# Networks and Systems
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## Lecture-53
## Properties: Division by 't, Initial value theorem, Final value theorem

We have seen some plurality in the relations between the Laplace transforms from the time functions. For example, differentiation time domain corresponds to multiplication by s in the frequency domain. Similarly, differentiation in the frequency domain corresponds to multiplication by t in the time domain to recall t f of t Laplace transform minus d by ds of f of s.

We have also seen that integration in time domain corresponds to, division by s frequency domain essentially, that is: integral f of t dt corresponds to f of s over s in a dual way, if you have f of t up on t it must corresponds to integration in the frequency domain this is the property which is not particularly useful. But it may be useful of this in some special situations just let us, look at this rule we have.

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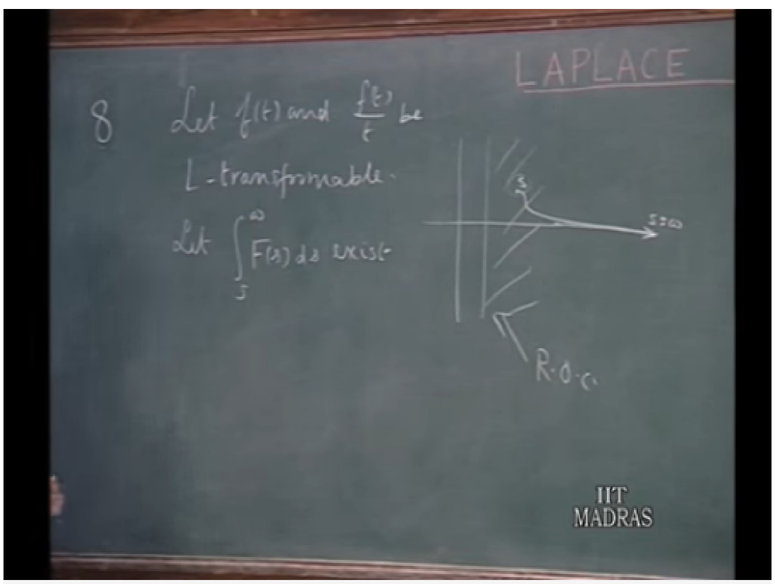


Let f of t and f of t up on t be Laplace transformable and let the integral s to infinity of f of s ds exist. So, this integral exist what we mean by, s to infinity is suppose this is the region of convergence of the Laplace transform of both f of t by f of t by t electively then we take any point s here and then integrate this f of s over some contour starting from point s in the in the convergence appear infinity. So, that the real part of the s goes to infinity; that means, then you take s is equal to infinity; that means, at least the real part of s is going to infinity.

So, 1 can possibly say that the point on the x axis how get infinity is the ending point at this.

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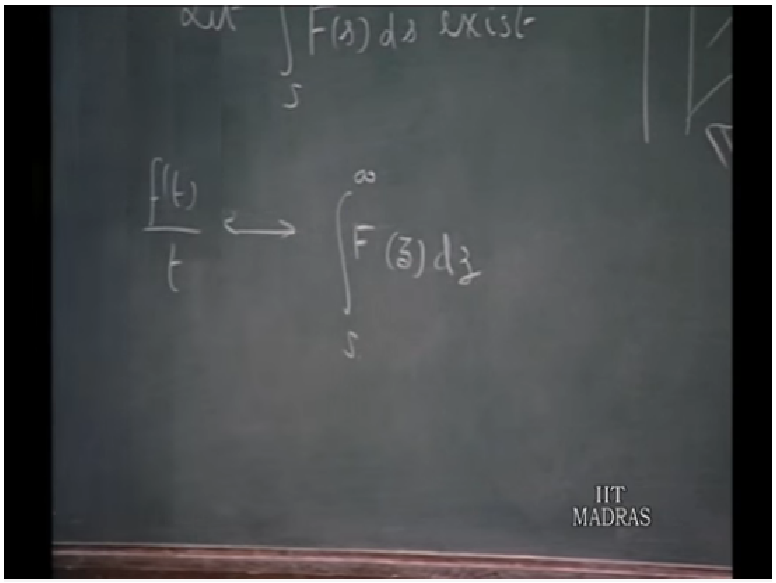


So, if you do that, then the theorem states that, f of t up on t as the Laplace transform s to infinity of f of s ds this is the rule and since, we are talking about s. So, in as well to make it clear it can be a dummy variable z f of z dz, where z is the complex variable and once you make the integration, it is a function of s because s is the limit here.

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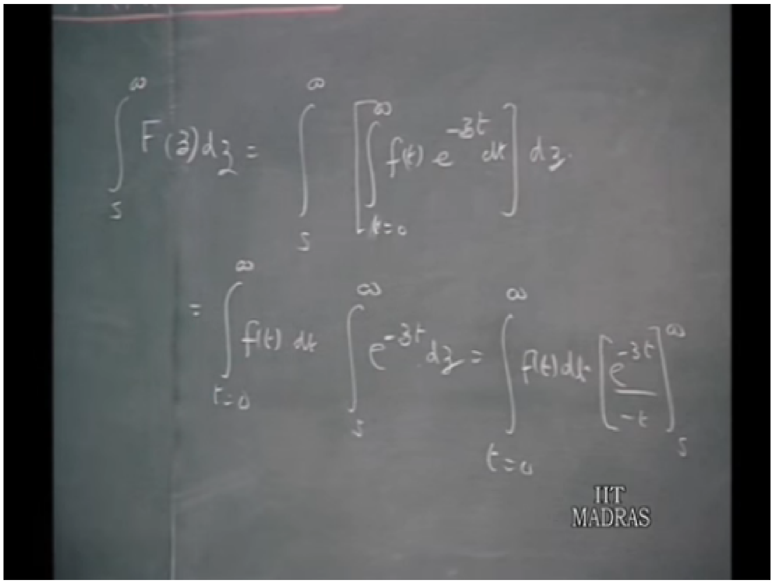


So, this will be a function of s proof of this is the again is the straight forward s to infinity of f of z dz can be written as, s to infinity f of z dz you can write this as this is the Laplace transform of f of t therefore, 0 to infinity f of t instead of f of s, I will writing f of z therefore, I must writing e to the power of minus zt dt this will be f of z the integration is with reference to t of course t equals 0 to infinity and dz and this first integration is with reference to t second integration is with reference to z.

