Radiation Heat Transfer
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Lecture - 30
Radiative Equilibrium with scattering

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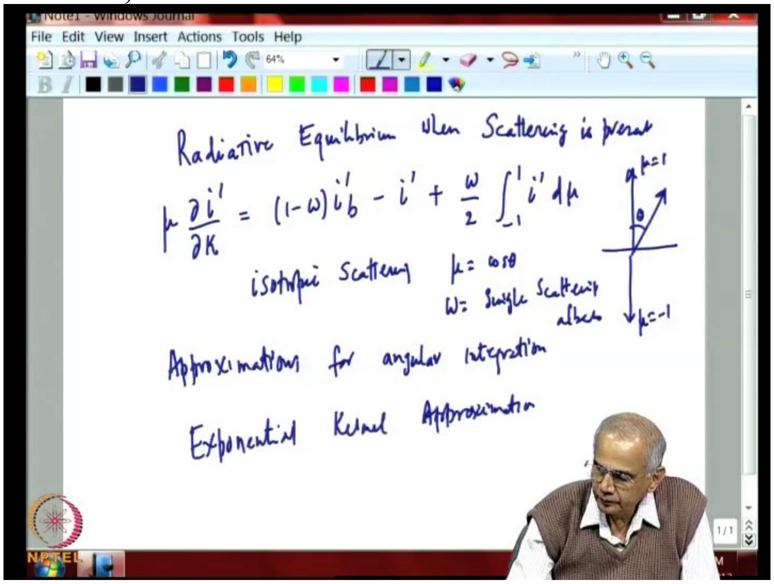

In this lecture we will look at radiative equilibrium, when scattering is present. If we recall from early lectures, the equation for radiative transfer with scattering would be like this. This is a relative change of intensity with optical depth. This is a single scattering albedo.
Here for simplicity we have looked at only isotropic case. We are looking only at isotropic scattering for the purpose of illustration. Here mu is nothing but cos theta. That is, this is the vertical coordinate and this is the direction of your radiation, this is theta. And, so you are going from plus one to minus one here. This mu equals minus one here and mu equals one here and omega is the single scattering albedo.

Now, this equation is more difficult to solve than the equation we have solved when omega was zero; pure absorption case. Where, this term was not there, this term was not there. We could integrate this equation quite easily and get the answer.
Now, the problem gets little more complicated because one more term with scattering, which involves the integral over the angle. There are various possible approximations for angular integration. If we recall, we discussed the Exponential Kernel approximation in

early lectures, which essentially involved replacing the rays travelling in various angle with a single ray at a specific angle. We saw that this approximation was quite useful. But, when we get into the problems with scattering the problem gets more complicated. We look for approximations, which simplify the problems right in the beginning.

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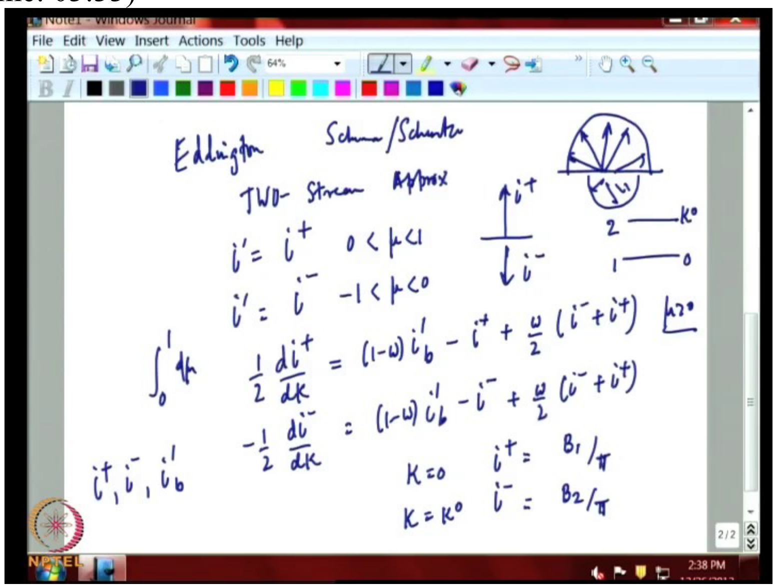

Now, one well-known approximation; which is, attributed to Eddington in some books or some books to Schuman and Schuster, involves assuming that the radiation is isotropic, but the upward and downward directions are not equal. That is, in many radiation problems their upward and downward intensities are usually not equal because it depends on from which surface the radiation emerges. But, within the hemisphere we can neglect the variation of the intensity with the angle. This is what is known as two stream approximation.

