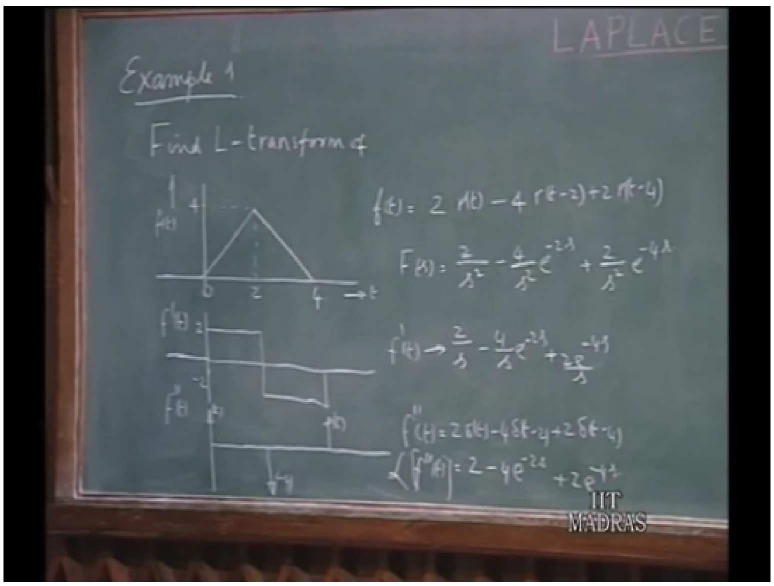
Networks and Systems
Prof. V.G.K. Murti
Department of Electronics & Communication Engineering
Indian Institute of Technology – Madras

Lecture-56
Examples of Laplace Transform

We had considered several interesting properties, of the Laplace Transformation in the last 2 lectures. So, it is a good time now, to work out a number of examples, which illustrate the application of these properties.

(Refer Slide Time: 00:36)



First example; Let us find the Laplace transform, of a function f of t which is, represented graphically in this form. This is t in seconds, f of t as the value of 4 units t equals 2 seconds. So, this is the function f of t whose Laplace transform we are required to find.

So, we can express this f of t as. From this point to this point it claims an amount of 4 units in 2 seconds; therefore, the slope is 2 units. It is the ramp function 0 for negative values of time. Therefore, 2 rt this describe the behavior from 0 to 2. At this point instead of claiming up rate of 2 units per second, it is climbing down at the rate of 2 units per second, because in 2 seconds, it comes down by 4 units.

Therefore the next slope must be minus 2; therefore, you must bring this positive slope of 2 units to a negative slope of 2 units. Therefore, you must introduce a negative ramp 4 units and then that ramp is introduced t equals 2 seconds. Therefore this minus 4 rt minus 2.

So, if these 2 ramps alone are operating, then this continuous to go down like this, but at this point the downward slope must be arrested, next slope of minus 2 must be arrested and then it's flat and out. Therefore, to introduce another slope of plus 2 units at r t equals 4. So, it is this 3 ramp functions which requires this characteristics.

Therefore, we can find f of s as 2 up on s square minus 4 upon s square would be the Laplace transformation, if it been 4 rt, but since it is delayed by 2 units you must introduce e to the power of minus 2 s. And this as the ramp of 2 units plus 2 over s square is the basic Laplace transform of the ramp function.

But it's delayed by 4 seconds; you must introduce e to the power of minus 4 s. So, that would be the Laplace transformation of this. it is illustrate you to obtain this same result, by differentiating in the f of t; may be once or twice and finding out the Laplace transform of the derivative functions, and then go back and see, how it agrees with this or not.

So, let us see, if I take the derivative of this f prime t, then for 0 to 2 seconds it has plus 2 units, slope of plus 2, and from 2 to 4 seconds it is the slope of minus 2. So, that would be the derivative function; 2 and minus 2, that is the derivative function, this is f prime t. Now, if I take the second derivative of this. Once again take the derivative of this.

