Networks and Systems
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Lecture-43

Application of Continuous-Time Fourier Transform to System Analysis

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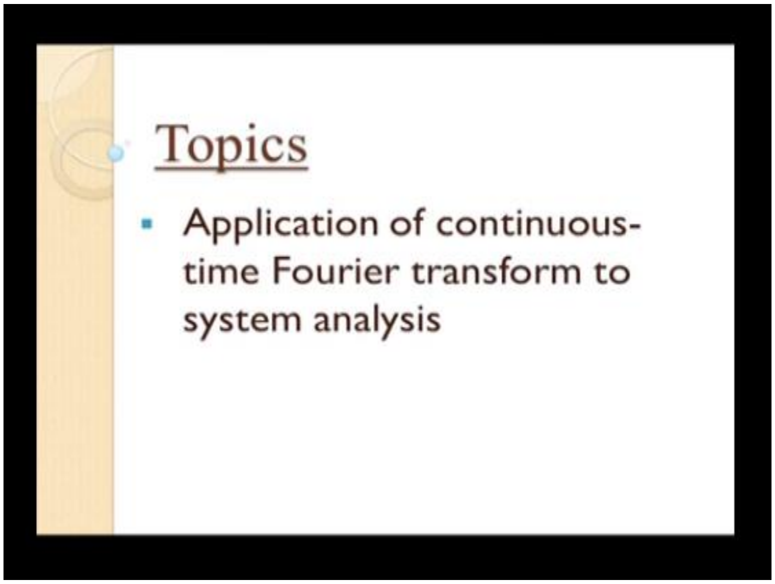

After having study the various properties of the Fourier Transforms. Let us now, discuss how this theory can be applied to the analysis of systems. So, we will like to see how the Fourier Transform theory can be used to find out the output of a system corresponding to various type of inputs.

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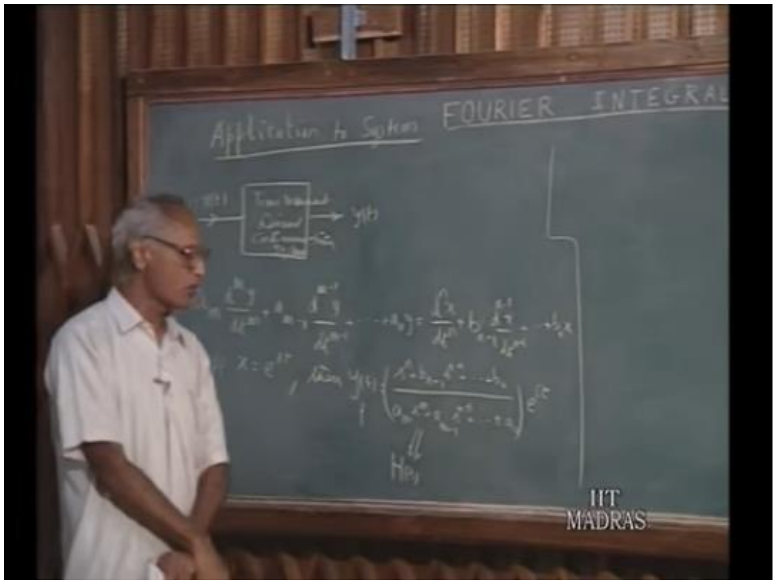

So, in the system terminology, if we represent the input and the output as x of t and y of t respectively, this is the linear continuous constant parameter system. In general the input and output are related by the differential equations of this type dm y by d tm plus a m minus 1 dm minus 1 y dt m minus 1 etc plus a not y that is: equal to d n x dt n plus b n minus 1 d n minus 1 x dt n minus 1 etc plus b not x.

This is the type of differential equations that, in general you have for a single input and single out put constant parameter time invariant, linear, continuous time system, in such a system if the input x is the e to the power of say st. Then the force responds will be obtained by taking the instead of d operator you multiply equivalent of s operator that means; is equal to s to the power of n b n minus 1 s to the power of n minus 1 etc plus b not divided by a m sm plus a m minus 1 s m minus 1 plus a n multiplied by e to the power of st.

Where this is the input and this is the corresponding output this is the force responds the particular integral solution of the differential equation and this we observed is called the system function of h of s. We are in particular interested in the system function evaluated when s equal to j omega.