Now, if you reverse the author of integration, I will say t equals 0 to infinity f of t. So, whatever, functions are there which are independent of z, I will pull them outside. So, f of t dt then s to infinity of you have got e to the power of zt dz and this will be, T equals 0 to infinity f of t dt this integral will be e to the power of minus zt you recall that, you are integrating with reference to z.

Therefore, t is the constant over minus t and with the limits s to infinity it is what you heard and in the upper limit e to the power of minus zt goes to 0 because you are taking this integration from s equals 0 to point at infinity along x axis that as therefore, for positive values of time because this integral involves only positive values of time therefore, when positive t is there and real part of z is goes to infinity, then this become 0 on the upper limit at the lower limit e to the power of minus s t by minus t.

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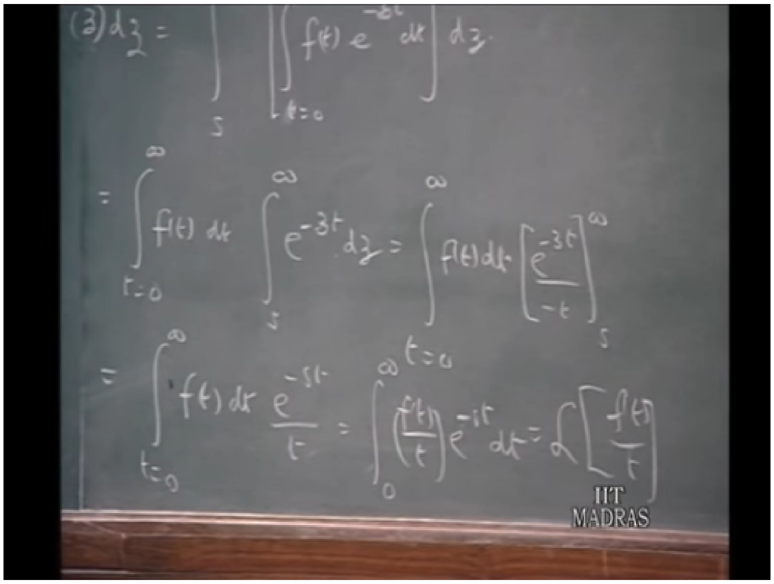

Therefore, this will be t equals 0 to infinity f of t dt. This will be e to the power of minus st by t because, we are taking the lower limit therefore, this minus sign is observed by another minus sign coming out in front therefore, this is what we are having this will be 0 to infinity of f of t up on t e to the power of minus st dt continuous is indeed the Laplace transform of f of t over t this is the Laplace transform of.

So, Laplace transform of f of t over t is this integral. So, division by t in the time domain correspond integration in the frequency domain just like; integration the time domain corresponds to division by s in the frequency domain as I mention this rule is not particularly useful to us in our context in over applications to networks. So, you will just record this as a dual rule to the rule for integration in time domain leave it at that.

In solution of networks transformations and systems sometimes we may not be interested in finding out the entire function of time from the Laplace transform variable which is Laplace transform which is available from the solution of network. We would be interest in finding out the initial value of f of t or it is derivative first derivative or second derivative without having to find out the entire function of time f of t.

Suppose, the f of s is given would like to know what is f of 0 plus or what is the initial value of the derivative of f with reference to time t equal 0 plus without our having to find out the entire f of t for all time t.

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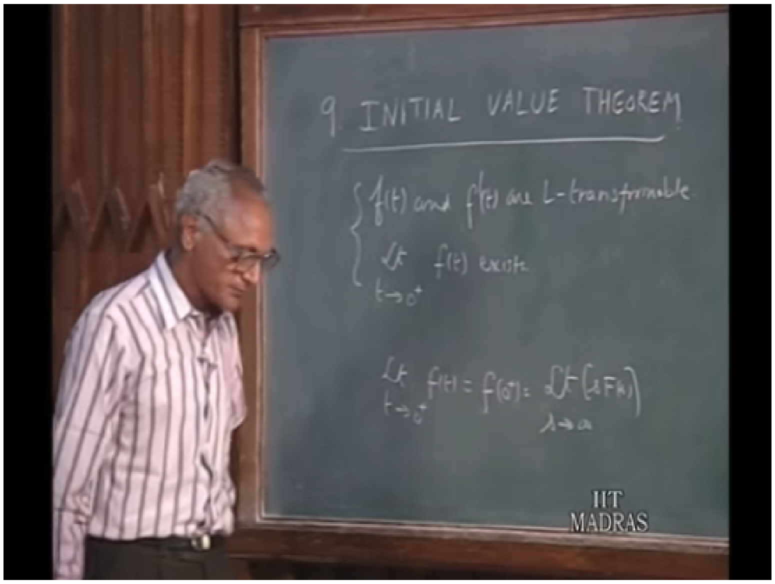


Now, this can be done using a property, how Laplace transforms which is known as the initial value of theorem. Now, let be conditions for this f of t and the derivative f f prime t are the Laplace transform. Laplace transform say gift for both this further, limit as t goes to 0 plus of f of t exists. Suppose, this conditions are fulfill then limit as t goes to 0 plus of f of t, which will write simply as f of 0 plus is given by limit as s tends to infinity of s times f of s this is the statement of the initial value of theorem.

That the initial value of the time function is given by limit as s tends to infinity of s times f of s. This limit is quite easy to evaluate because once, to have a rational function as s tends to infinity you have to take the ratio the 2 lead in coefficients in the numerator and denominator that will be limit and that is equal to f of 0 plus.