Here, what we are assuming is that, we can assume that all the streams going upwards are independent of angle; we can treat them as the single angle. All streams going downwards are isotropic in the lower hemisphere and they can be treated to be at one angle. But, the upward and downward intensities are not same.

This source is dividing the radiation problem which has multiple streams; multiple directions into two directions; up and down. Now, this is a very useful place to begin because once we have understood the basic way of doing two stream approximations. We can also then do four streams, six streams, eight streams, sixteen streams, thirty two streams. We can go on extending the problem to more and more streams. But, the basic principle is same as the two stream model.

In the two stream model, what you are doing is the intensity in the upward direction can be called i plus, which is valid for mu from zero to one, which is essentially in this coordinate. The intensity i prime is equal to i minus. That involves all downward going streams. By replacing this by a single number i plus and this by single number i minus half to integration. Let us do integration.

We take the equation in the last section. Integrate first, zero to one d mu in the upward direction. When you do that, you will get the following result. The first term involves integration of only i prime which is independent angle in upward quadrant according to the assumption. We are integrating over mu, so this will be half; first term. Then, in the second term there is no angular dependence, so it will remain same. The third term again will not change because it is independent of angle. The last term because it is constant with mu, it will become i minus plus i plus. This is for the upward region, so mu greater than zero. For mu less than zero, we have another equation which is minus half d i minus one by d kappa. It is equal to distance in change. It does not change, i plus, this is i minus plus omega two i minus plus i plus. The main difference that came in this integration over angle is in the first term on the left hand side; where, mu d mu integrated to plus half in the upward quadrant and mu d mu integrated to minus half in the lower quadrant. We have two equations and there are three unknowns in the problem; i plus, i minus and i prime b. The three unknowns are i plus, i minus and i prime b. We have two equations. We will have to look at a third equation. Before that, let us understand one boundary condition; at kappa is equal to zero below boundary. The upward going to radiate has to be equal to radiation B one pi. We have presently calculated model; zero and kappa zero. This is one and this is two.

In the top plate, the downward radiation has to be nothing but B by two pi. We know that boundary conditions depend on i plus and i minus at one point. We have now two equations, but three unknowns. We have to generate one more condition, and that we will very soon realize we need fluxes. This is the intensity radiation of fluxes.

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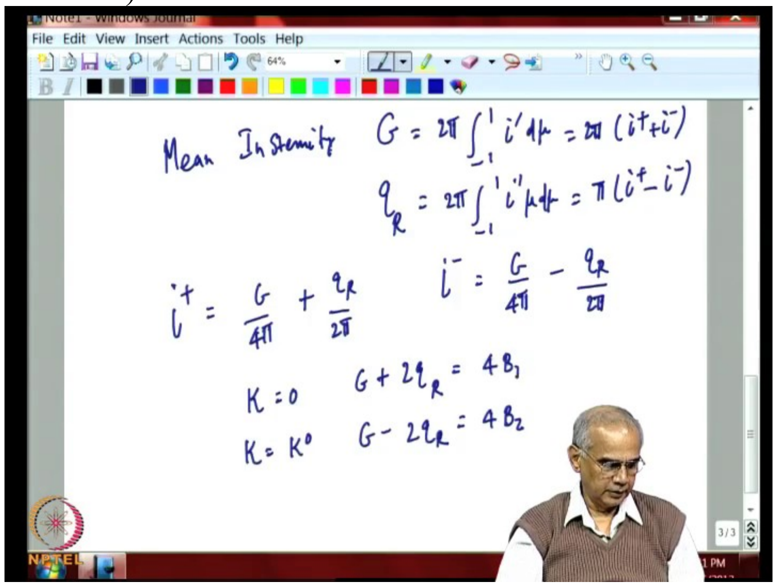

But before we do that, we need to define certain quantities. First quantity we define is the mean intensity G. G is by definition, average of i prime. In this simple Simon and Schuster or Eddington approximation, i prime is one value in the upper quadrant, upper hemisphere and another in lower hemisphere. This one will ultimately be nothing but i plus plus i minus.

Now the flux, radiative flux, which we know all along is nothing but two pi minus one to one i prime mu d mu. This we know from the definition of radiative flux with the very first class. Now, this has to be integrated separately in upper hemisphere, in the lower hemisphere. This will give us pi into i plus plus i minus for i plus, minus i minus. Sign plays a very important role here.