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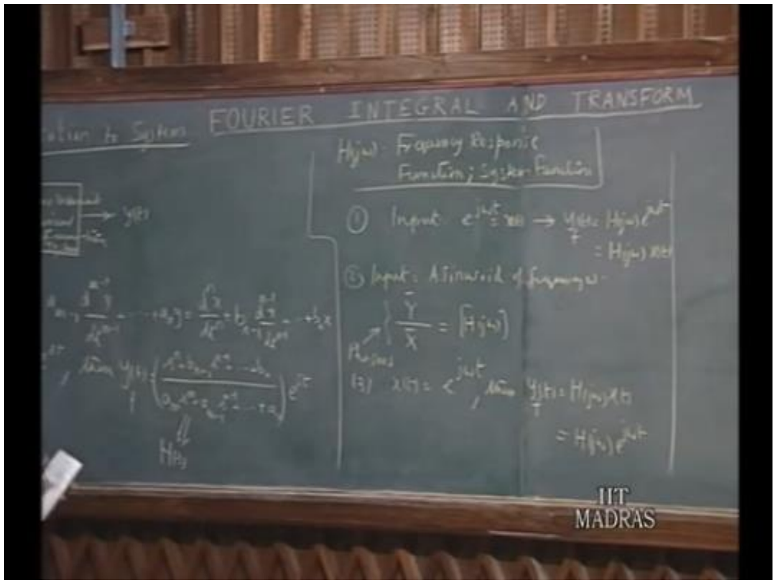

So, this is h of s h of j omega is the frequency response function also called system function so, often this is: itself called system function and this particular function this will make use of in the Fourier transforms applications to systems. Now, let us see the implications of the system functions 1 suppose, the input is e to the power of j omega t then this is xt it means the force response y f of t the particular integral is the solution of this differential equation.

The force response will be h of j omega multiplied by e to the power of j omega t that means; h of j omega multiplied x of t. So, the force response and the input are related by multiplicative constant h of j omega which is independent of time that is: why I am calling this as a constant.

A particular case a input is the sinusoid then the output phasor and the input phasor these are phasors whether the voltages are currents may be in the case of electrical network or in terms of general systems you can define phasors in the same manner both are sinusoid function of time this is given by h of j omega once again where, the sinusoid of frequency omega.

So, under steady state the output sinusoid and the input sinusoid their phasors are related by this h of j omega. So, that is the significance of the system function h of j omega, if you have an input xt which is an exponential function e to the power of j omega t then the force response y of t is h of j omega multiplied by e to the power of j omega t these after all both are related to each other.

And it is this property, which we are going to make use of in our Fourier Transform theory because we are picking up any given x of t as the summation infinite summation of time functions of this type. So, this is what we really want make use of.

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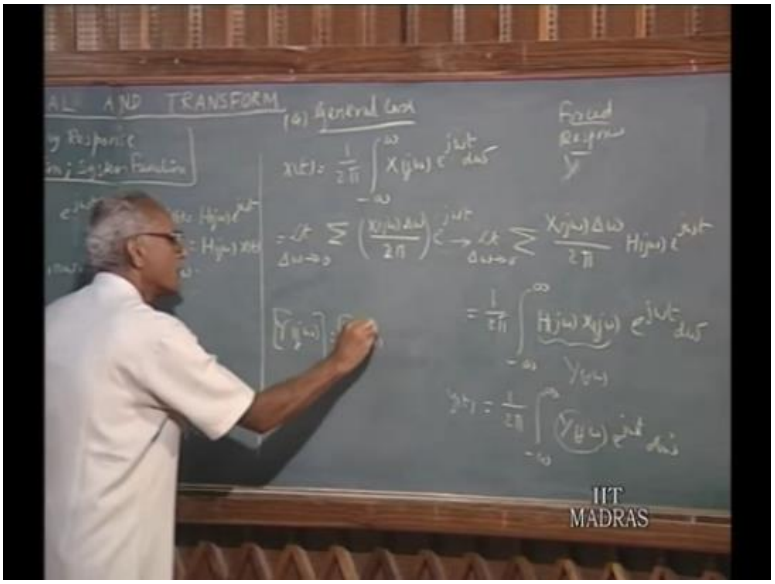

So, in a general case we have x of t is related to its Fourier Transform as 1 over 2 pi minus infinity to plus infinity of x j omega e to the power of j omega t d omega you recall that, this integral arose from considering a number of elementary signals of coefficient density f j omega lasting from minus infinity to plus infinity.