You have here we are thinking of this is 0 therefore, this is also 0. Therefore, there is an impulse of 2 units here; 0 to 2, 2 units impulse. And at this point it jump from plus 2 to minus 2; therefore, there is a negative impulse of 4 units of minus 4 is the strength of the impulse. At this point again it claims up by 2 units therefore, this will be 2.

So, you can write f double prime t, that is the second derivative of the function of time. F double prime t as 2 delta t minus 4 delta t minus 2, because that is an impulse standing a t equals 2 seconds, plus another impulse function of 2 units at t equals 4. This is the second derivative of this.

The second derivative now, consist purely impulse function therefore, its Laplace transform, is easy to find out. So, Laplace transform of the second derivative is, delta t as a Laplace transform 1. So, it is 2 minus 4 delayed impulse minus e to the minus 4 e to the power of minus 2 s, and this is again a delta function.

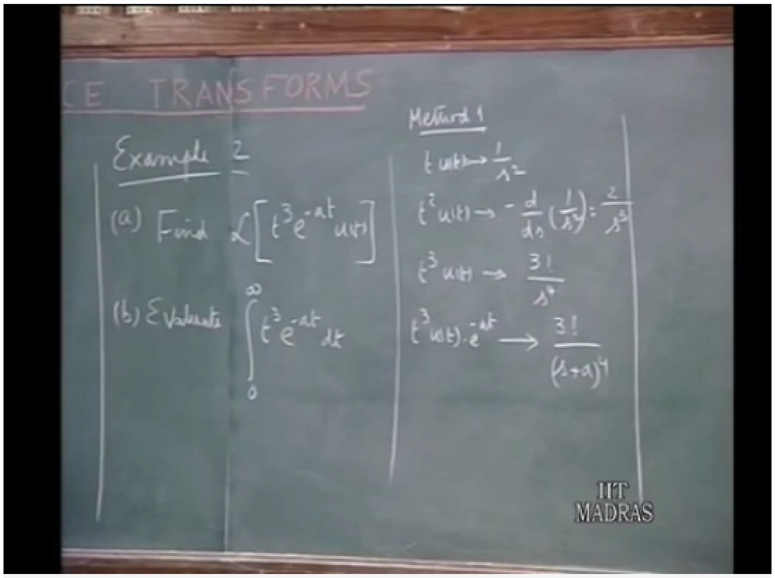
Therefore, this is 2, and but delayed impulse therefore, you must be introduce a term e power minus 4 s. So, that is the Laplace transform of the second derivative. To find the Laplace transform of the first derivative; that is this is obtained by integrating this. So, if you know the Laplace transform of this function, the integral will have this same Laplace transform divided by s.

Therefore, this will be adding the Laplace transform 2 up on s minus 4 up on s e to the power of minus 2 s plus 2 times e to the power of minus 4 s over s. In addition, normally we have the integral value of this function at t equal 0 minus. In this case it is 0 therefore, that does not appear here.

Now, once we have the Laplace transform of this. The Laplace transform of the integral that is the Laplace transform of this, is obtained by multiplying this by 1 over s once again, and that is what we had. So, this also illustrates the possibility, that when you want to find Laplace transform of functions like this.

You may find them directly, but alternately you can take the derivatives, and find the derivatives Laplace transform easily found out. Then you can use that information to find out the Laplace transform of the original function, with 2 alternative approaches. It also illustrates the rule for integration that we have already discussed.

(Refer Slide Time: 07:38)