Given these two, we can rewrite i plus as G by four pi plus q R by two pi and i minus as G by four pi minus q R by two pi. So, what this tells us is that the upward and downward intensities are different because there is a radiative flux in this system. The difference between upward and downward is not same because of all the radiative flux.

Now, the boundary condition we wrote last time can be now written in terms of these quantities. We will get at kappa equal to zero, G plus two q R. It will come out as four B one. At the top surface these are G minus two q R equal to four B two. If we go back to the equation for i plus and i minus zero down, we can now rephrase some in terms of i plus and i minus. If we do that, we will get the following result. Now, we are writing everything in terms of q s and G s.

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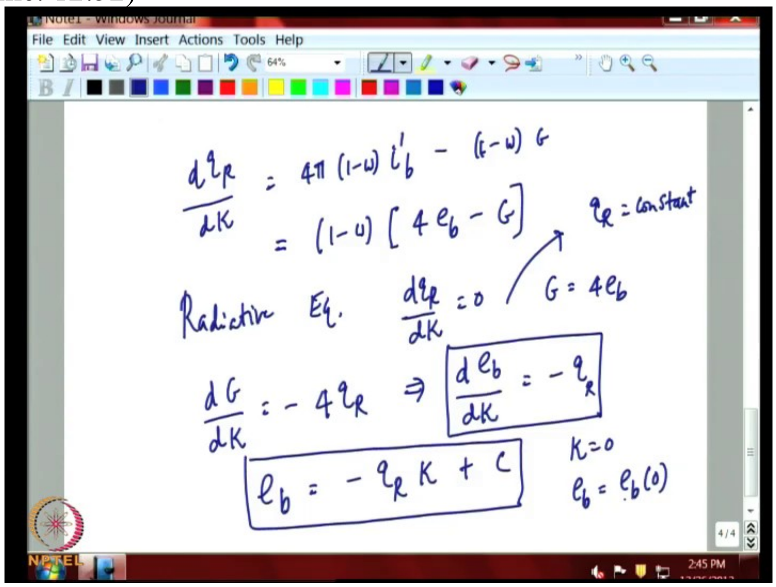

We will have d q R d kappa is equal to four pi one minus omega i b prime minus one minus omega G. This is the first term in terms of radiative flux. If we consider the case of radiative equilibrium, which is what we are trying to focus on; before we do that, let me simplify a little bit, so that we will make the point very clear. We can see here one minus omega becomes common here. We have four pi i prime b or we can write it four e b minus G.

So, under radiative equilibrium which is a focus right now, we know that flux cannot vary with space and so G is equal to four e b is the solution. It tends to the four times the emissivity power of black body emission at the temperature of the medium. This is a very nice result; because if you go back and look at the relation between G and q, you are showing that G equals 4 e b. We also know that d G d kappa is equal to minus four q R. This implies d e b d kappa is minus q. Very simple result we have got, which will be the basis of our discussion.

Now, this result is now very important; because in radiative equilibrium, notice that q R is a constant. If they are constant, then integrate the equation to write e b is equal to minus q R kappa, plus a constant. This result relates the emissive power of the gas. This depends on gas temperature, this quantity, to the radiative flux, optical depth and a constant. Except for this constant, we realize the fact that kappa is equal to zero e b is equal to whatever the gas temperature at that point.

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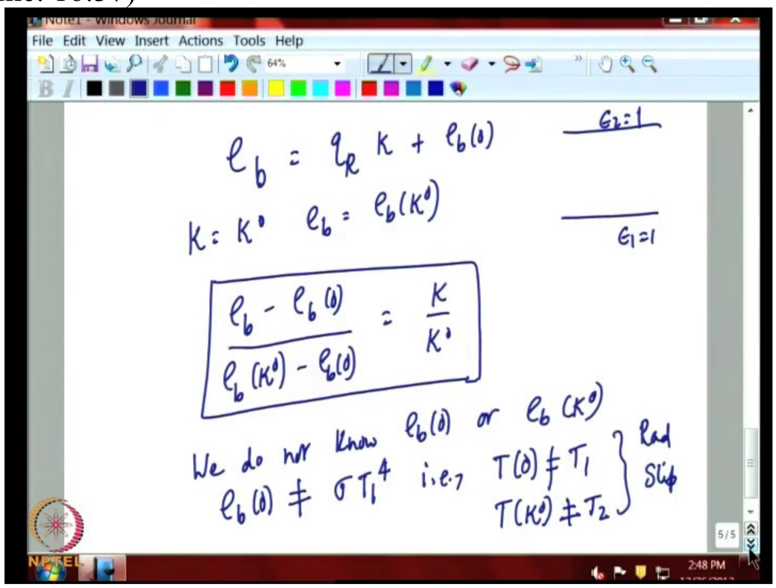