So, in fact we said this is: limit as delta omega goes to 0 a huge a lot of summation of huge a lot of terms x j omega delta omega over 2 pi this is being the coefficient density and delta omega over 2 pi is being width of the elementary slice of the spectrum that, we are taking this is: the coefficient and this is the function of time which takes. And if you take all such signals add them up you get this x of t that is, the general input.

So, general input xt it can be a thought of as the summation of elementary signals of this type each with an amplitude like this and this omega is running from plus infinity to minus infinity. Now from this you conclude since, if it is e to the power of j omega t the force response is h of j omega times e to the power of j omega t all we are doing is e to the power of omega t you have an particular amplitude certain coefficient.

So, the force response will be x j omega delta omega over 2 pi times h of j omega e to the power of j omega and once again here also, we have that limit this is the response for the single term but, we have whole a lot such terms limit as delta omega goes to 0. Since, this integral is viewed as the limit of such quantities the limit of such quantity can also viewed as the limit of such quantities the limit of such quantities can also viewed as the integral.

This, will be equal to 1 over 2 pi from minus infinity plus infinity of h j omega times x j omega e to the power of j omega t and we observed that this is: in the where you form that, we have for its inverse fourier transformation. So, if the response of this is y of t then this can be written as this is your j omega therefore, y of j omega will be this is y of j omega then this will be equal to y of t.

Therefore, y of t which is given by this quantity can be a thought of as 1 over 2 pi minus infinity to plus infinity this is y j omega e to the power of j omega t d omega. So, that means y j omega so, if this is the inverse Fourier Transform of y of t y j omega must be equal to this. So, the conclusion is that the Fourier Transform of the output quantity equals the Fourier Transform of the input multiplied by the system.

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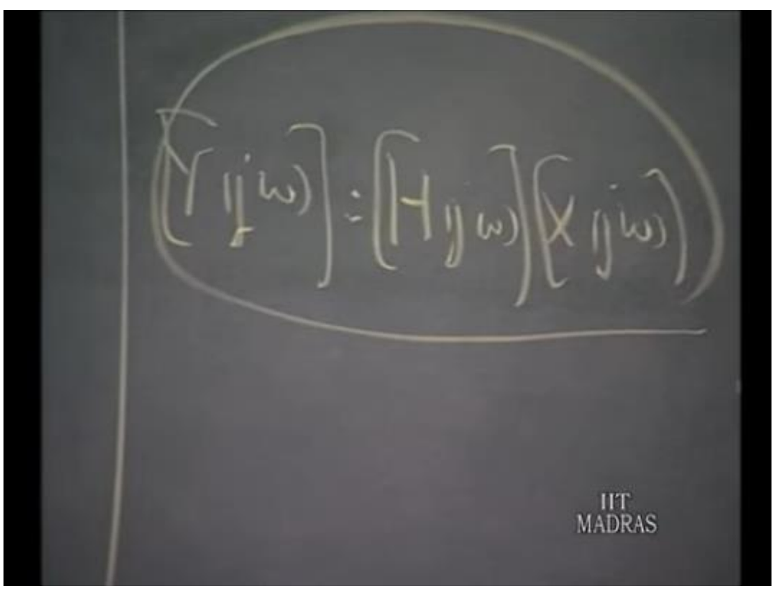

So, this is the very important rule the Fourier Transform of the output system here y of j omega will be equal to the Fourier Transform of the input multiplied, by the system function with h of j omega.

