at every point, there is a meaning for this amplitude. So, this is the continuous spectrum and the dc value here it is A d. And x axis in terms of omega as before, every where is the omega, this is n omega 0 of course here.

This is n delta omega 0 and here, the running variable is the omega. Right, so what we have here is, when arrangement where you have every frequency presents. But, the coefficient density corresponding to each frequency is different. That is dictated that is given by the spectrum.

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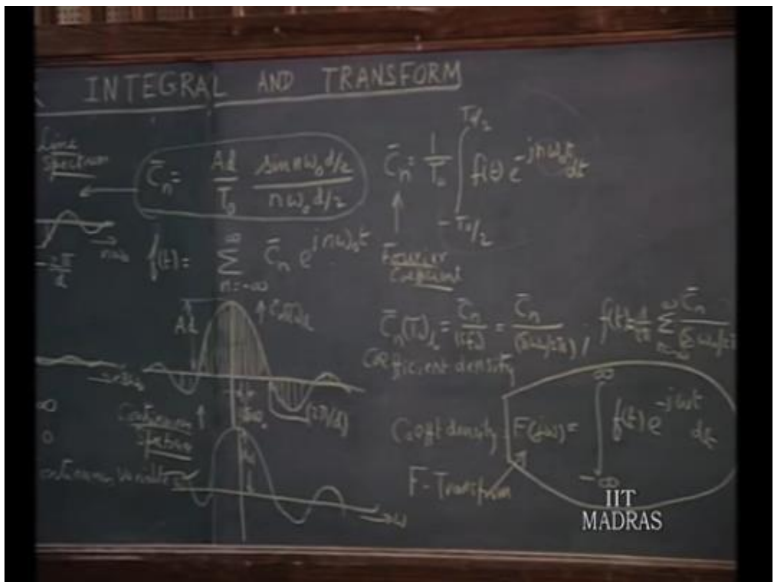

So, let us see what happens, the coefficient density is now, C n times T 0, wherever we are having is called now F of j omega. That is F of j omega. How do we get now F of j omega, here you observe, that in the C n is given by this formula, the coefficient density is obtained by multiplying C n by T 0.

So, once, you multiply C n by T 0, what you are having is, the integral relation minus T 0 2 plus T 0 upon 2, f of t, e to the power of minus j n omega 0 t, d t. So, in our case it happens, because T 0 is multiplied by this. So, what you here is, it is this formula, applicable to the periodic case.

And what is the period T 0 by minus T 0 upon 2 and plus T 0 upon 2 are infinity. So, you are integrating F of t multiplied by something over a complete period, which is extended from minus infinity to plus infinity, f of t and e to the power of minus j n. N omega 0 now, in the first case, it is replaced by n delta omega 0 here. And when you go to a periodic case, n delta omega 0 is a continuous running variable omega 0.

Therefore, this will be the f, e to the power of minus j omega 0 d t. So, the coefficient density in the A periodic case will be f j omega, which is minus infinity to plus infinity of f of t, e to the power of minus j omega 0 t d t. And this is called the Fourier transform of f of t. This is called the Fourier transform of f of t.

Now, what is the corresponding result for the Fourier series that means, Fourier transform is an equivalent of what we had for c of n. The formula to getting the Fourier coefficient, we had some formula. The corresponding formula here is, the Fourier transform, instead of C n, we deal with F of j omega. Why I will explain for a moment later, about the notation.

Now, what is the result we have corresponding to this, f of t in the periodic case is expressed as the sum of different exponential functions. Now, what is that we have here, f of t as you recall in the case of large period is this. So, now, we have to take the limit of this as delta omega 0 becomes goes to 0. So, limit of f of t, limit of this series as delta omega 0 goes to 0, means we have to take the integration. After all, C n omega 0 by 2 pi, f of j omega, this is F of j omega.

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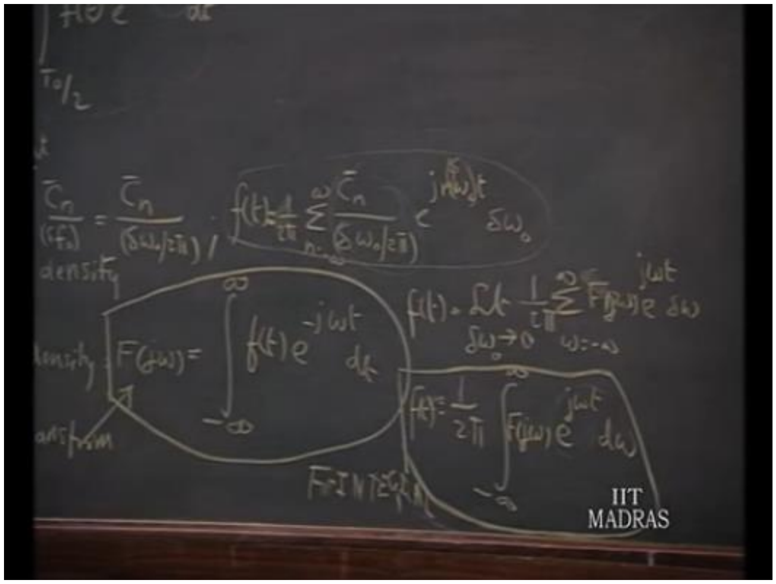

Therefore, we have to write here as the f of t, write it here. Limit as delta 0 omega 0 goes to 0 of 1 over 2 pi, the summation of C n over delta omega 0 over 2 pi, we call that f of j omega. So, f of j omega, e to the power of j omega t, because n omega 0 becomes, n delta omega 0 becomes.

