Networks and Systems
Prof. V.G.K. Murti
Department of Electronics & Communication Engineering
Indian Institute of Technology – Madras

Lecture-52
More Properties of Laplace Transform

The rule we have considered just now, is that if f of t is multiplied by exponential function of time that corresponds to a shift in the frequency domain who have a dual rule which corresponds to shift in time domain and that corresponds to multiplication by the exponential function in the frequency domain.

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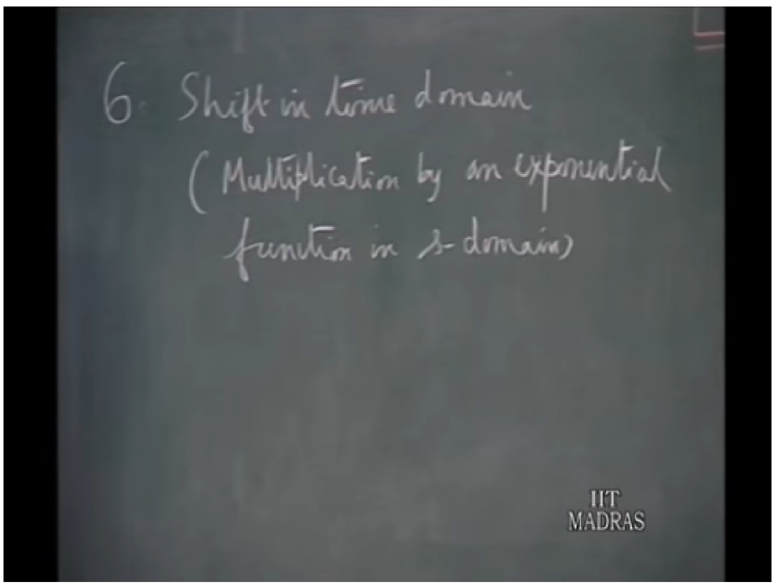

This is corresponds to multiplication by an exponential function in the frequency domain that is the complex frequency domain or s domain. Let me, first put down the rule and then we will try to demonstrate that.

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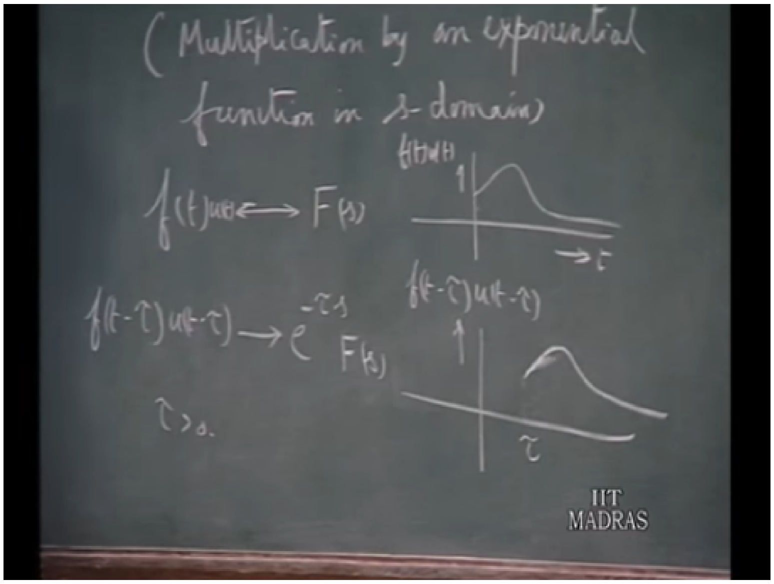

Suppose f of t and f of s form a transform pair. And let me assume that I will also put u of t to make sure that it is a causal function. Suppose, I have this is f of t ut now let me, say that I translate this I delay this function of time by an amount of tau seconds. So, you are starting from t equal 0 it starts from here and it goes like this so this is tau. The same function is delayed among tau therefore, we can describe this as f of t minus tau u of t minus tau, this is the description of this function.

Everything is delayed by toe seconds whatever, value this functions takes at t1 this will take at this t1 plus tau second in the seconds is delayed. So, what is the Laplace transform given the Laplace transform of this is f of s who ask the question what is the Laplace transform of this.

So, f of t minus tau u of t minus tau has the Laplace transform it hands out to be e to the power of minus tau as f of s. So, a shift in the time domain corresponds to multiplication by exponential factor in the frequency domain delay by tau seconds involves multiplication the frequency domain by e to the power of minus tau s. We take tau to be greater than 0 that is this corresponds to delay.