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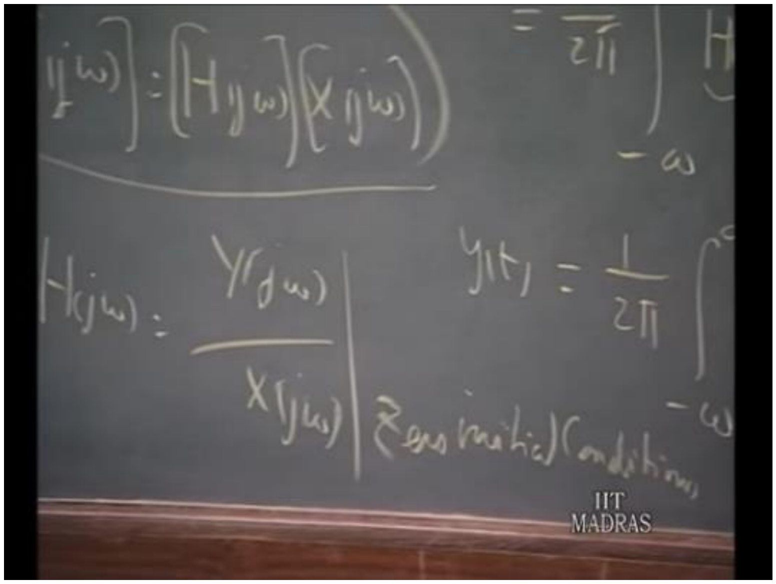

Now, 1 point is we have been taking about the force response particular integral solution and we explain later that, this is indeed will be the total solution because including the transience for the time being assume that, this is. So, I will explain it later point of time we can say the output will be equal to input Fourier out put transform will be input Fourier Transform with h of j omega of h of j omega can be thought of as the Fourier Transform output.

The Fourier Transform of the input with 0 initial condition I will talk about that later with 0 initial condition. So this is another way in which we can find out the response of a linear time invariant continuous system for a given excitation of the given input all you have to do is from the given input find it is Fourier Transform.

You know the calculate the system function and determine whatever, manner you have at your disposal multiply these 2 that will give you the Fourier Transform of the output. Find out the inverse Fourier Transform that, will give the response quantity that is how it goes 1 more property of the this system function is: if the input happens to be a delta function a unit impulse then, we know that x j omega equals 1.

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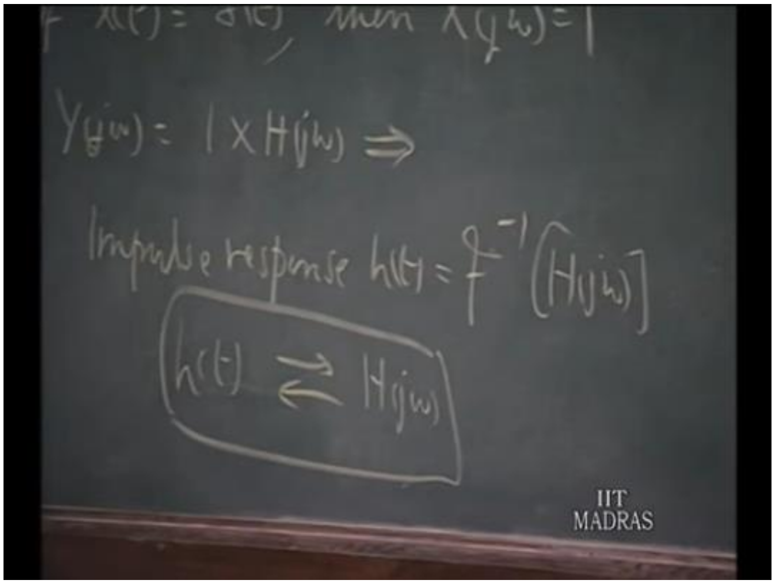

Therefore, the output Fourier Transform y j omega equals 1 multiplied by h of j omega that means; the impulse response, if call impulse response h of t that is the inverse Fourier Transform because under the impulse excitation the output has the Fourier Transform h of j omega.

Therefore, the time function corresponding to this will be inverse Fourier Transform amplitude or to put in other words the impulse response h of t and the system function form a transform pair this is indeed a very useful result. System function is the Fourier Transform of the impulse response or the impulse response is the inverse Fourier Transform of the system function.

Now, this idea ties in nicely with our convolution property you recall that earlier, I mention that in our introductory lecture, I mention that the output can be thought of as the convolution of the impulse response on the input.