The result that we will get is this e b is equal to q R kappa plus e b zero. We could also, of course recognize the fact that at kappa is at the top; e b has to be equal to whatever is the gas temperature to the top of that emissive power. Substitute all these, then you get a simple result, which says that e b minus e b of zero by e b of kappa zero minus e b of zero is equal to kappa by kappa zero. This is a very simple result which we could have written at quite easily.

Let us say that kappa equal to zero, the temperature of the gas is same as temperature of the gas touching the bottom surface; if kappa is equal to kappa naught, the temperature of the gas is same as temperature of the gas from the top surface. But, all the decision is simple and elegant. It is not that useful because we do not know these; we do not know e b zero or e b kappa zero. It is important to recognize that in the radiation problems that e b zero is not equal to sigma T one to the power of four, assuming both sides are black.

Even if they both are black, the black body emissive of the bottom surface need not be equal to the gas black body emissive because the temperature of the gas at T is equal to zero; that is, temperature of the gas at T equal to zero is not equal to T one. Similarly, temperature of the gas at the top is not equal to T two; this is because of radiation slip. Because of the radiation slip, we priory do not know what e b, e b zero are. We need to find ways to calculate this quantity from basic fluxes. The question arises as to how we do this calculation, how do we relate the radiation temperature of the gas at the bottom with the surface temperature. This slip has to be calculated. We have done this earlier in a different context, but now we have to do it in the context of this problem.

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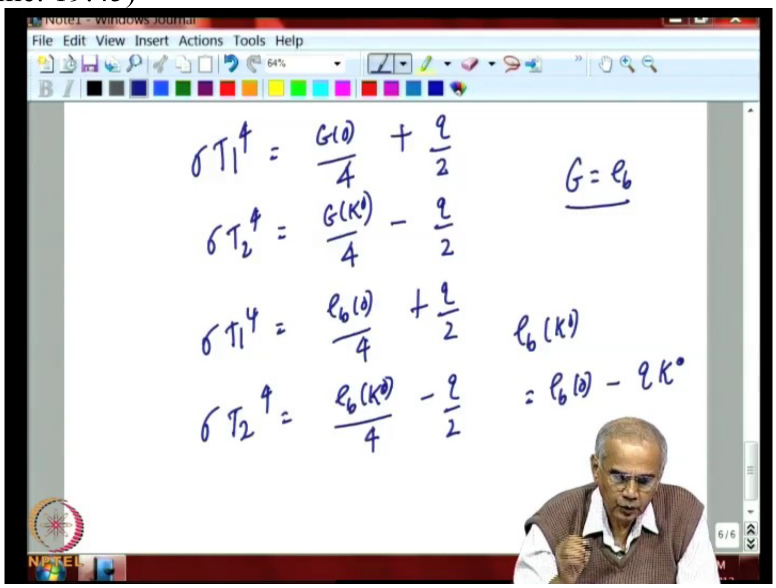

From the boundary condition we have to recognize the following relationship. Sigma T one to the power of four is nothing but G o by four plus q by two and sigma T two to the power of four, this is from the boundary condition; sigma T two to the power of four is equal to G zero by four minus q by two. Now, we know that G is equal to e b. That is the condition from radiative equilibrium. We also know that the solution of the equation that, e b of kappa zero is equal to e b of zero plus, minus q into kappa zero. This is coming from the direct solution of the rate transfer equation. Now if we substitute these here; so, let us go to next page.