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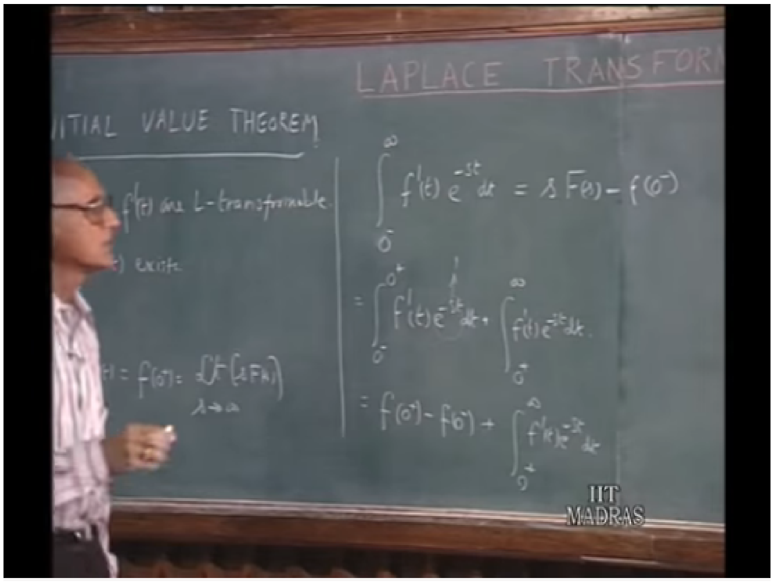


Now, what is the proof for this. Let us say, we are trying to find out the Laplace transform of f prime p. So, if f prime p e to the power of minus st dt 0 to infinity 0 minus to infinity that is: the Laplace transform of the derivative of f of t according to what, we had already discussed this will be s f of s minus f 0 minus according to the rule for finding the derivative Laplace transform of the derivative of the time function this.

What we have held the Laplace transform of the derivative of this is equal to s f of s minus f 0 minus. Now, this can be this integral can be split in 2 parts 0 minus to 0 plus of f prime t e to the power of minus st dt plus 0 plus to infinity of f prime t e to the power of minus st that, of course is equal to s f of s minus f 0 minus.

Now, as for as this integral is concerned this evaluated over a very tiny intervals from 0 minus to 0 plus. So, the value of t in this portion is equal to 0 therefore, this is equal to 1 in this interval 0 minus to 0 plus is essentially 0 f e to the power of minus st, where t equals 0 is equal to 1 therefore, we are really integrated in f prime t dt from 0 minus to 0 plus. So, the value of this will be f 0 plus minus f 0 minus that is: what we are having in addition you have been this additional function 0 plus 2 infinity f prime t e to the power of minus st dt.

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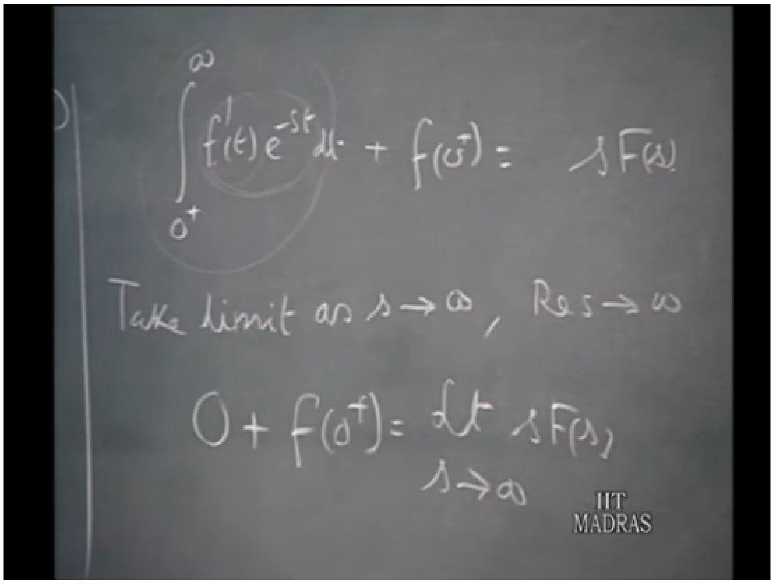


So, when you combine this 2, if f 0 minus gets cancelled out then you are having equating this 2 you get 0 plus to infinity of f prime t e to the power of minus st dt equals plus f 0 plus equals s times f of s. This is also the thing which we would have straight away written because Laplace transform of f prime t starting from 0 plus onwards. We said Laplace transform of that is: f of s minus f 0 plus something which, we already observed when we are taking about differentiation rule.

So, you do not have did not have derived, but, my purpose in doing in this fashion is something, which will explain a little later. Now, in this let us, take the limit take, limit as s goes to infinity. Now, we take the limit as s goes to infinity such that, the real part of s goes to infinity positive infinity that, is along the say along the real axis for example, then when you take the s goes to infinity because f prime t is Laplace transformable you are taking s going to infinity and you are talking about positive values of time this is the exponential order.

Therefore, this will become 0 this becomes 0 as s goes to infinity for positive t such the real part of s goes to infinity therefore, this becomes 0 therefore, this integrant becomes 0. So, this will be 0, if you do that this will be 0 and you have f0 plus equals limit as s goes to infinity of s f of s that is, what we want to prove. Now, so s must tends to infinity such that the real part of s goes to infinity.