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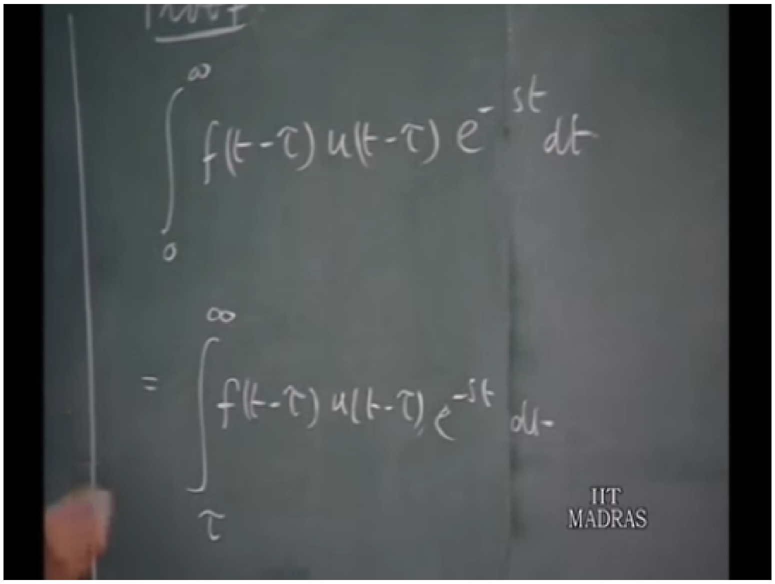

Now, the proof of this is quit straight forward. So, we are interested in filling the Laplace transform of f of t minus tau u of t minus tau and this to be multiplied by the e to the power of minus st dt that is what you are interested in final. Now since, the variable of integration is t as long as t is less than tau u of t minus tau is zero.

So, as long as t is less than tau the argument is ready to this makes the 0 therefore, I can as well start the integration from tau infinity of f of minus tau u of t minus tau e to the power of minus st dt. Because, this factor made the integrate 0 for values of t up to equal tau therefore, f of t minus tau u t minus tau e to the power of minus st dt and for values of t larger than tau than this is going to be 1.

So, you may as well leave it like that or you may even if you want to keep it in no harm in there.

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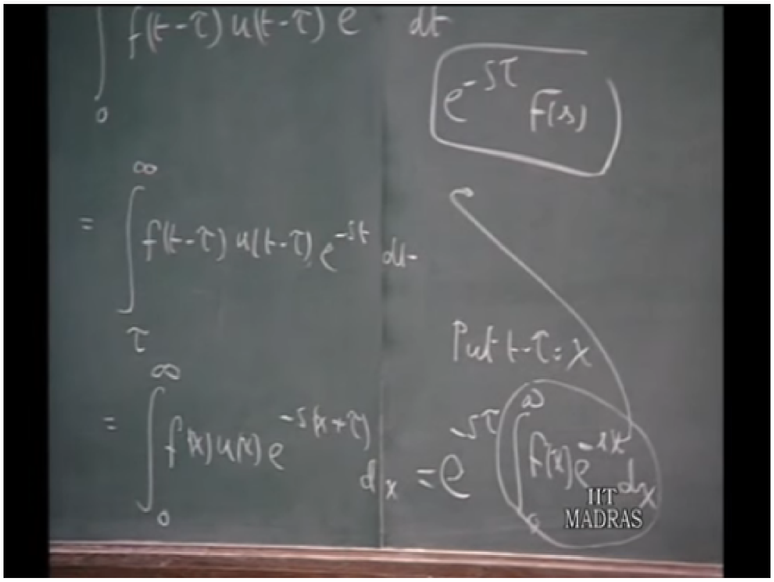

Now, let me put here p minus tau as let us say x then, this is f of x u of x e to the power of minus s t equals tau plus x x plus tau and dt equals dx. And the range of integration now when t becomes tau x become 0. So, it start from 0 and when t equals infinity x also be infinity because, tau is finite that is what you are having now, this integral involves the variation of integration x this tau is the constant.

Therefore, I can write this as e to the power of minus s tau 0 to infinity of f of x e to the power of minus sx dx and what you are having here is indeed you can also consider this 0 to infinity f of t to the power of minus st dt. So, that is the Laplace transform of f of t. Therefore, the result is e to the power of minus s tau this is what we mentioned here as the Laplace transform of f of t minus tau that is the proof of this.

This particular rule is very very important because, it comes in very handy in finding out the Laplace transforms how various typical functions that we handle like, pulse functions and so on as will take some examples to illustrate this. Let us, see let work out some examples.

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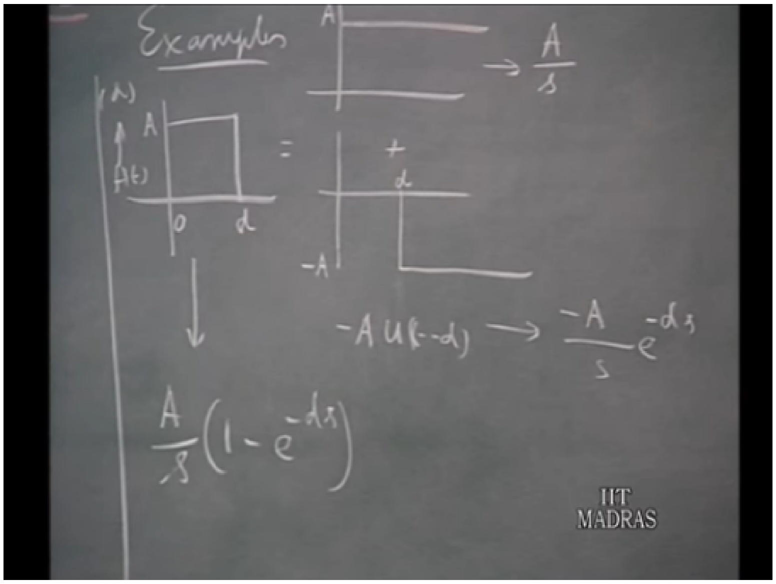