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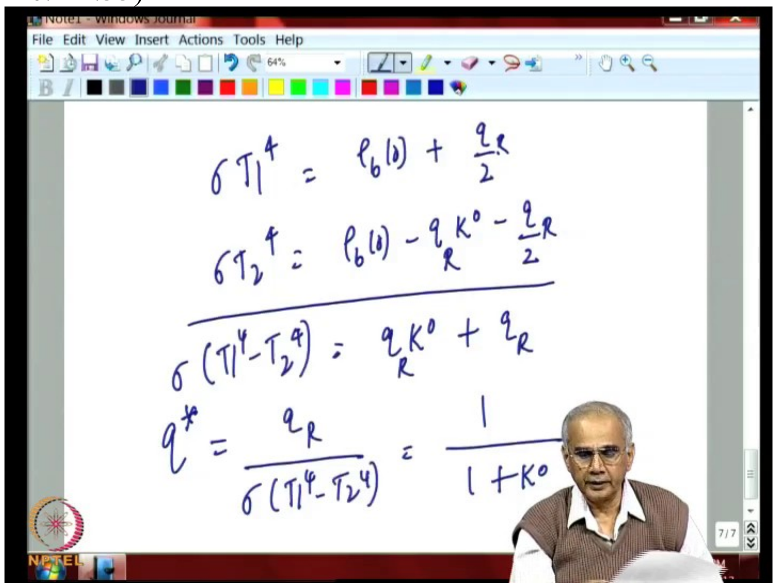

We will say sigma T one to the power of four is equal to e b of zero plus q by two. This we know already. Now, the second way we are going to change it to read like; we replace e b of kappa zero with e b of zero and q kappa zero minus q by two. Now, we want to find what q is. We subtract here. We will get sigma T one to the power of four minus T two to the power of four. This cancels out. This acts q. So, q kappa zero plus q. We get a very neat result. We are getting result that q R by sigma T one to the power of four minus T two to the power of four is equal to one plus kappa zero one by one plus kappa zero. Now, if we recall the solution of the radiative equilibrium problem, we did many lectures ago with the exponential approximation. At that time, we call this quantity as q star, a non dimensional q. The result, we want to recall to you is the result we got there. The result we got there is not quite same as the result here. It is not identical but quite close.

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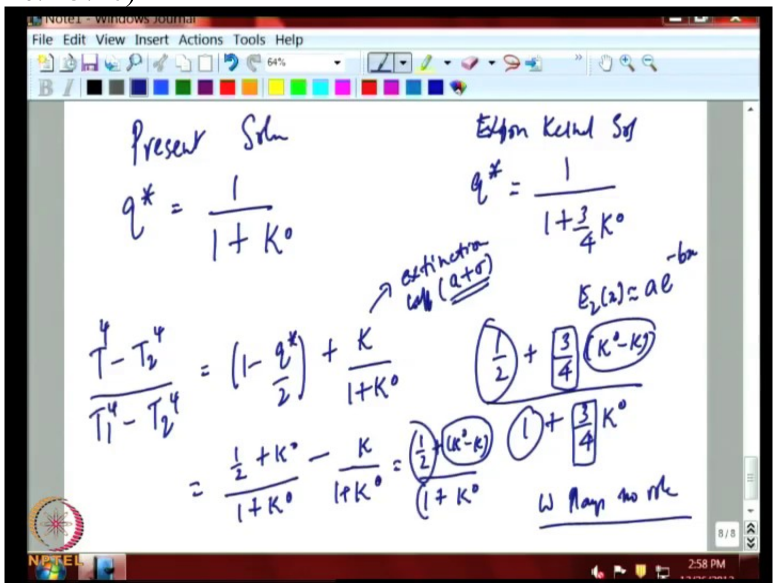

Let us now compare the present solution and the Exponential Kernel solution. The q star, here is one by one plus kappa zero; here we got q star is equal to one by one plus three-fourth kappa zero. There is slight difference in the coefficient. If we recall our discussion of the Kernel approximation, we said that this coefficient is very much sensitive to the approximation we made for E two of x. We said, depending on the choice of a and b we can get different results.

Once we make an angular averaging and approximation, then you must realize that the answer is sensitive to the choice you make. This result is not identical to the result we got earlier only because of the angular approximation. Then, we look at the solution for temperature. Temperature is T minus T two to the power of four. Here, we will get after substituting this quantity as one minus q star by two plus k by one plus kappa zero. Along this side if you recall, it was half plus three-fourth kappa zero minus kappa by one plus three-fourth kappa zero. Now, these two may not look alike. But, they are quite close in their functional forms, so that the two methods of solving the radiative equilibrium problem; one, using the kernel approximation and the other using the Simon and Schuster or Eddington approximation, give very similar results. These two will be quite alike.
This can be made to look somewhat similar to that by looking at; substitute this q star we will get half plus kappa zero by one plus kappa zero plus k by kappa zero. If we want one

plus kappa, we can change it to make it to look very close to the other result. By and large results are not very different. Actually there are minus sign.