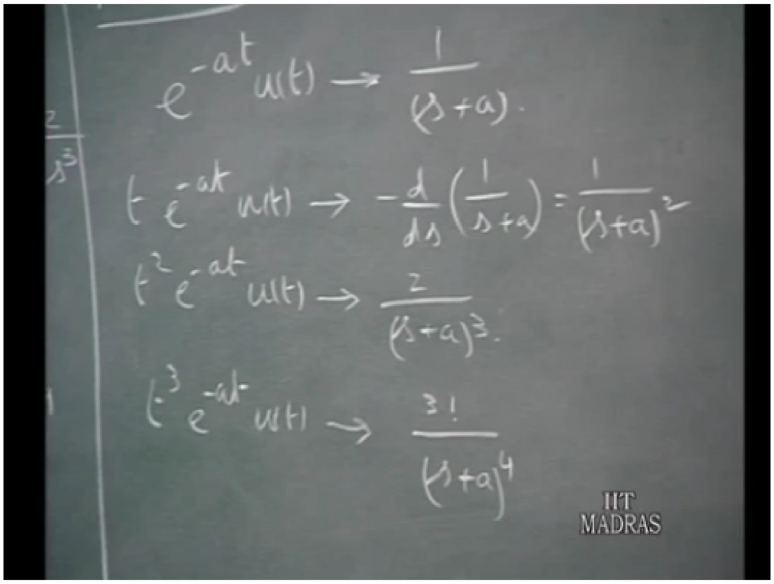


Let us take a second example; a, find the Laplace transforms t cube e to the power of minus at ut. And after having found that out, evaluate 0 to infinity of t cube e to the power of minus at ut. This is the question that is asked. Now, to find this out, we can. Let us say method 1. We start with Laplace transform of t. We know t ut that is, 1 over s square. t square ut minus d by ds of 1 over s square, because this is multiplied by t.

Multiplication in the by t the time domain corresponds to the negative of the derivative of the s domain. So, minus d by ds of 1 over s square, which of course is, 2 upon s cube. Then likewise you can carry this 1 more step t cube ut will be, once again you can take the derivative of this. This will become 3 factorial by s to the power of 4; that is t cube ut. But now if f of t the Laplace transform of f of s. f of t multiplied by e to the power of minus at the Laplace transform of that is the obtained, by substituting s plus k for s.

This is something which we already discussed. Therefore, t cube ut multiplied by e to the power of minus at, is Laplace transform is obtained, by substituting the s plus a for s. So, this will be 3 factorial or 6 over s plus a to the power of 4; that is what it is.

(Refer Slide Time: 10:21)

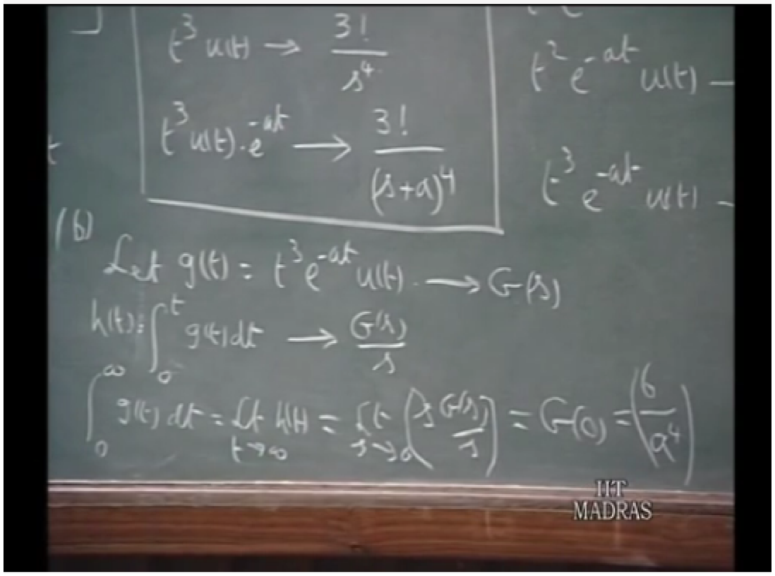


Now, let us do this by taking another approach; method 2. Let me start with the Laplace transform of e to the power of minus at ut. So, e to the power of minus at ut, as the Laplace transform 1 over s plus a. Now t multiplied by e to the power of minus at ut, its once again minus d by ds, 1 over s plus a that will be 1 over s plus a whole square.

Then likewise t square e to the power of minus at ut; that will give me, by the same process taking the second derivative, again the derivative of this we reference to s and putting a negative sign, it becomes 2 over s plus a whole cube. And t cube e to the power of minus at ut will fetch me 2 times 3 factorial over s plus a power 4. So, that is the same result that we obtained earlier so this problem over this out.