Let us, consider first of all a pulse function 0 to d it exits from 0 to d value a here this is f of t. Now, you can always consider this pulse to be a step function of amplitude a starting a t equals 0 plus another step function of amplitude of minus a starting at time p equal d. If you add this 2 step functions: 1 regular step function starting a t equal 0, another a delayed step function starting a t equals d adding of this 2 you will get this.

And we know the Laplace transform of this will be this is a not a unit step function. But the step function of amplitude or magnitude a therefore, the Laplace transform of this is a over s. Now, the Laplace transform of this is how do you describe of this is minus a u of t minus d this is the expression for this second function where the first function is a u of t. And therefore, it has the Laplace transform all we are now doing is step t delaying this seconds.

Therefore, the Laplace transform u of t minus d is 1 over s e to the power of minus ds you have to multiplying this a minus a. Therefore, the Laplace transform of this is minus a by s this would have been the Laplace transform if this is delayed did not occur.

Because, the delay we have e to the power of minus ds therefore, the Laplace transform of this is the sum of these 2. So, you have a up on s to 1 minus e to the power of minus ds. So, you are observe that whenever we have distinguish is in the function. Then, type of pulse functions like, this they can quit conveniently handle by using this property of the shift in time domain what it difference is make transform domain is all that means, is we have multiplying by exponential factor e to the power of minus s times the delay that is involved.

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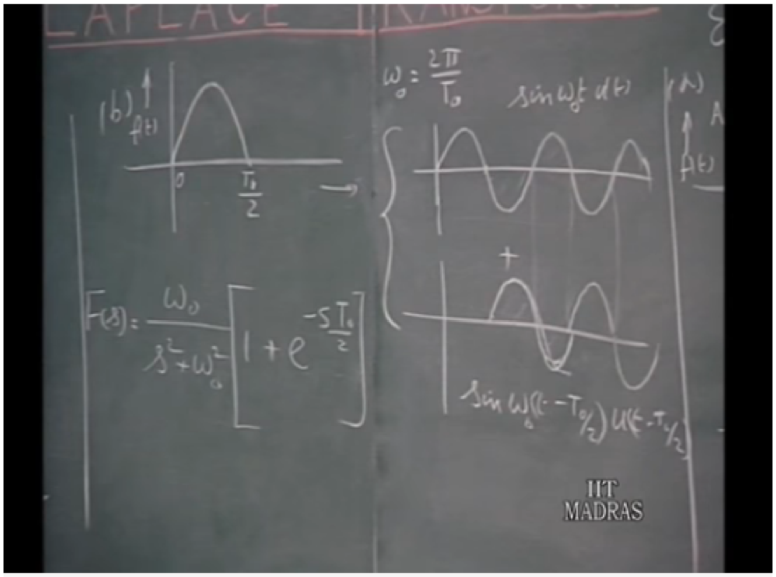

Let me, take another example b: Suppose I have 1 half cycle of a sinusoid just this 0 to t0 up on 2 and of course, will assume that omega 0 equal 2 pie up on t 0. So, you want to find out Laplace transform of this a single loop of a sinusoidal. Now, this can be considered to be a regular sinusoidal omega 0 t ut plus.

Suppose, I delay this sinusoidal by 1 half period so I get another sinusoidal like this and this would be sin omega 0 t minus t 0 up on 2 delayed by the amount u of t minus t 0 up on 2 that is delayed sinusoidal. Now, if you add this 2 this negative loop is cancel buy the positive loop, this positive loop have canceled by the negative loop and so on.

If you add this 2 this is what you get write. And therefore, if you add this Laplace transforms are have this 2, you will get the Laplace transform of this by the linearity principle if you talked about. Therefore, the Laplace transform of this single loop will be the Laplace transform of this which is omega 0 over s square plus omega 0 square plus the Laplace transform of this which is obtained by multiplying the Laplace transform of this by e to the power of s times minus s times delay that is involved.

Therefore, e to the power of minus s t0 up on 2 that is what we have that is the Laplace transform of the single loop you do not have to do any integration the advantage of knowing all the properties is that whenever, you are having functions of this type if you cleverly manipulate this, you do not have to carry out this integration to find out the Laplace transform from the Laplace transforms the known functions you can construct the Laplace transform of this purely to algebraic integers as shown here.