Actually we can write this as half in plus kappa naught minus kappa by one plus kappa naught. Now, the main thing we must appreciate here is the fact that, the only difference really is that these two are similar. Those are also same as that. The only difference is in the coefficient. The coefficient is somewhat different because in the Kernel approximation, we can relate this to three-fourth to the choice of a and b in the Exponential Kernel approximation. But, the results are not really substantially different.

Now to highlight this point of view, we will adopt another method; wherein we will solve the same problem, but slightly different approach which involves making a slightly different approximation to the problem. Before we go for the notice that we started with the problem isotropic scattering, notice that omega plays no role here. In radiative equilibrium, the single scattering albedo plays no role, however these kappa is now the extinction coefficient. Hence involves both the absorption coefficient and isotropic scattering coefficient.

In the radiative equilibrium problem the single scattering albedo does not play a role. The scattering process is incorporated in the definition of the extinction coefficient, which is sum of absorption and scattering. But, later you will see that when we deal with problems which are not in radiative equilibrium, omega will play a very specific role in the solution. So, whether omega is important or not important in the problem depends upon, whether you are dealing with pure radiative equilibrium or we are dealing with the problem with either in conduction or convection that also plays a role, then omega also plays a role. Let us now look at another solution, so that we fully understand the nature of the solution.

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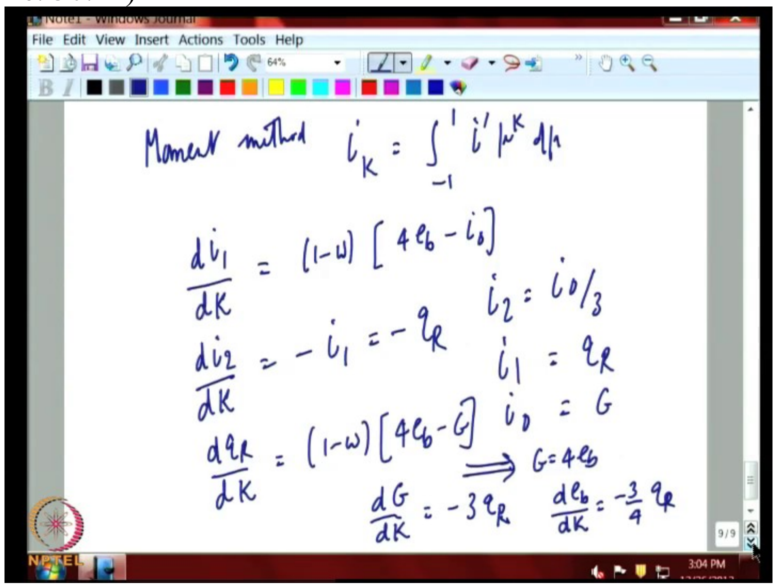

This is called moment method. In the moment method, you look at various moments of the intensity. For example, general k th moment of intensity is defined as minus one plus one i prime mu to the power k d mu. This is the k th moment of intensity. When k is zero we have the mean intensity; when k is one we have the flux; when the k is two control radiative flux and pressure and so on.

The moment method is an elegant approach to the same problem of angular variation. But, now we have an approach by which we can go for high higher moments to get more and more equations. If we take this zeroth moment which we had already done, we will follow the previous notation; this small i. We saw that d i one by d k is equal to one minus omega into four e b and i zero. This we have already seen.

Now, the new result you are going to derive is by taking the second moment at k equals two and relate i two to i one. If you use the Simon and Schuster approximation, one can show that i two will be nothing but i zero by three. That is, in this equation k equals two; your assumed intensity is uniform in the upper hemisphere; uniform and a different value in the lower hemisphere. If we integrate, you will get this answer. We have two equations and two unknowns, but our interest is in still radiative equilibrium.

We recognize the fact that i one is nothing but radiative flux; i zero is nothing but the mean intensity. We go back to our result which we got earlier who says d q R by d kappa is equal to this is same as what you got last time. Then, on radiative equilibrium you say G equals four e b. But, if we look at the second equation for G, we get a very interesting result which says that, there is a mistake. We just remove this now and write; i two is i zero by three and this i one is minus q R.

We substitute for i two, i zero by three and then use G here; we get the result; here it says that d G d kappa is minus 3 q R. Now, we want to compare this with the previous approximation somewhat differently, we got d g d kappa as minus four q R. That is the difference in the manner in which we will do the angular integration of the intensity. Here, we invoked higher order approximation; really, one was a lower order approximation. So, since G equals four e b, We can see straightaway that d e b d kappa will be minus three-fourth of q R.