Taking 2 different approaches; starting with the Laplace transform of t cube ut first, and the Laplace transform e to the power of minus at first. These are 2 alternative, we are doing this. Now, for the second question.

(Refer Slide Time: 12:08)



Now let b; let g of t be t cube e to the power of minus at ut. It has got the Laplace transform; let us say g of s, which is this. Now, if I take the integral from 0 minus to t of gt dt we can say 0, because there is not. The 0 plus value of g of t is also going to be 0 therefore, whether you take 0 minus or 0 plus makes no difference.

So, I simply write 0 to t of gt dt. This get the Laplace transform gs upon s, because the initial value of the integral is taken to be 0, therefore, it is gs up on s. Now, what is 0 to infinity of g of suppose this is ht or some, what we want is not 0 to t, but 0 to infinity. Therefore, 0 to infinity of g of t dt can be regarded as, the final value limit as t telling to infinity of h of t.

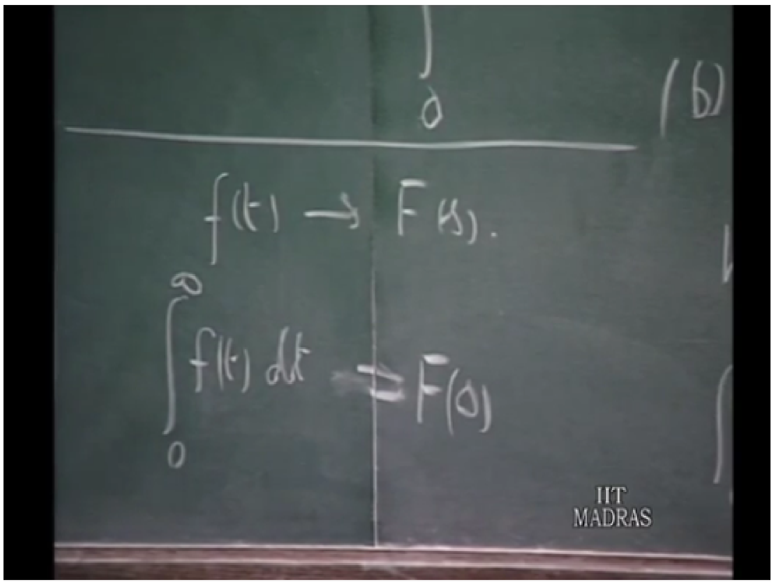
So, the final value of this function, who wants to find out this function, has got this Laplace transform gs over s. To find out the final value of this function. This is equal to limit as s tends to 0, of s times g of s over s; that is the final value theorem. You know particular function as the Laplace transform gs over s, or whatever it is, h of sit is if you like.

Then the final value of h of t h infinity is obtained by, taking the limit as goes to 0 of s times h of s. Now, in this case s times g of s over s. So, this indeed s cancels out g of 0.

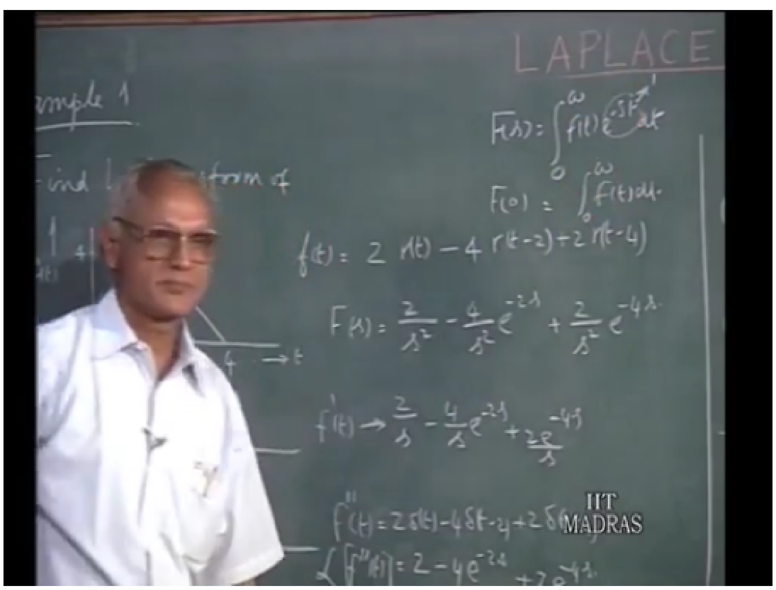
So, the final value of integral, from 0 to t gt dt it happens to be a simply g of 0, where g of s is the Laplace transform of this quantity. In our k g of s is equal to this.