Here, also delta omega 0 in the large case, n delta omega 0 becomes omega and then, this is delta omega. And the frequency omega, because n from minus infinity to plus infinity. That means, omega is minus infinity to plus infinity. So, thus really becomes 1 over 2 pi. So, that summation in the limit becomes integration from minus infinity to plus infinity and omega f of t e to the power of j omega t d omega, this is f of t.

So, what we have this is f of omega I am sorry, this must be F of j omega the same. So, instead of Fourier series for f of t, instead of f of t is being thought of the summation of several exponential functions like this. What we are really having is, f of t is consisting of similar exponential functions, e to the power of j omega 0 t, where omega can take any value from minus infinity to plus infinity.

Each of these is multiplied by a coefficient, which is F j omega d omega by 2 pi. And you are integrating that over the complete omega axis from minus infinity to plus infinity, this is f of t.

And this is called Fourier integral, this is called Fourier integral. So, instead of series, you have an integral and instead of a formula like this for calculating the Fourier coefficient.

You are calculating the Fourier coefficient density, which is referred to as a Fourier transform. Now, a few words about this, why we get this 2 pi here. This 2 pi arises, because we are calculating the Fourier density, coefficient density in terms of frequency rather than omega 0.

This is how it is conventionally defined, therefore, because we are talking about the coefficient density. In terms of having the denominator frequency rather than in hertz rather than radius per second, we get this 2 pi term. And secondly, we call this F of j omega instead of f of omega. Because, later on, we like to relate the Laplace transform, where this f of j omega can be a thought of as a special case f of s, we deal with Laplace transform domain.

Therefore, it would be convenient for us to put this f as j omega, because after all, when you talk about the exponential terms here. Every j is accompanied by every frequency is accompanied by j. So, j omega would be all right. You can think of f of omega as well, but f of j omega would be more convenient for us to use.

Now, let me quickly summaries, what we have done up to this stage. We started with a periodic function. The formula for finding the Fourier coefficients for the periodic function is this, we know general function. And once, we have the various coefficients, we can construct the f of t by combining all such exponential terms from minus infinity to plus infinity. This is the Fourier series and this is the integral; that we use to calculating the Fourier coefficient.

Taking a particular case of a periodic pulse strain, we showed that it has the spectrum like this and the characteristic of the spectrum is, that is the line spectrum. You have coefficients only at integral multiples of the basic frequency. Therefore, this spacing between 2 adjutants lines is the fundamental frequency omega 0 in radians per second.

Now, then we enquired as to what happens, if we keeps this particular pulse constant at the center around the origin and allow this T 0 to become larger and larger. So, that is the transition, before we go make T 0 equal to infinity to see the trends. So, as T 0 becomes larger and larger, we observe that the amplitude of this various C n coefficients become smaller and smaller.

So, the spectrum, line spectrum we had becomes shallower and shallower. It will be of this order. It will be like this and the height will diminish as T 0 increases for them. But, the another fact, that we noticed was, the spacing between the lines now. Instead of omega 0, it was the fundamental frequency, earlier becomes smaller and smaller.

Because, as T 0 increases omega 0 becomes smaller and smaller. But, this is not meaningful to us in the limit, when T 0 becomes infinity, because when we are moving towards the direction in which the period is considered to be infinitely large. So, if the period is going to infinity large, this whole thing become 0. And this C n will become 0.

And therefore, we will now, no way of knowing, what is the relative sizes of two different frequency components. So, to beat this problem, what we said was, instead talking about the Fourier coefficient C n, let us talk about Fourier coefficient density, which is C n divided by delta f 0. So, if you request this new quantity, which is C n times T 0 the period.

Then, immediately the trouble that arose, because T 0 being the denominator will vanish. And then, we had C n T 0 will be minus T 0 upon 2, T 0 upon 2, f of t, e to the power of j omega 0 t d t. And immediately this spectrum would be like this, corresponding to this periodic pulse strain.

So, you have depending upon the pulse width and amplitude, you have certain value A d. And this no longer, as period becomes larger and larger. This no longer has to become smaller, A d will remain the same, the line will become closer. And the spacing between two adjutant sides, delta omega 0, which is the new fundamental frequency, which is quite small.

And we can put f of t as 1 over 2 pi times, this coefficient density times e to the power of j n delta omega 0 t. I am writing this as times delta omega 0. So, the Fourier series now will be in this form. But, instead of C n, we are talking about we are talking in terms of coefficient density. Now, at this stage, we take the limit as T 0 becomes infinitely large.

So, if T 0 becomes infinitely large, we observed that delta omega 0 become smaller and smaller. That means, the line, the spacing between the lines becomes 0 in the limit; you have a frequency component at every point on the omega axis. So, what we started as a line spectrum now becomes continuous spectrum. That means, almost every single frequency is presented.

Of course, almost everywhere, because at this point, we can say this is 0. But, some other, nearly everywhere, you have the frequency component present. And what we are plotting now is 0 anymore the Fourier coefficient C n, but the coefficient density. And we give the new name for the coefficient density, we call that of j omega the Fourier transform, which is given by the formula minus infinity to plus infinity of f of t, e to the power of minus j omega t d t.

That is F of j omega. To recover the original function f of t from the f of j omega. Instead of the Fourier series, we will have now, take the limit as the summation as delta omega tends to 0 of 1 over 2 pi times the Fourier coefficient density, times e to the power of j omega t delta omega. And that leads us to the formula like this, which is called the Fourier integral. So, basically, then we have when we are dealing with Fourier transform for a periodic function.

We have two basic integral to deal with, from the given f of t, we can generate this Fourier transform, which is really Fourier transform coefficient density as explained here. And once, we have the Fourier transform, we can recover the Fourier function of time through this integral. These are the two basic integrals for characteristic of the Fourier integral theory.

Now, we will look into the meaning of the coefficient density and the similarities. And the contrast between the periodic function and a periodic function in a greater detail at this point of time.