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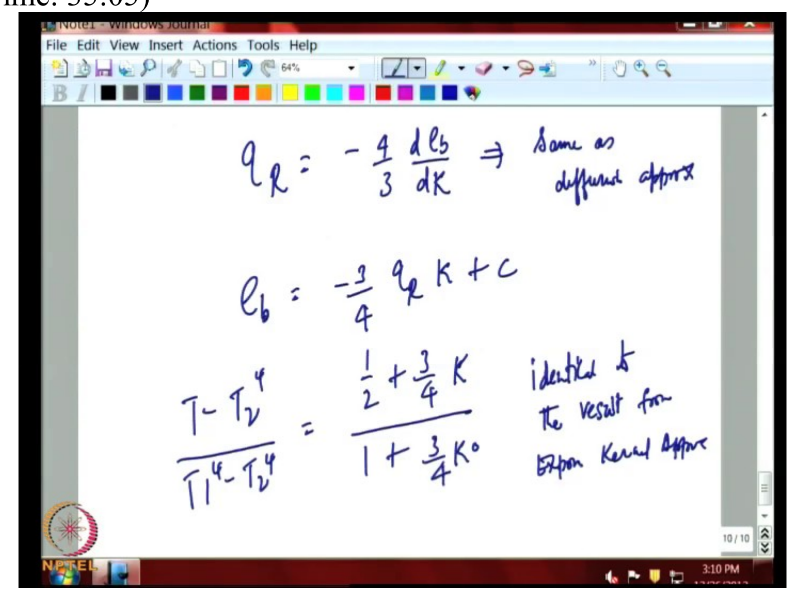

We can rewrite this as q R as minus four-third d e b d kappa. Now, this is what we did earlier as the diffusion approximation. Earlier, the result was obtained in the optically thick limit. In the optically thick limit we argued that q R as minus four-third d e b d kappa. Through that we bought in the definition of Rosseland mean absorption coefficient. But, now we are getting the same result without really invoking the optical thick limit. Now, you must remember that some of the approximation we have made regarding angular variation gets better and better as you go to thick limit. Although we did not explicitly invoke the thick limit, in reality some other approximations we have used here is strictly valid on the thick limit. Now, we get the integral equation. We have done this many time before. So, e b will be equal to minus three-fourth q R kappa plus a constant. Given the boundary condition that we have, which you know about, which are same as what we did just now there is no difference. We will get the following result, which is identical to the result from Exponential Kernel approximation.

We have got this result of the temperature variation between two plates, within two black plates by Kernel method. But, the result is identical to what we obtained by the Kernel approximation. This is happening because in the angular integration, the kind of approximation made is generally similar to the approximation we made when we dealt with Exponential Kernel approximation ok.

This implies that this temperature distribution within the gas will go as a plus b kappa zero and c plus d, a plus b kappa. So, a plus b kappa by c plus d kappa zero. So a, b, c, d, really depend upon the details of the approximations which you have made about the angular dependence of the intensity. For many applications those details do not matter ever it is three-fourth or one. Really, it should not matter that much. But, we do recall that this particular solution which is obtained through Kernel approximation agreed very closely with the exact numerical solution. The choice of certain approximation leads to a more accurate final result.

What we have shown so far is the fact that the solution can be obtained in many ways; either in Kernel approximation or Simon Schuster or Eddington approximation or the method of moments. All the three are really, what they doing are really giving you various ways to approximate the angular dependence of some intensity. These approximation solutions are useful, if we know what kind of angular variation is important. As we saw, when we compare with the exact solution, this can be within few percent of the reviewed solutions. Therefore, such approximations are very very useful in the applications that we encounter in real application. Hence, these approximations are extremely useful.

Now, the extension of these two, the situation where there is heat transfer by conduction.

Now if we recall that the pure radiative equilibrium problem had a slip at the boundary which is a quite common result, we encounter in the radiation transfer. Radiation transfer is the basic equation; integral equation. Hence, we cannot impose the condition that the temperature of the gas at the wall is equal to the wall temperature. We must allow for the fact that there is some slip. This is an inevitable consequence of a problem in which we have neglected all processes, except radiation. When we do that, the problem does not allow one to assume that the temperature of the gas at the wall is the wall temperature.