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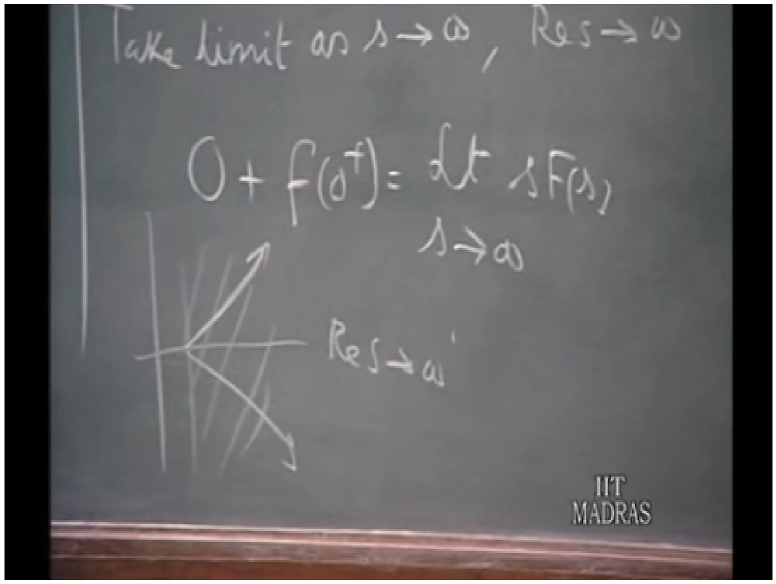

That means, if there is a reason of origins that we are having here whatever, it is s goes to infinity either in this direction or this direction. So, that the real part of s must go to infinity that also a important; that means, you must take the value of s going to infinity either in the first quadrant at the fourth part. So, real part of s must goes to infinity that is something which we have to keep in mind.

Now, for this theorem to be valid so f t and f prime t it must be Laplace transform already mentioned limit st goes to 0 plus f of t should exist; that means, you cannot have the Laplace transform of this. For example, if you a constant for example, some an impulse for an example from origin then you cannot have this Laplace transform for this mean the initial value theorem will not apply for that.

Now, the reason why I started with this integration from 0 minus a whether you define the Laplace transforms, starting from 0 minus or 0 plus the initial value theorem will always give you limit as s to infinity of s f of s f of 0 plus only it cant give you f of 0 minus sometimes people may like to think that, if the Laplace transform is defined as 0 plus to infinity of f of t e to the power of minus st then if that, f of s to take s s f of s take the limit as s tends to infinity will give 0 plus.

If you are take in the Laplace transform f of s to be starting from 0 minus then the limit as extent to have f of s s tends to infinity s times f of s give f of 0 minus no whether, you define the Laplace transform as starting from 0 minus or 0 plus the initial value can give you only this condition f of 0 plus only it cant give you f 0 minus. So, that is; why I want to make that, very clear that is; why I started with the define definition of Laplace transforms starting from 0 minus. So, even if you start from 0 minus the initial value theorem will be only 0 plus.

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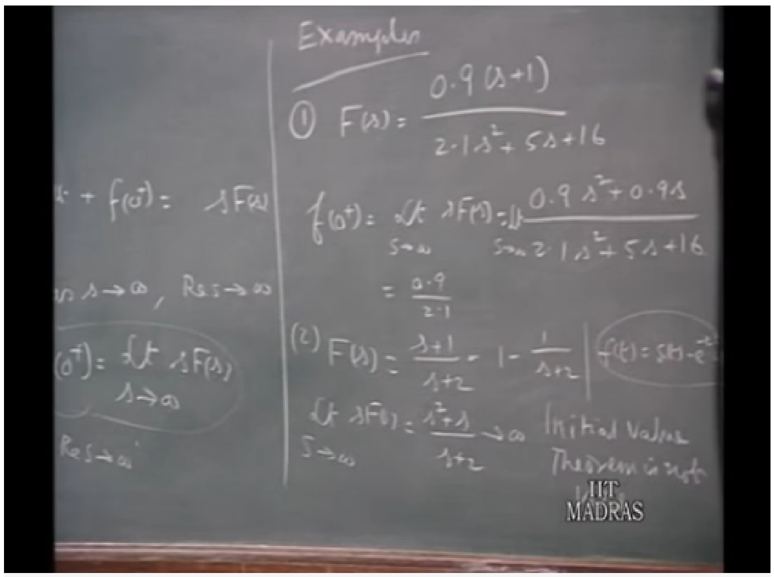


Let me, give you few examples 1: suppose I had f of s 0.9 s plus 1 0.1 s square plus 5 s plus 16. So, this is f of s and through various techniques which we already are which we discuss later you can find f of s f of t. But suppose, we are not interesting finding f of t who want to know what is f 0 plus the function of time as the origin is approach from the positive sign f 0 plus.

For this, we do not have to find out the f of t this initial value theorem tells us that: limit as s tends to infinity of s times f of s which is: 0.9 s square plus 0.9 s divided by 2.01 s square plus 5 s plus 16. And as, I said when you take limit as s tends to infinity as s f of s we can take s to be approaching infinity along the positive x axis.

So; that means, we can take the only the ratio of the 2 leading coefficients because this becomes insignificantly compared the first term as become larger and therefore, 5 s plus 16 becomes insignificantly become 2.1 s square therefore, as a s goes to infinity of again here put limit as s goes to infinity of this will be simply point 9 divided by 2.1 that is all whatever, that may be.

So, initial value theorem will enable as to find out the initial value at t equal 0 plus have a function of time from its Laplace transform without are having to go through the finding out f of t from this. Let us take another example f of s is s plus 1 over s plus 2. Now, limit as s 5 tends to infinity of s times f of s equals s square plus s over s plus 2 and this goes to infinity because s goes to infinity s square up on s will become essential equal to s that, goes to infinity.

Now, why lets this condition limit as t 0 plus f of t exists that; condition will not be that be violated; that means, this is actually what will happen, this as an impulse 1 minus 1 over s plus 2 f of s is 1 minus 1 over s plus 2; that means, f of t here is delta t minus e to the power of minus zt ut.

So, because this delta function, this initial value theorem does not is not valid here initial value theorem is not valid. So, you will say that whenever, delta function exist at the origin we will not have the initial value theorem applicable in such equations because it leads to infinity whatever, it might be. So, it is not very useful unless you talk about infinity of magnitude because that; take makes it little more complicated.