So, the final value; that means, when s is equal to 0; this is 3 factorial 6 upon a to the power 4. So, that is the answer for second part 6 upon a to the power of 4. In other words what we are having is, the principle that we are using here is.

(Refer Slide Time: 14:56)



(Refer Slide Time: 15:26)

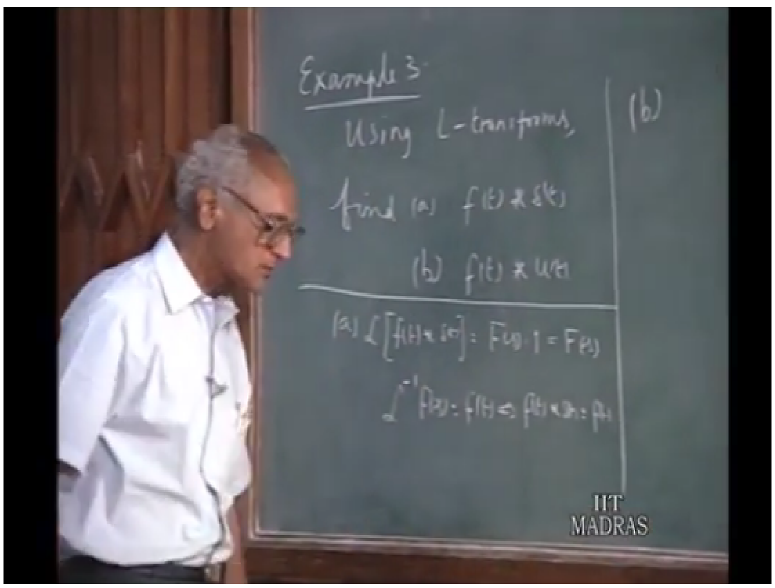


If f of t has the Laplace transform of f of s. What we are saying is, 0 to infinity of f of t dt is equal to f 0; that is what the principle that we used here. This is quit easy to see why it is. So, because after all f of s equals 0 to infinity of f of t e to the power of minus st dt. When you substitute s equals 0 in this, this becomes f of 0, and this is 0 to infinity of f of t dt, because this become 1.

When s is equal to 0 this becomes 1, and that is the same thing that we used here. So, all this are tied up in some fashion are other, but what we wanted to do here is, to illustrate the rule for integration, and then there also want to illustrate the application of the final value theorem in working out the solution.

As third example, let us consider the convolution of function of time with delta function or unit step function.

(Refer Slide Time: 16:13)



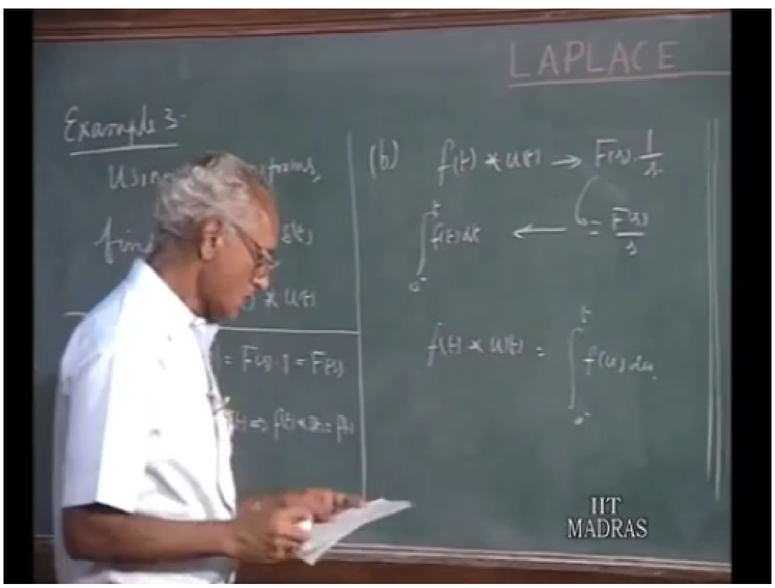
So, using Laplace transforms, find what this functions among to; f of t convolve with delta t first. B, f of t convolves with u of t. If you recall, the results we already know. When we talked about the convolution property in the introductory lectures, we said whenever a function is convolved with delta t that function is reproduce itself.