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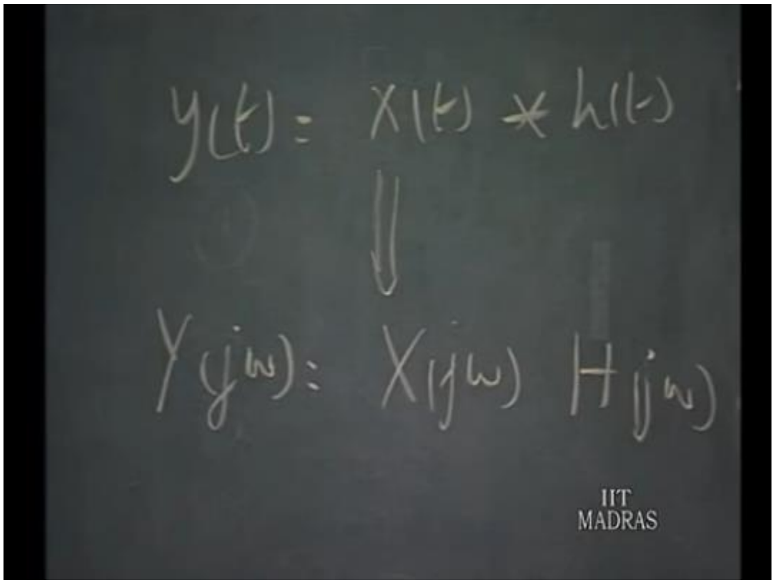

So, you recall that out put y of t is the input xt convolved with impulses and from the Fourier Transform theory we know that, when you have 2 type of functions convolved the Fourier Transform is obtained by taking the product of the Fourier Transform of the 2 functions which are convolved.

So, that is exactly what we had here y j omega h j omega times h j omega equals to of this so, these 2 going to right. Now, with this basic concepts relating to general system let us, apply these techniques let us apply this to a electrical network problem, where we like to calculate the output for different type of excitations.

Let us now, apply these concepts to evaluation of the transience in a particular network through an example, now what we are really doing is the given excitation input function is thought of as a number of elementary sinusoids starting from minus infinity and extending to whatever, point of time at which evaluate the response.

Now, if a sinusoid is given as the input network the force response is also a sinusoid and we are trying to find out the summation of the force response to the individual sinusoids. Now let us see, if the system does not have any initial energy to start we assume all capacitors are uncharged all injects do not have any 0 current to start with any non zero current to start with then all these sinusoidal inputs will give corresponding sinusoid responses and we like to add them up.

That is what we get by finding out y j omega as h of j omega times x of j omega where x of j omega h of omega is the system function which is obtained under steady state sinusoidal conditions. Now, in a differential equation we have both the complementary solution and the steady state solution which is steady state solution is being the particular integral solution.

Now by adding up all these elementary responses, we are adding up all the force response of the particular integrals solutions that we have from these various input functions. Now, what about complementary function since, we are starting our time the application of these inputs and t equals minus infinity and if the system has got a natural response which dies out with time which has got the natural response boards of negative values for the real parts.

That is: natural frequencies have negative real parts then what happens is starting from minus infinity they decay down to 0 when we are talking about the response calculating the response at finite time particularly from t equal to 0 and beyond and therefore, all the natural responses would have died out.

And what we are now dealing with is only the force response, that is: why total response is obtained by adding up the force responses of all these elementary sinusoids which are started minus infinity and continued maintaining their amplitude ride through the period up to plus infinity the natural responses would have died down.

This is true for all system whose natural response decay with time there may be certain difficulties where the natural response do not died out with time natural responses have a system natural frequencies have 0 real part. That means they have sinusoids of dc responses as natural responses then there will be some problems but, we will not be dealing with such cases as far as this course is concerned.

We assume that this system has a stable system with all the natural responses are being decaying with time in exponential manner. And then because they have started at minus infinity by the time we come to 0 in the decay down and it also can be shown that the inverse Fourier Transform of the response function through the contour integration on the complex plane it can be shown that suppose, you want to apply the excitation at time t equals to 0 and then want to know the response from t equals onwards for positive t.

That means the function has the excitation is 0 for negative t and we would also like to have the response also 0 for negative t because it is a causal system and if you take the inverse Fourier Transform of y of j omega it trans out to for such systems as we have studied the response for negative t is going to be 0.

Therefore, this is quite in accordance with the causality principle of the system and therefore, there is no necessity for a natural response terms to appear in the final solution. So, the overall to sum up them if you apply the Fourier Transform theory and then apply the steady state methods after all f j omega is the response is the ratio of the response to the input phasor under steady state conditions.

So, we are applying the steady state condition theory and I apply and obtain the natural the total transients response even though, we are applying steady state theory because all the natural response add up to 0 for negative values of time and then decay by the time we start counting our response taking account of the responses for positive t and.