So, we will say whenever, this limit leads to infinity, which means f of s impulse functions here it will not the initial value theorem will not be valid. So, let us now consider the dual rule of this which gives the final value of a function of time without our having to find out the inverse Laplace transformation. We have seen in the initial value theorem that, the behavior of the f of s at s equals infinity essentially, downs the value of of t equal 0 as a dual to this the behavior of f of s at s equals 0 will essentially decade the value of f of t, when t goes to infinity. And that is, given by this statement of what is called the final value theorem.

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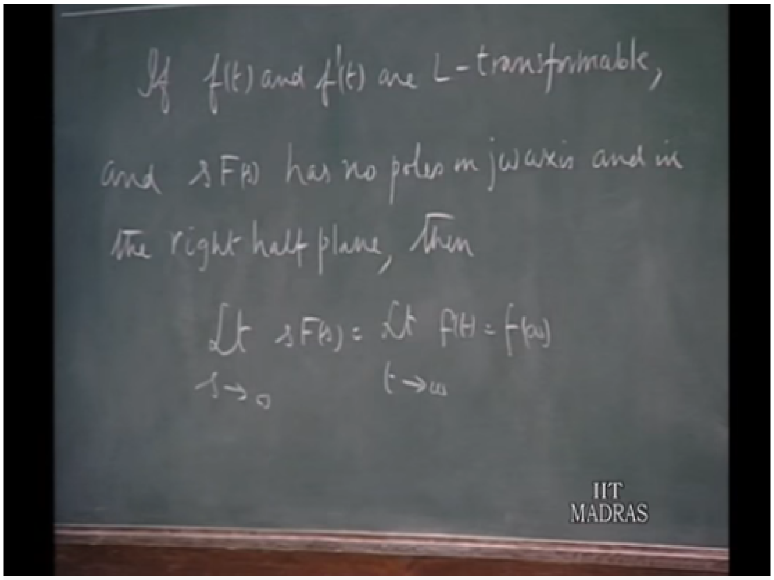


It is says like this, if f of t and f prime t are Laplace transformable both are Laplace transformable. And s times f of s has low poles on j omega axis and in the right half plane I will write this the right half plane. Then limit as s tends to 0 of s times f of s equals limit as t tends to infinity f of t or you can say that is the, final value of the time function f of t. So, you take the limit as goes to 0 f times f of s that will give the final value of the function of time f infinity.

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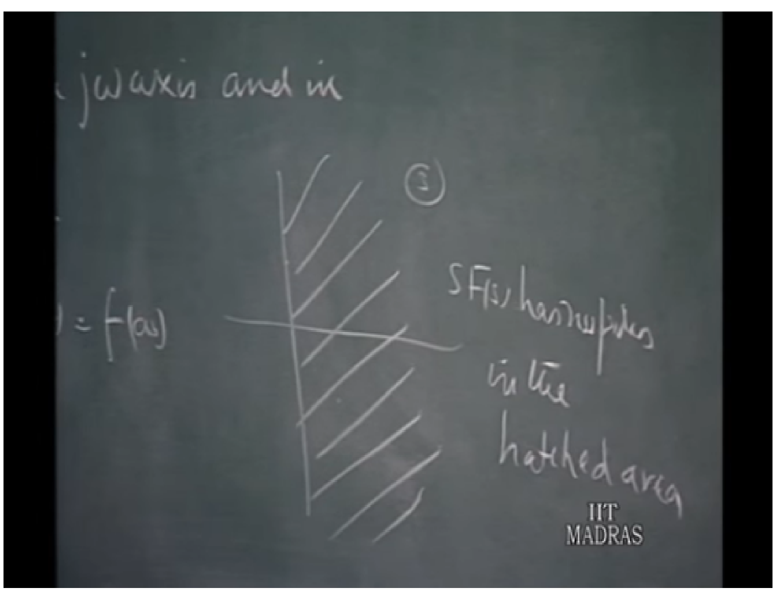

Now, we say s f of s has no poles in the j omega axis in the right half plane; that means, if you plot in the complex plane this must be the region of enlisted of s times f of s. So, s of s f s times f of s has no poles in the hatched area, if you have a pole on the imaginary x axis or in the right half plane then the particular theorem is no longer valid.

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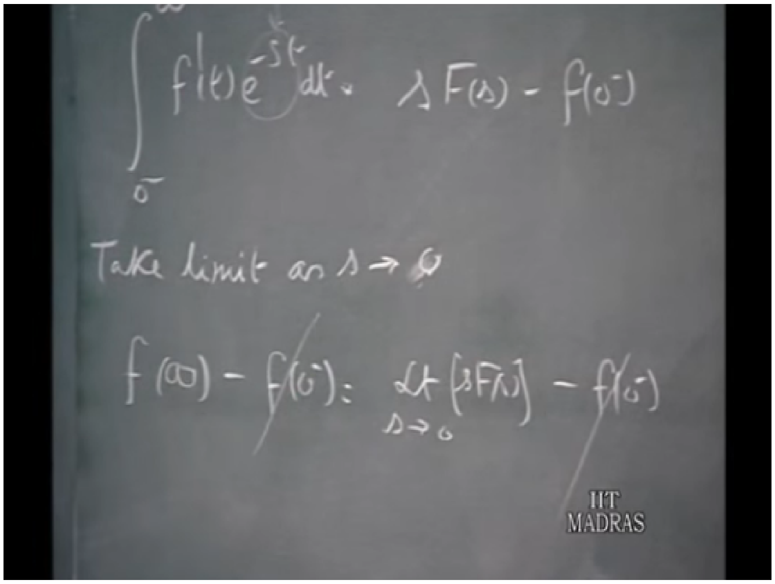


Proof is again straight forward you have 0 minus to infinity of f prime t e to the power of minus st dt that is: the Laplace transform of the derivative of f of t is s times f of s minus f 0 minus according to our rule. Now, take limit as s goes to 0 this will be then s goes to 0 this will become 1 this will become 1. So; that means, you have essentially you are integrating f prime t from 0 minus to infinity.