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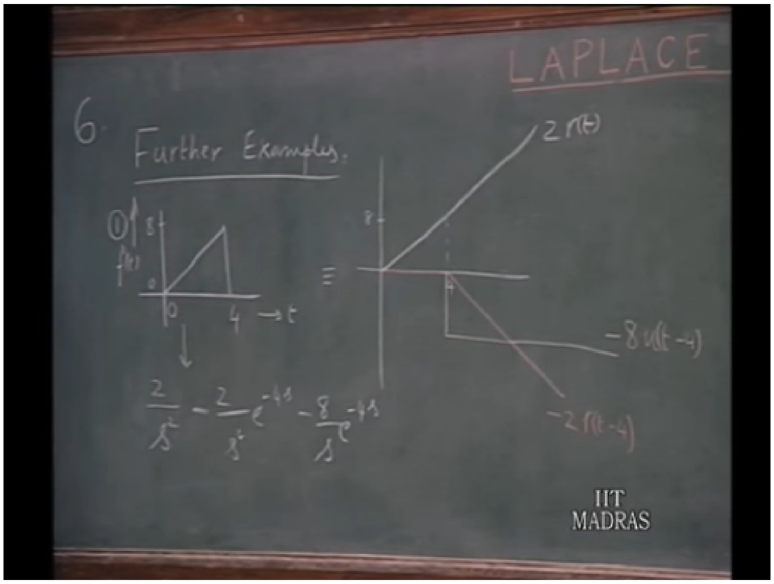

So let us, move on let me consider 2 further examples: 1 suppose, I have a triangular pulse function 0 to 4 seconds it reaches the values of 8 as t equals 4 and you like to find out the Laplace transform of this. Now, once again you do not have to any integration because, do you know the Laplace transform of ramp function and f functions.

So, you like to decompose this function in terms of appropriate step and ramp functions Let us, see how we do that suppose we describe this portion by a ramp function then you have a ramp starting at 0 reaching the value 8 at time t equals 4 and this would be described as 8 units it jumped by 8 units in 4 seconds.

Therefore, it is a slope of 2 so, this is 2 times r t that would be the ramp. Now, this ramp describes the actual behavior of f of t up to t equals 4 seconds. But at this point we must nullify whatever is left beyond 4. So, to do that we introduce a negative ramp here at 4 seconds of the same slope so that, the growth is arrested.

Therefore, this would be having the negative slope of minus 2 and then, its pair are t equals 4 seconds. So, that would be description of this curve minus 2 r t minus 4 would be negative going ramp with the slope of magnitude 2 starting that t equal 4. If you added this 2 you what your get is this 1 plus a constant here because, whatever increase here is cancel by the increase here.

So, what is left by adding this 2 curve is not only, this triangular pulse. But also a kind of constant a step here. So, we should cancel the step as well and the magnitude of the step is 8 minutes and starts it 4. So, in addition we need to have a step here a negative step. So, this will be described as minus 8 u of t minus 4 so if you added this 3 characteristics. Then, you would get this pulse you have a ramp function to start with negative ramp growth.

Then, afterwards among to which ever going to must alps be canceled out that will be minus 8 u to the power of t minus 4. Now, you know the Laplace transform of this 3 functions. So, the Laplace transform of this would be the Laplace transformer of the ramp which is 2 up on square. And then, the Laplace transform of this ramp is if this is 2 up on s square, this is 2 up on s square times e to the power of minus 4s.

Because, the same ramp with an negative sign delayed by 4 seconds. Therefore, minus 2 up on s square e to the power of minus 4s that would be the Laplace transform of this delayed ramp the Laplace transform of this step function is minus 8 it would have been started a t equals 0 it would have been minus 8 by s.

But since has been delayed 4 seconds you have to write minus 8 by s e to the power of minus 4s that is what you are having 2 up on s square times 1 minus e to the power of minus 4s minus 8s power e to the power of minus 4 this is the how it goes.

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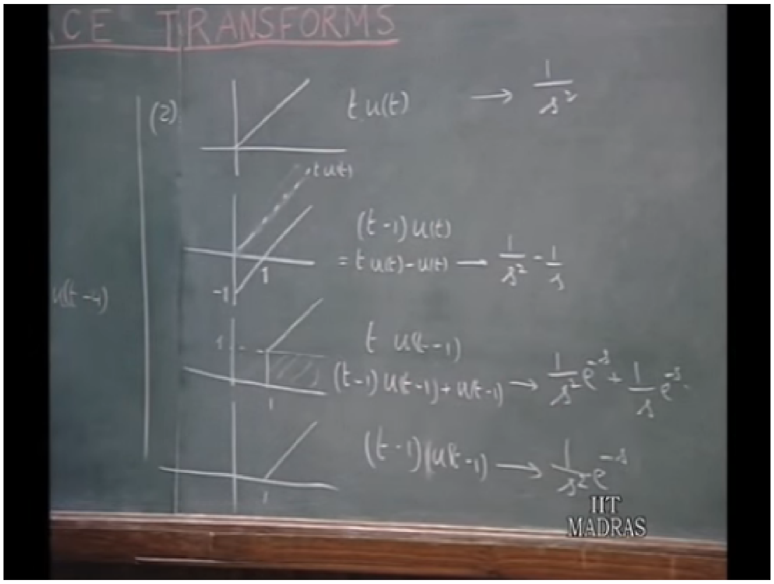

Second example now, you must be careful when we are talking about the rotational for the delay time function that you f of t minus tau times u of t minus tau use an proper expression for the delay time function to distinguish between other possibilities. Let me, take a set of example this would be tut suppose f of t is t then, this is but suppose I write t minus 1 u of t what does it mean?