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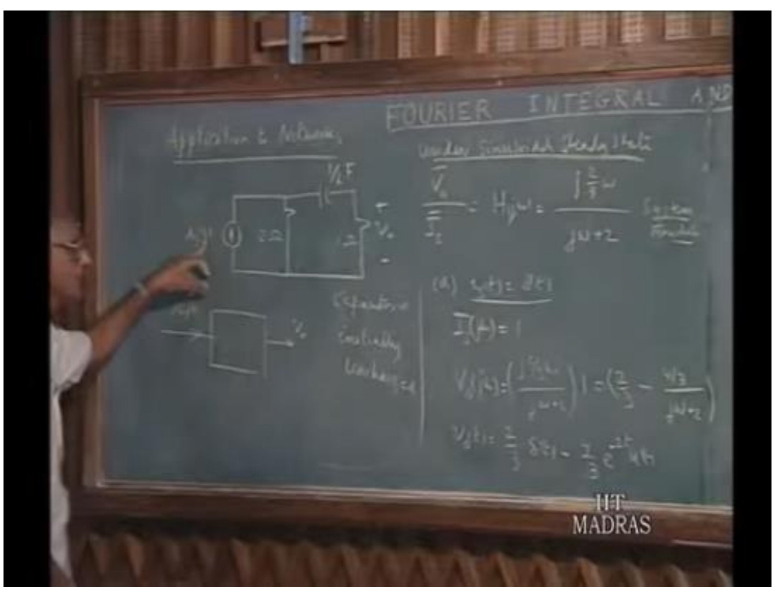

So, let us take an example, to illustrate this suppose, I have a current source I s of t which is given to a circuit like this, where this is 2 Ohm resister 1 by 6 Faradays capacitor and 1 Ohm resister and this is my out put that I desire. And this can be viewed as the system where I s of t is the input and v not is the output so, from steady state theory, we would like to find out what is h of j omega for this what is the ratio for the phasor v not to phasor I s

So, v not phasor over I s phasor under steady state under sinusoidal steady state this is our h of j omega once, you identify this as the output and this as the input we have the output of the input the system can be defined v not over I s you can work this out and so this is equal to j 2 by 3 omega divided by j omega plus 2.

Impedance and impedances you calculate the ratio this phasor the input phasor the response to any kind of excitation I s of t that, we have we assume that the capacitor is initially uncharged. So, let us take first a case where I s of t is the delta function then the input is the impulse function therefore, the input the Fourier Transform of the input equals to 1.

Therefore, the output Fourier Transform v not of j omega equals the system function which is equals to j 2 by 3 omega over j omega plus 2 times the input Fourier Transform which is equal to 1 and this I can write this as now, I have a function like this, I must find out the inverse Fourier Transform I must put this in a recognizable form.

So, this I can write this as 2 up on 3 minus 4 up on 3 over j omega plus 2 this is what we call a partial fraction expansion, we talk about this in greater detail, when we go to Laplace Transform but, we can easily see that this is equal to this. I try to break up this into 2 parts each of which can be recognized to be the Fourier Transform of known functions that is the general idea.

Therefore, v not t is obtained as two-thirds, two-thirds in the transform domain therefore, in time domain it two-third of delta t two-third of delta t and the Fourier Transform of that is 4 up on 3 e to the power of minus 2 t ut, because 1 over j omega plus 2 corresponds to e to the power of minus 2 t ut and you have minus 4 up on 3 as the coefficient. So, this is the total response of the system, when the input is a unit impulse.