So, when you are integrating f prime t this becomes f of t you are taking the limits between infinity and 0; that means, f infinity minus f 0 minus this is, what you are getting here and on the other hand you are having limit as s goes to infinity s goes to 0 of s times f of s minus f 0 minus. So, if you take the limit as s goes to 0 on both sides this is: what you result what results.

And you cancel this 2 terms this is: what you are having f infinity as t goes to infinite limit as t goes to infinity of f of t, if you call f infinity is limit s goes to 0 of s times f of s. This is again a dual rule to what we had earlier reserved as initial value theorem.

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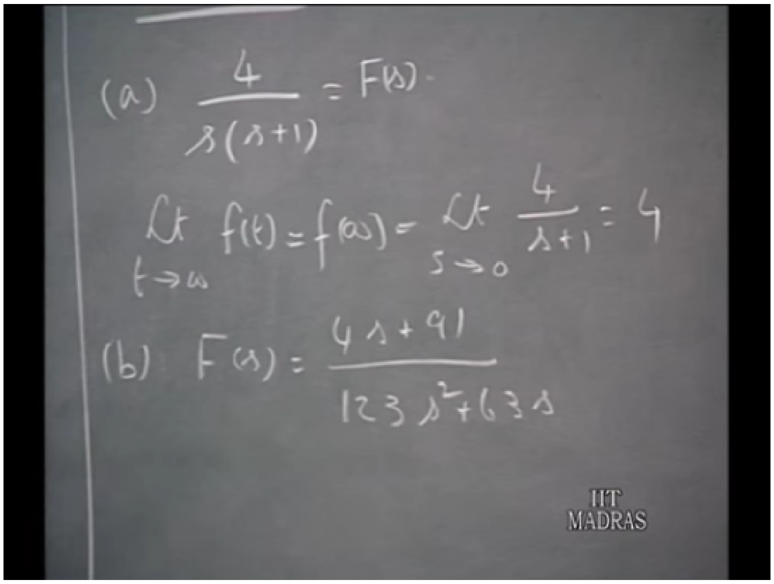


Examples a: 4 upon s over s plus 1 this is f of s. Now, we will like to know whether we can find f infinity without having our finding out f of t. So, limit as t goes to infinity of f of t which we write for continence as f infinity simply this will be limit as s goes to 0 of s times f of s which means 4 up on s plus 1 this is 4 that is all. So, the final value of the time function is 4 units, we did not have for really find out f of t.

Let us take another example b this tells as at the advantage of theorem is without having our 5 to find out a particular analytical expression for f of t from f of s sometimes can be complicated, we can straight away find out the final value without having to go through this intermediate step of finding f of t. Let us take, f of s as 4 s plus 91 divided by 123 s square plus 63 s once, again we wish to show that, f infinity can be found out without having explicit you can find ft with this accrued numbers.

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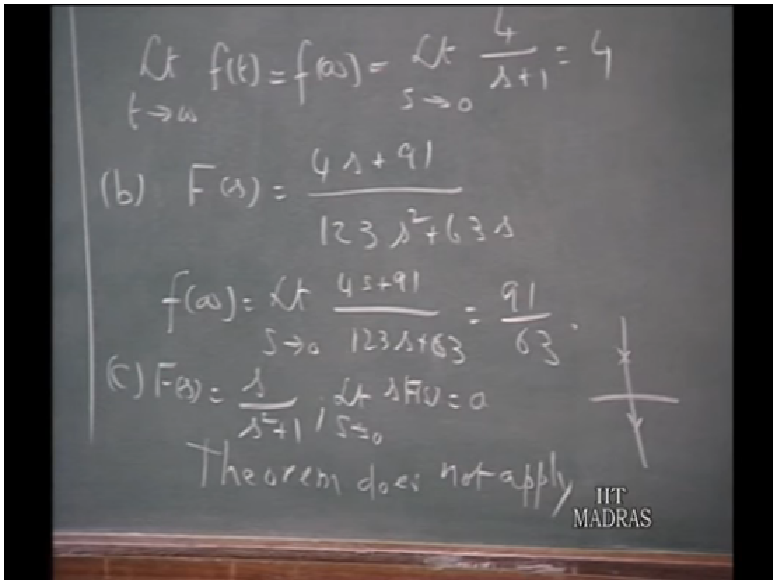


So, f infinity now, is limit as s tends to 0 of s times this which is 4 s plus 91 divided by 123 s plus 3 because this is multiply by a s this square will be come with s and that other s square dropped out this becomes 63. So, this becomes 91 by 63 third example f of s equals s over s square plus 1 suppose, you this is of course cos t you will know that cos t ut.

The final value theorem, if you like to apply for this you must multiply this by s and then take the limit as s goes to 0 this will tell you limit as s goes to 0 of s times f of s is of course 0. But then what is the final value of cos t we can say, what it will be, it will becomes going on oscillating between plus 1 and minus 1 write.

So, if you take the average of that it may be 0, but, then the point 2 observe here is this does not satisfy the statement of the theorem s f of s has no poles and j omega axis. So, if you take s times f of s, this will be s squared over s squared plus 1 s square plus 1 s squared over s square plus 1 will have poles at plus or minus 0 1 therefore, initial the theorem does not apply.

So, the final value theorem does not apply in this case and therefore, whatever limit you will get may or may not be true therefore, we cannot use this wherever, s times f of s poles either in j omega axis or right half plane.

So, this is situation the theorem is not well similarly, if you have 1 over s square plus 1 than also the theorem, is not well because for the same reason that, sin t you cant find out the final value of sin t we can only assume that 0 will average, but, that is not very regress statement. We shall now, consider another important property, which is useful in system studies particularly when you want to find out the response to an arbitrary input, when you know the response to an impulse input through the medium of convolution integrals.