This is not a delayed function because, u of t is not been replaced by u of t minus 1 t minus 1 ut would be this 1 this will be at p equals 0 it would have been minus 1 that is what it could be the same slope of course. But it does not start t equals 1 this is not this is not delayed version of this important recognizer that.

Suppose, I have t ut minus 1 so it is this is t ut, but the it t is multiplied by ut minus 1 that means, the function t is multiplied by a step which is starting a t equals 1 at this point. Therefore, this would have been this would have been t, but this portion as in cut of. So, that would be t ut minus 1.

So, this is 1 and that is t ut on the other hand, if you have t minus 1 ut minus 1 this would be whatever sequence of values this had it takes 1 second later therefore, it is a proper delayed function this same function is delayed by 1 second therefore. So, you see you have presumably this delayed function but there not the delayed function of this.

So, you must be careful indistinguishing between the various function that covered by this expressions. Now, the Laplace transform of this is of course 1 over s square very clear now this is the delayed function therefore, you can immediately right 1 over s square e to the power of delay is 1 second that 1 over s square e to the power of minus s there is problem of more that.

Now, what about this t minus 1 ut can be written as t ut minus t ut; that means, you have t ut u remove 1 unit value at every point of time that this is what we get t ut this would be parallel to this. So, if you remove u of t from that this curve is as therefore, Laplace transform of this is 1 over s square minus 1 over s Laplace transform of t ut os 1 over s square Laplace transform of ut is 1 over s.

So, as for as this is concerned t ut minus 1 you can think of this as t minus 1 ut minus 1 plus ut minus 1 add this 2 that is what you get. So, t minus 1 ut minus 1 is a proper delayed function of this. Therefore, this is 1 over s square e to power of minus s and ut minus 1 then, we have the Laplace transform 1 over s e to the power minus s.

So, you can see that this 3 functions now different Laplace transforms they are not the same time functions of course. Therefore, we except Laplace transform to different you can also see this as this can be thought of as a delayed step plus a delayed ramp that is what we have delayed step right. Now, before we proceed to a discussion of another property let me, caution you about 1 possible pitfall that 1 might fall into.

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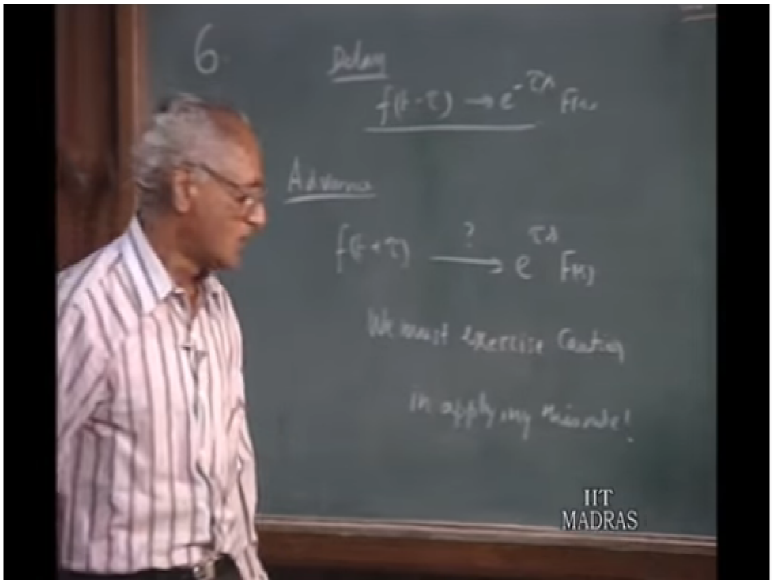

We agreed that, if the function f of t is delayed by tau seconds f of t minus tau this will have to the power of minus tau as f of s that is the Laplace transform of this. Now the question that, ask you what happens to advance in time. Suppose, we have a function f of t you advance it so that means, f of t plus tau; tau is once again consider to be positive does it mean that this will be e to the power of tau s f of s.

So, this the question that you like to ask we must exercise caution in applying this rule why? When you advance a time function what happens is when, you push some part to the time function to the negative values of time. Then, a part of the function can get last therefore, it is not a if delayed them the same functional variation whatever we had f of t ut continuous.

But you advance it in the negative direction that is the negative direction of a time that means, the part of the function can be get last. So, therefore, whatever is left may not have a complete replica of the original f of t ut therefore, this rule may or may not be work.

So, 1 must be carefully this as for as Fourier transforms are confirm the delay if the delay are advance tau seconds is multiplied by the e to the power of j omega tau or e to the power of minus j omega tau as the case may be without any difficulty. Because, the

Fourier transforms angle time functions which exists from negative values of time as well, but not Laplace transforms.