That means f of t star delta t f of t itself, because the delta t stands this functions as its moves along, and it reach at any particular point, the value of the product of delta t and that f of t. The displace delta t and f of t will be f of, the value the function at the particular point of time that is the magnitude the impulse.

And when you integrate that will be f of tau times f of tau whatever it is, wherever it is situated. Therefore, the convolution of this will be must result in f of t itself. And when we talked about the integration rule, under Fourier transforms, recall that f of t convolved with u of t, has been shown to be the integral of f of t.

We will see this result here, using Laplace transform. So, a, we know that the Laplace transform of f of t convolved with delta t, is the product of the Laplace transforms of the 2 individual functions. This is f of s multiplied by 1. The Laplace transform of delta t is 1; therefore, this is f of s. And the inverse Laplace transform of f of s, is equal to f of t. Therefore, f of t star delta t is f of t itself; that is what we already know.

(Refer Slide Time: 18:33)

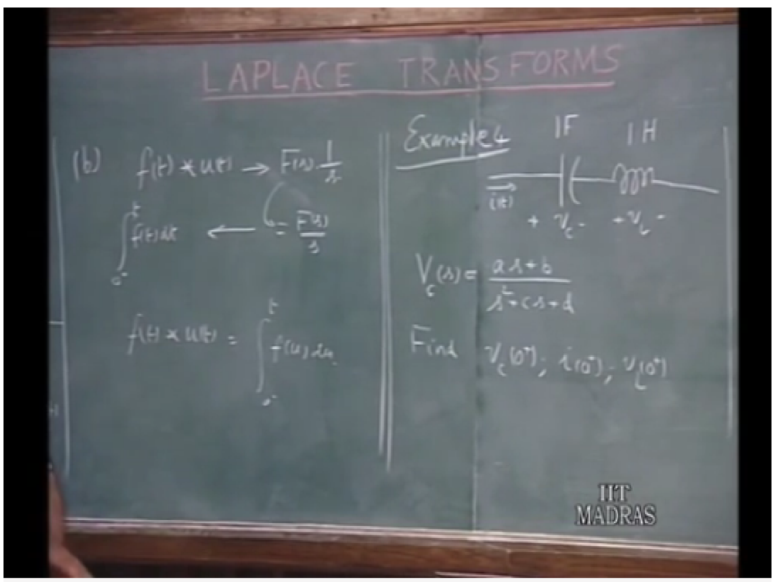


Second question; f of t convolved with u of t, as for its Laplace transform f of s multiplied by 1 by s. So, this is f of s over s, and we know that f of over s, has the inverse Laplace transform 0 minus to t of f of t dt. So, when we convolve with f of t with unit f function, it is the integration of ft.

In the Fourier transform theory, you may taken this from minus infinitive onwards, but in the Laplace transformation because you are dealing with causal time functions. We take the limit from 0 minus to t, because f of t is assumed to be 0 for negative values of time; that is the difference.

So, the integral of f of t dt from 0 to t, is f of s over s, and that is indeed the product of Laplace transform of these u function. Therefore, we conclude that f of t star ut is 0 to t of fu du if you wish, because it is dummy variable integral of f of t. assuming that f of t is the causal time function; therefore, we are starting the limit integration from 0 minus onwards. These results are already known, but this is only a confirmation of results that we already derived, in earlier context.