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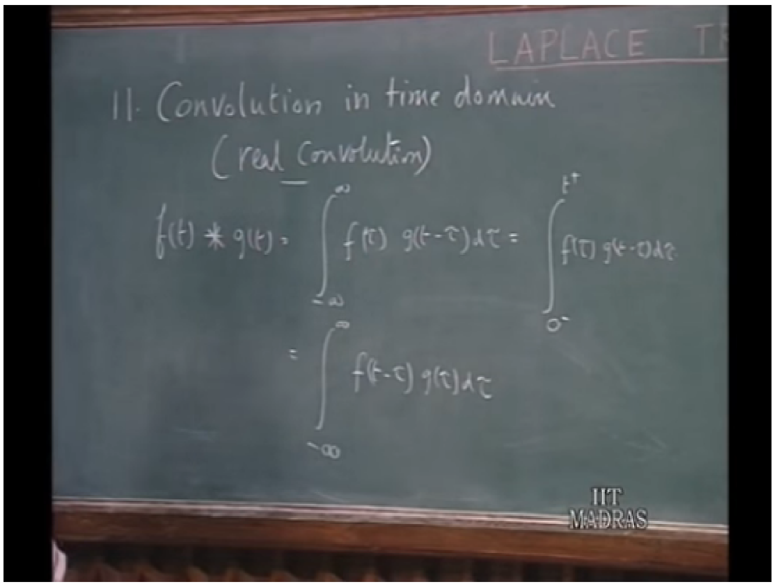


So, you would like to know, how the convolution in time domain how it transforms itself in the s domain. Convolution in time domain, this is sometimes referred to as real convolution. This is said to be real convolution in the sense that carrying out this convolution in time domain which is the real variable not in the transform domain which is the complex value.

So, this is real convolution you recall that ft convolve it g of t is the definition is f of tau g of t minus tau d tau you can also write this as ft minus tau g tau d tau this are both are equal and you take, the variation of tau the minus infinity to plus infinity in the general case. And for causal functions, where ft and gt are causal time functions, you take the limits because, f tau as the value only from tau equals 0 onwards possibly f of tau as f of t as an impulse the origin will take 0 minus.

Then, when tau exceeds t this becomes negative therefore, there is no point in carrying out integration. Beyond t equals tau they are tau equals t therefore, we take this as t then f tau gt minus tau and take care of possibility of an impulse, if g of t the origin we take this as t. So, this is what we are having by means of the meaning of f of t convolve with g of t.
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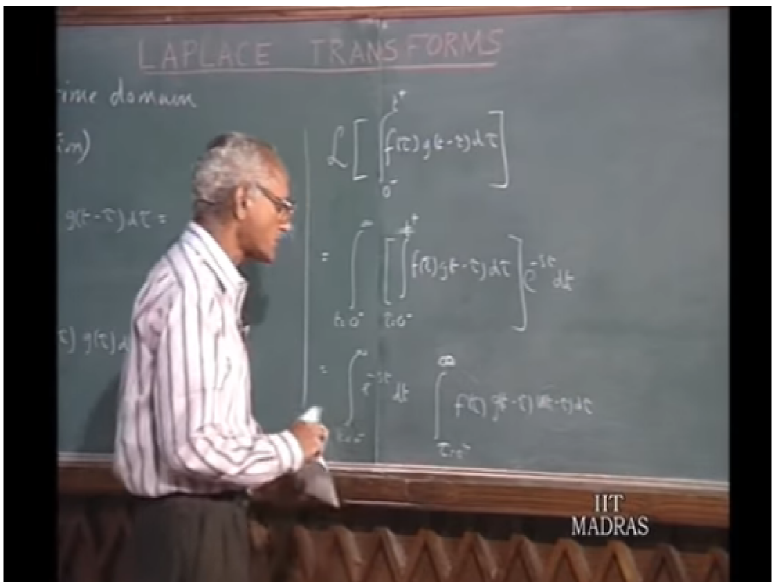


Now, let us try to find out the Laplace transform of this. So, Laplace transform of f tau t minus tau d tau 0 minus to t plus that is: what we are seeking. So, this will be 0 minus to infinity of this quantity 0 minus to infinity a t plus f tau gt minus tau d tau and this integration is carried out on tau. And whatever the results is there function of t you multiplied by e to the power of minus st dt this integration is with reference to t.

So, this is the definition of Laplace transform of the convolution of the 2 time functions ft and gt which are assumed to be causal. Now, I would like to do the integration must with reference to t and then go to tau later therefore, to do that I must, I have also a function t plus s. So, to avoid that what can do is I will have e to the power of minus st dt because I would like to do the integration with reference to I will do the integration with reference to tau itself start with toe equal 0 plus to t plus.

Now f of tau gt minus tau suppose, I introduce ut minus tau d tau t equals 0 minus to infinity. Now, ut minus tau is going to be 0. When tau exceeds t, when tau exceeds t ut minus tau is 0 therefore, this integrant will become 0 when tau exceeds t therefore, I have instant having t plus I can as well infinity. Now, that I introduce the symbol ut minus tau here I may as well take the limit of integration of tau from 0 to infinity this is 0 minus infinity.

So, to ensure that; the same integral valid even here same the values of the 2 integrals are the same I will introduced purposely ut minus tau and take the limit as up to infinity because the value of ut minus tau is going to be 0 for tau greater than t the reason why I did this because, we will like to relate them to the Laplace transform integrals therefore, the limits must be 0 to infinity. So, in order to have that kind of property, I have to purposely introduce this alright f of t minus t d tau.