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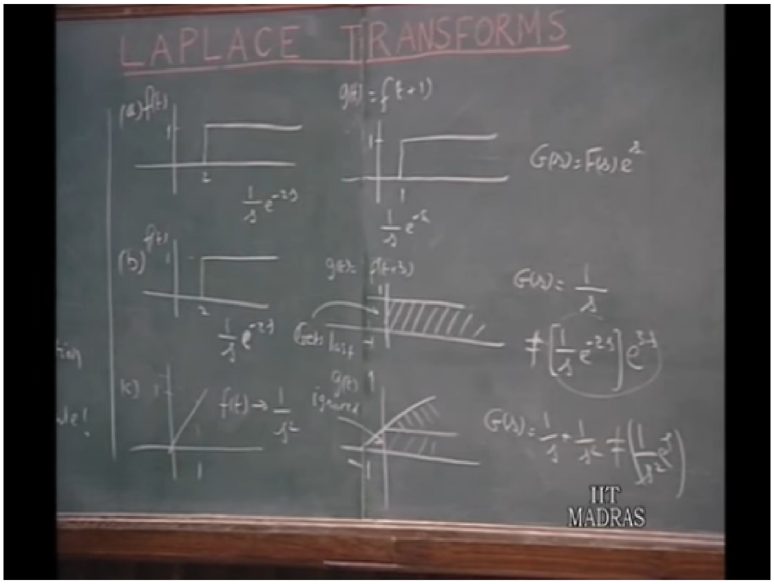

So, let me illustrate by means of an example suppose f of t is like this is 2 seconds and then, I have let us say g of t which is obtained as f of t plus 1. So, f of t is a delayed step function starting a 2 units and g of t is advanced by 1 second is f of t plus. The Laplace transform of this is obviously, 1 over s e to the power of minus 2 s the Laplace transform of this is 1 over s e to the power of minus s.

Therefore, we can see in this case the g of s is f of s multiplied by the e to the power of s corresponding to 1 second advance therefore, this rule works quit well here. But because, when you are advancing them as nothing of this function is getting loss right. Now, suppose you have the same f of t and now I say g of t now is f of t plus 3 that means, this is advance by 3 seconds.

That means, it is like this when you take the Laplace transform of g of t however you ignore this portion for negative values time and take account only of this. So, this portion is gets neglected it gets lost this portion. So, we would say g of s is a Laplace transform this of course I take this all the period magnitude.

So, we will take the this the Laplace transform of unit f function this is 1 over s. Now, f of s is 1 over s e to the power of minus 2 s the function is advanced by 3 3 units therefore, this is not equal to 1 over s e to the power of minus 2 s which is the Laplace transform of f of s the advance corresponds to t s.

So, this would be what would have expect if the same rule applied for advance functions as it would applied for delayed functions. So, if here multiplied by e to the power of 3 s you got 1 over s e to the power of s, but it is not equal to 1 over s. Therefore, this is not equal to this; this is in correct expression for this as for the Laplace transform is concern.

Let me, take a third example suppose you have f of t which is a ramp function. So, f of t here will give me 1 over s square. Suppose I deal advance this by 1 second so, this I will call it g of t. Now, once again now you are considering a Laplace transform of this, this portion is ignored and you are considering already this portion and that is equal to a step function plus a ramp function of low.

Therefore, the Laplace transform of the g of s would be 1 over s plus 1 over s square and this is certainly not equal to 1 over s square multiplied by e to the power of s corresponding to the advance therefore this is not equal to this. So, when we are advance in time functions 1 has to be very careful whether, this particular rule like this will apply or not.

If the advance correspond to a situation where no portion of the original function of s has been truncated as it has been in this case then that rule will apply. But in other cases like this which is more usually the situation this rule will not apply. So, as for this rule we talked about primarily intended for delay time functions. When, it is advanced 1 has to take it with a pinch of salt.

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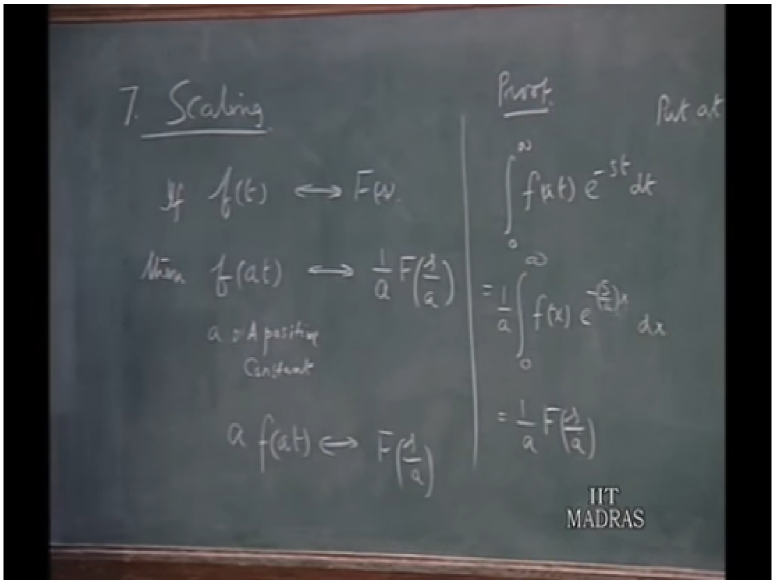

The next property will talk about is scaling what we mean by this is f of t and f of s form Laplace transform pair. Then, f of at where a is a positive constant that means, you scale the time events we make them appear faster f of at this corresponds to a Laplace transform 1 over a f of s over a is the rule.