(Refer Slide Time: 20:11)

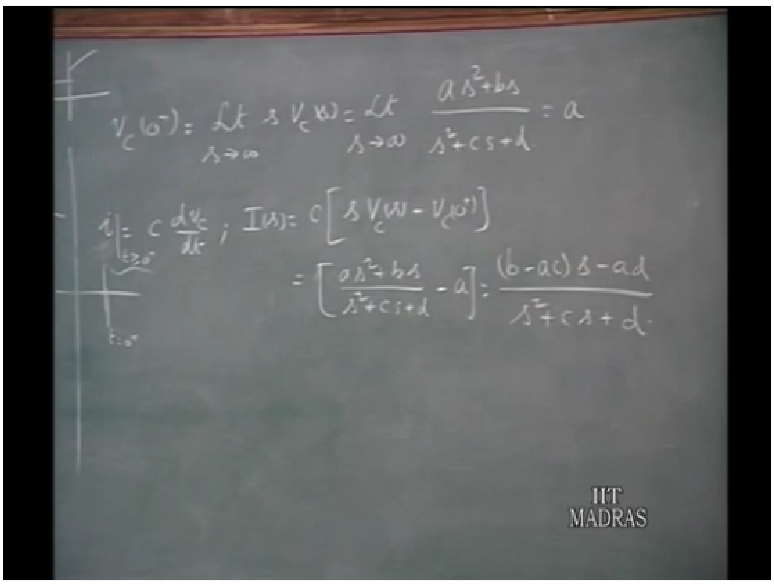


Example 4; Let us consider now, a section of a circuit, in which we are inductance l of 1 henry and a capacitor of 1 farad, and let the voltage is across the 2 elements we described as vc and vl. In some circuit analysis, we have obtained vc; the Laplace transform of vc of t Laplace transform. The voltage across the capacitor is found out to be as plus b over s square plus cs plus d, through some analysis we have obtained this.

Using this information we are asked to find, vc 0 plus, the voltage of the capacitor immediately after 0. The current in the circuit 0 plus is the current it. And the voltage across the inductor 0 plus. These are the 3 quantities that are require to be found out. Now, since we are interested in finding out the 0 plus values in all this situations. We can assume that the Laplace transform of that we are talking about.

The Laplace transform defining integral start from 0 plus 0 onwards, not from 0 minus, because you are after all interesting 0 plus value. That means, we ignore any jumps in functions from 0 minus to 0 plus. We assume that all our functions starts from 0 itself, and therefore, we can use the 0 plus value, instead of 0 minus value wherever it's necessary.

(Refer Slide Time: 22:25)



So, let us see vc 0 plus the initial value of the capacitor voltage, is limit as s goes to infinity of s times vc of s; that is initial value theorem. Therefore, this will be limit as s goes to infinity, of as plus b upon; say s square plus cs plus d multiplied by s. This must be multiplied by s and take the limit as s goes to infinity. Therefore, that will be as square plus bs over s square plus cs plus d.

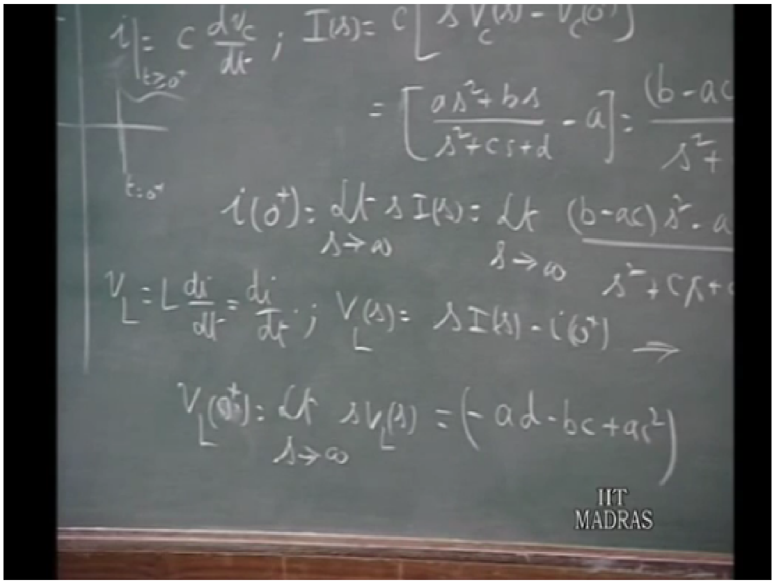
And when you take the limit as s goes to infinity and as I mentioned in the last lecture. We can take s going to infinity in the, along the real axis, in the positive time axis. Therefore, we can ratio of the 2 leading coefficients only, because all the other terms where in significance cs plus d, hence with significance in s square, bs becomes negligibly small compare with as squared. Therefore, it is a ratio the 2 leading coefficients; therefore, this is a.