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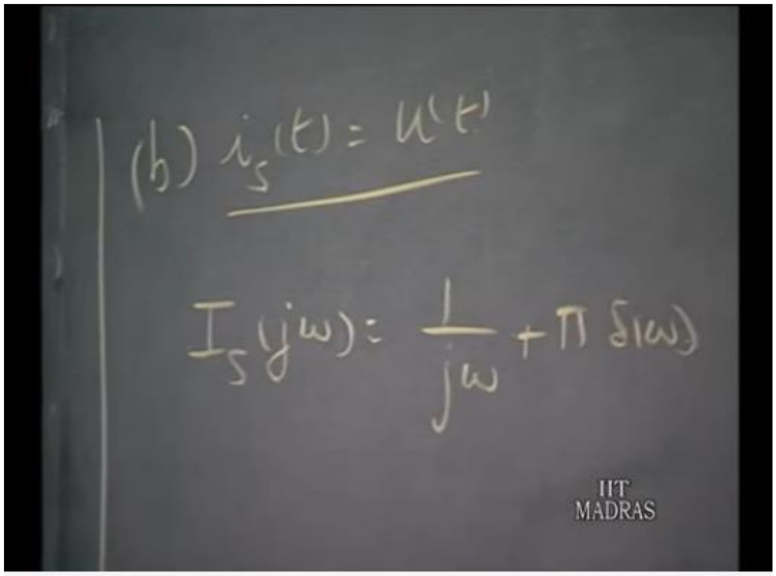

A second case suppose, I s of t is the ut then I s j omega the input Fourier Transform is 1 over j omega plus pi delta omega.

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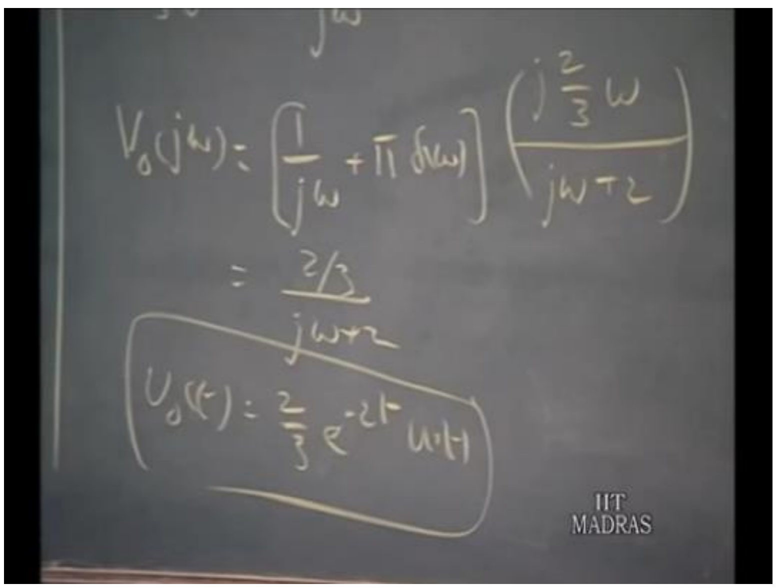

The output the v not j omega is 1 over j omega plus pi delta omega multiplied by this system function, which is j two-thirds of omega divided by j omega plus 2 and you can simplify this and show this to be equal to you see pi delta omega is multiplied by this become 0 because the whole function is 0 at omega equal to 0.

Therefore, this does not contribute to any term in the final result this will turn out to be 2 by 3 j omega plus 2 or v not of t equals two-thirds e to the power of minus 2 t ut. So, that is how you can calculate the transient perform of network using the fourier integral Fourier Transform concepts.

To summarize in this particular lecture, what we have done is we started out with the formula for integration and we saw how the integration rule is compatible with the differentiation rule in particular. We said any function f of t integrated from minus infinity to t is equivalent to multiplication in the transform domain by term 1 over j omega plus pi delta omega. And in particular when the transform f of j omega is 0, when omega equal to 0 the second term drops out.

Then, we saw how the Fourier Transform theory can be applied to general system studies and in the general system studies frequency response function h of j omega plays very important role as far the Fourier Transform is concerned. And h of omega is the value of the phasor of the output the phasor of the input at the particular frequency.

So, under using steady state method you can evaluate the h of j omega or either experimentally or through the analytical calculation. Once, you have h of j omega the output Fourier Transform and the input Fourier Transform are related by the system function h of j omega and I mentioned that, even though this is the ratio of the force response the Fourier Transform of the force response to the Fourier Transform of the input.

In many particular cases, many cases which are common interest by the natural frequencies decay, natural response decay with time this also gives us the total solution. And we took up a specific example to illustrate this idea and we took a r c network and found out the response of the r c network.

Response in the sense the output occurs 2 terminals calculated for 2 different excitations 1 is an impulse excitation, the other is step excitation and this shows that, the Fourier Transform theory can be applied to a evaluation of the transients and network, even though the Fourier Transform is essentially related to steady state behavior of networks. It can be made use of evaluate transients as well.