So, scaling in time domain also corresponds to scaling in the frequency domain, but in the inverse fashion. F of you make events faster that will be slower and wise versa, But of course with the multiplying factor here proof 0 to infinity of f of at e to the power of minus st dt that will can be written as if you put at as x this will be f of x e to the power of minus s x up on a dt equals dx up on a.

So, put this once and when t equals 0 x is 0 when t equals infinity x is also infinity. So, always having this f of x instead of e to the power of minus x you have x up on a or s up on a instead of s t you have s up on a time c x s up on a time p x is the done the variable we can replace by t as well.

Therefore, what we are having here is instead of s you are really having s up on a times this x which is the variable of integral. So obviously, this is the Laplace this is if f of x is the Laplace transform of f of t this would be f of s for a. Therefore, 1 over a f of s by a that is what we have do here or you can put this a times f of at as the Laplace transform of f of s over a it is the same thing.

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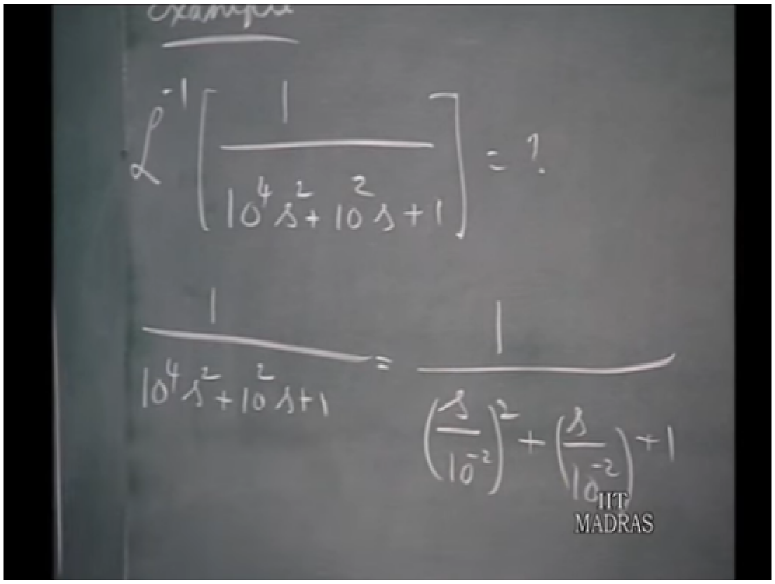

Now, this particular rule comes in handle sometimes in numerical working like this that means, illustrate the given example. When, you have large numerical constant sometimes we can avoid this by suitable scaling. Find the inverse Laplace transform of 1 over 10 to the power of 4 s square plus 10 to the power of s plus 1 of course I ask this question.

What is the inverse Laplace transform of this? Who out to find this out? So, you can scale this in the frequency domain in this fashion 1 over 10 to the power of 4 s square plus 10 to the power of s plus 1 can be written as 1 over s up on 10 to the power of minus 2 whole square.

Suppose, I divide this by the same thing s square divide by 10 to the power of 4 or 10 to the power of 4 you can have the denominator is 10 to the power of minus 4 this same thing and especially in this fashion. And then, this can be written s again s up on 10 to the power of minus 2 which is same thing as 10 square s plus 1.

So, if I had scale this s by the factor so that s up on 10 to the power of minus 2 is replace by s then, this is 1 over s square plus s plus 1. We know the Laplace transform 1 over s square plus s plus 1 once we know that, we can find the inverse Laplace transform of this by applying this scaling rule.

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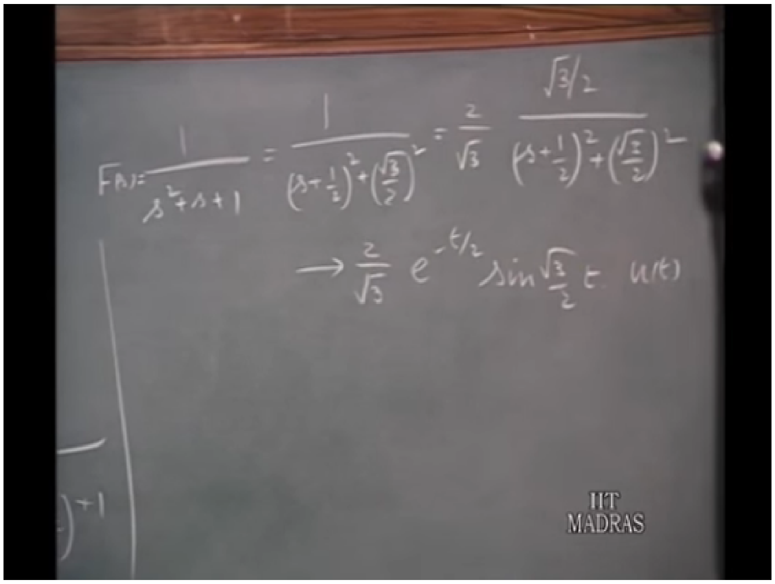