So, the immediately after 0, the capacitor voltage at 0 plus as you approach 0 from the positive direction will have value a. Now, what about the current? We know the capacitance i equals c dvc dt; that is the relation between the current in a capacitor, and the voltage across the capacitor for any general variables it and vc. Therefore, i equals c d vc dt. Therefore, we can say the Laplace transform of the current i of s is c times.

If vc as the Laplace transform vc of s, the derivative d vc by dt is s times vc of s minus. Now, we are taking stock of all values from starting from 0 onwards, 0 plus onwards. Therefore, I may write here vc omega plus. If you are starting all over account in from 0 minus you would put it as 0 minus; that means, I am trying to take the value variation of current. Suppose this variation of current. I am taking the star from 0 plus onwards only. I am not considering the jump from if any from here to here.

That means there is any jump in vc; suppose vc 0 minus is this and vc 0 plus is this, the current would have impulse here. But those impulses I am ignoring. I am talking about the variation of the current; i for t greater than r equal to 0 plus. Only that is the type of current which i am interested. So, that is why i am putting this 0 plus.

So, this will be see in our cases 1 farad. So, this will be s times vc of s. vc of s is a s square plus as square plus bs over s square plus cs plus d. Therefore, s times vc of s is as square plus bs over s square plus cs plus d minus vc 0 plus as just be evaluated that is a therefore, when you do this, then it become b minus ac times s, minus ad divided by s square plus cs plus d; that is the Laplace transform of the current i of s.
(Refer Slide Time: 26:09)



To find out i 0 plus, we apply the initial value of theorem, and say that this is equal to limit as s goes to infinity of s times i of s, which means limit as s goes to infinity of b minus ac s square minus ad times s divided by s square plus cs plus d, which by the same arguments that before, is the ratio of the 2 leading coefficients, which will be b minus ac.

So, this is the answer so the initial value of the current b minus ac mps. This is the Laplace transform of voltage across the capacitor. Now, again to find out the value, initial value of the voltage across the inductor vl; vl equals l di dt. In our case the inductance value l is 1 henry therefore, this is di dt, which now tells us that vl of s is s times i of s minus i 0 plus.

Once again we are taking the variation the voltage that was inductance from t equals 0 plus onwards, where ignoring any changes from 0 minus to 0 plus values. So, we again multiply the expression for the Laplace transform of it which is this 1 multiply this by s substitute, and remove from that i 0 minus; which is be minus ac. if you do that.

I will not go into the details of this this will turned out to be give some expression here, and then vl infinity is limit as s goes to I am sorry vl; 0 plus that is what you are interested, limit as s goes to infinity of s times vl of s. So, after carrying out this work, and take this limit as s goes to infinity of s times vl of s you will get the answer minus ad minus bc plus ac square.

So, that so many rules is the value of the voltage across the inductor. These examples shows that once we have the Laplace transform of quantity find out the initial values of different quantities associated with the capacitor voltage in the capacitors. You really did not have to carry out the numerical the value of the capacitor voltage vc t at all in symbolically in terms of ad dc where evaluate this.

So, in other words if the particularly and will the numbers say 97 35 9.5 whatever it is. You do not have to do the numerical work, try to find out real value of t, and from that vc of t and then try to find out i of t, and then from that vl of t, and then find the initial values of this respective quantities. You can straight away use the initial value theorem, and get all the initial value of the quantities which you are interested in.