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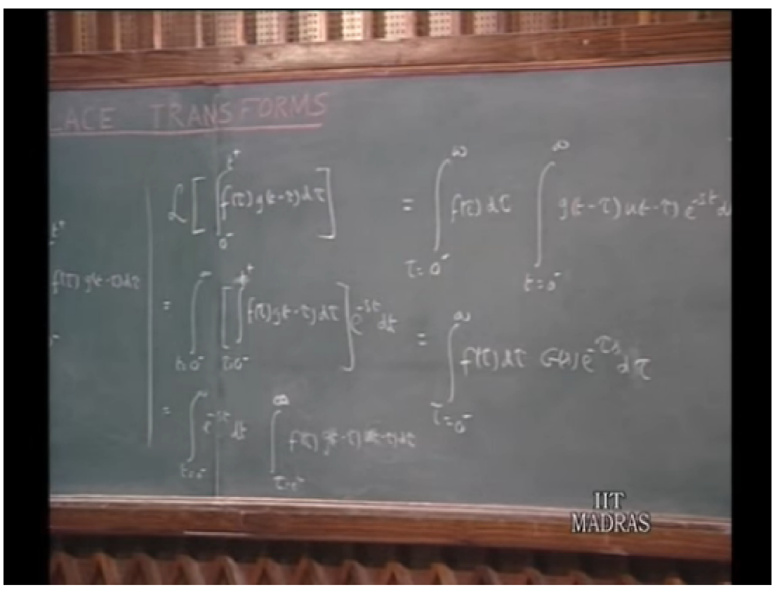


Now, this will be equal to 0 minus to infinity. Now I would like to interchange the limits of integration this is originally the first integration is refers to tau next integration with reference to t. Now, suppose I reverse the roles. So, I do the integration with f of tau later and then do the integration with reference to t first.

So, whatever we are having as constants whenever, we integrating with reference to t must be got outside which is f tau for 1 example, d tau can be brought outside. And all functions which are involved with t are to be taken into account in the first integration that will be gt minus tau ut minus tau e to the power of minus st dt. Now, this is the Laplace transform of the delayed function of gt delayed by tau seconds.

If the Laplace transform of g of t is g of s Laplace transform of the delayed time function t minus tau is e to the power of minus s tau time g of s if, g of t has the Laplace transform g of s g of t ut as the Laplace transform g of s g of t minus tau ut minus tau as Laplace transform g of s time e to the power of minus tau s that is; something which we already k. Now, therefore, we can write this as tau equal 0 minus to infinity of f tau d tau g of s times e to the power of minus s tau d tau sorry d tau come from here. Now, in this g of s is the constant.

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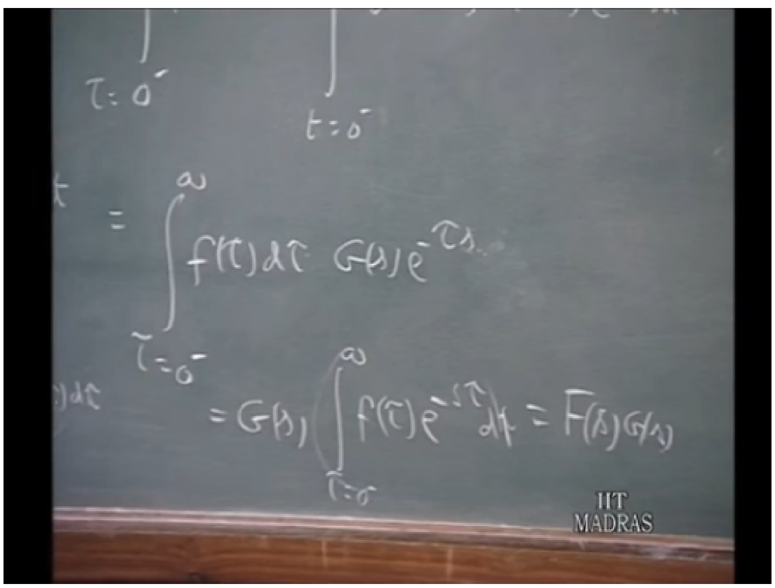


Therefore, g of s times tau equal 0 minus to infinity of f tau e to the power of minus s tau d tau, I have writing d tau twice here is not necessary. And this is indeed f of s f of t e to the power of minus t dt integrated from t equal 0 minus to infinity is f of s instead of t we have the dummy variable tau therefore, this is f of s multiplied by g of s.

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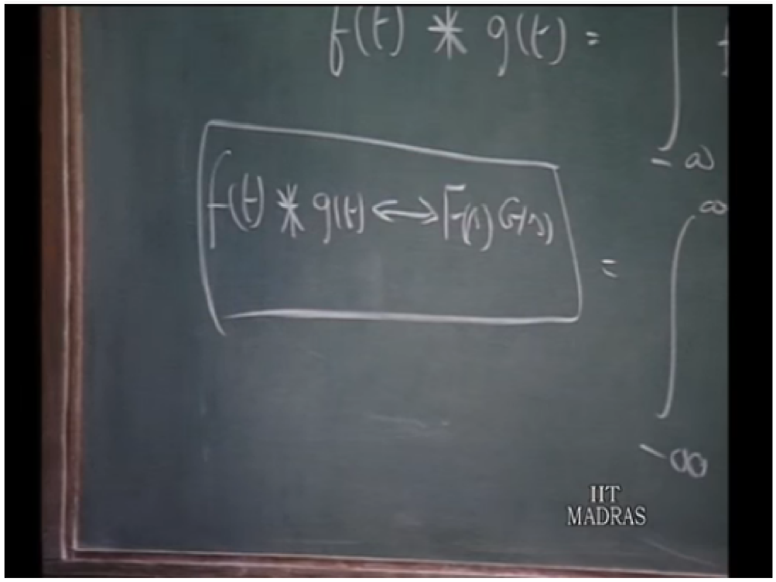


So, the neat result that we are having here is finally, that ft star gt star the convolution of 2 time functions ft star gt star as the Laplace transform the product of the 2 corresponding Laplace transform. So, this is the important facility convolution, the time domain corresponds to multiplication of the respective transforms in the frequency domain are: the complex frequency domain s domain this is quit neat.

And this is a rule which will be useful for our as a side in a system studies, where when we know the impulse response h of t to find out the response for arbitrary function of time all we have to do is the Laplace transform of the output will know to be the product of the Laplace transform. So, the impulse response and the input function.