Let us, see what we how we do that if you this had been simply 1 over s square plus s plus 1. Instead of this s up on 10 to the power of minus 2 you would have written this as we have already mention as s plus half whole square plus root 3 up on 2 whole square and this would have been 2 up on root 3 times root 3 up on 2 over s plus half whole square plus root 3 up on 2.

Why I put this root 3 up on 2 here is because, this is form omega 0 over s plus alpha square plus omega 0 square. So, this particular Laplace transform would be adding its inverse Laplace transform 2 up on 3 e to the power of minus t up on 2 sin root 3 up on 2 that is the of course, u of t will always t this is the inverse Laplace transform of this. But now, if this is f of s what we have having here is instead of s we have s up on 10 to the power on minus 2.

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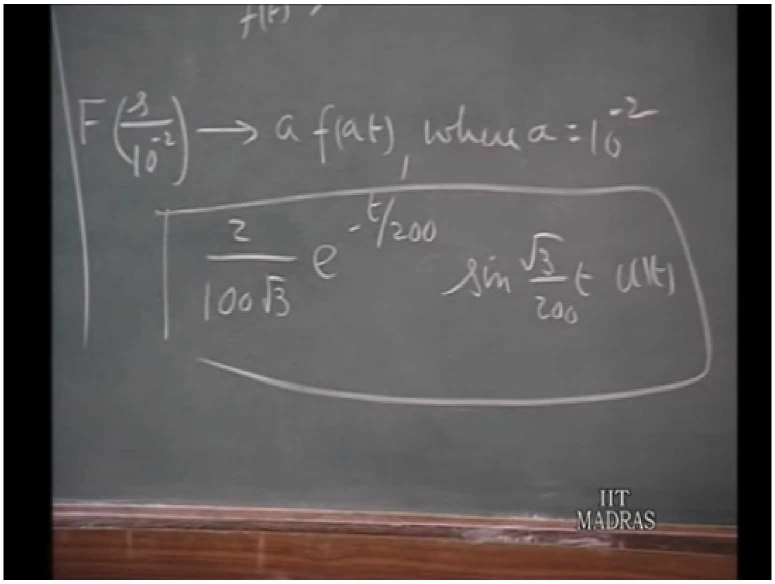

So, we have f s over 10 to the power of minus 2 that is what this expression is. Now, if you go to this rule here f of s up on a will be a times f of at a in our case ten to the power of minus 2. So, since we know f of t we can find out a times f of at. This corresponds to a times this is f of t if this is f of t the a times f of at where a equals 10 to the power of minus 2.

So, if you substitute that you have 10 to the power of minus 2 2 up on 100 root 3 e to the power of minus t up on 200 at sin root 3 up on 200 that is the inverse Laplace transform of this. So, this scaling rule helps as to make the coefficients change the scale of the coefficients. So, that you can handle convenient numbers instead of sometimes and will be the numbers like this we can often advantage all this rule.

Remember this, in accordance with what we discuss in the case of Fourier transform. If you make events faster in time, this is equivalent to increasing the frequency. That is if you scale t in 1 direction the frequency goes in other direction, if t is replaced by 2 t s is replaced by s up on 2. So, if you compress events time to take them accrue at a faster time the frequency spelt from increases.

So, you need to encounter higher frequencies on the other hand if you spread out in the time domain then, the frequency domain its gets compressed the higher frequency components come become smaller. So, in this lecture we covered 3 important properties of the Laplace transform: the first 1 we considered what when f of t function of time is multiplied by the exponential function of time e to the power of minus alpha t.

For example, the in the frequency domain it corresponds to a shift in the frequency variable. That is s will be change is to s up on alpha. And the second important property if discussed was when, you shift a time function in time delayed by tau seconds the corresponding effect the transform domain is multiplying multiplication by f of s by e to the power of minus s tau.

And we saw the ramp in this case of this, if you try to use this to apply to advanced functions we said, when you advance the time function tau seconds it may or may not correspond to a f of s which is multiplied by e to the power of s tau. In some case, it may in most of the cases may not because part pf the time function may get lost in the process.

Lastly, we consider the scaling in time domain which corresponds to also a scaling in frequency domain if you scaling up in time domain with to be scaling down in the frequency domain by factor a of course, there will be also change in the multiplying constant. In all this 3 cases we have considered various examples which illustrate the usefulness of these properties. When, you are evaluating either the forward Laplace transforms or inverse Laplace transforms we shall pick up at this point in the next lecture.