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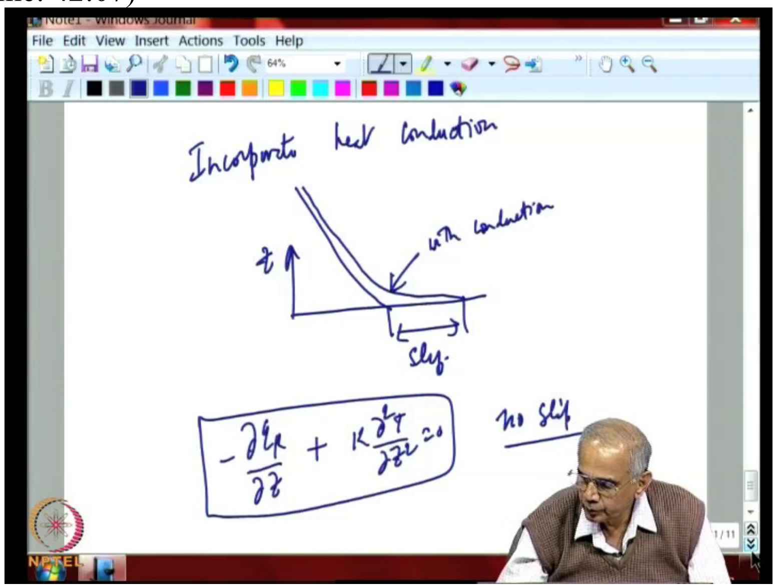

One way to get around is to incorporate heat conduction. The heat conduction is always there. We have neglected it so far or to simplify the problem; which is always there. The role of heat conduction which is diffusion phenomena is actually to smooth the temperature profile. We expect that the solution we got for pure radiative equilibrium, which had a slip, now will become continuous and not discontinuous. The smoothening of the sharp near the radiant wall will be done by conduction. The conduction in heat transfer will ensure that the temperature very near the wall is smoothened by the diffusion process, that is, near wall conduction.

If this is vertical coordinate z, then remind you that the net diverging radiative flux has to be equal to the conduction flux. This requirement has to be satisfied by the invoking of energy equation, but remember that this term can be quite small compared to this term, the first term, except very very close to the ground; where, the second derivative of temperature can be quite large. But, this diffusion term or conduction term is essential to avoid slip. Now there will be no slip, once we have slip incorporated the diffusion term. The diffusion process will ensure that you have a continuous temperature profile.

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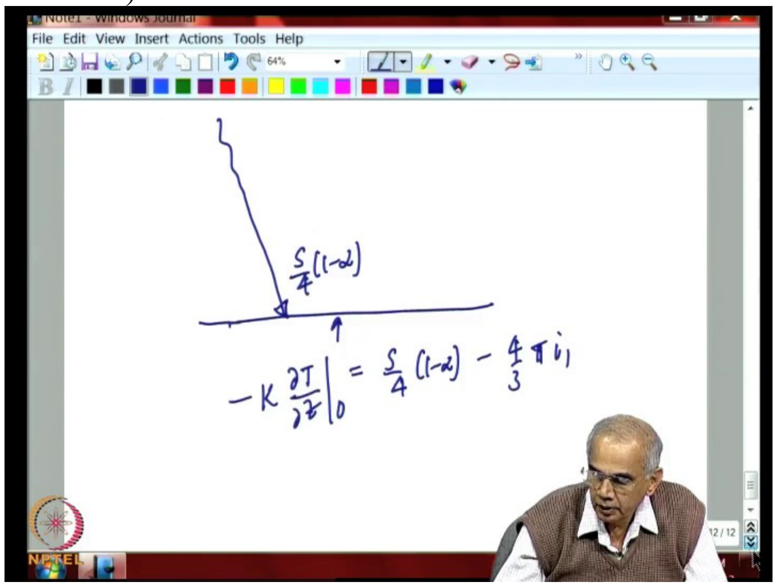

Now, we want to relate the heat flux by conduction and by radiation; is equal to what is coming here. We have a simple model of earth's atmosphere neglecting the absorption of the atmosphere. So, what is coming in really is S by four into one minus alpha; that is, incoming solid radiation after energy is lost by reflection to the space. All the remaining energy that we are assuming is not absorbed by the atmosphere, but the surface. In this case, we can write down the energy balance at the wall, at this wall, has to be equal to energy absorbed in the entire column really, minus four by three pi i one.

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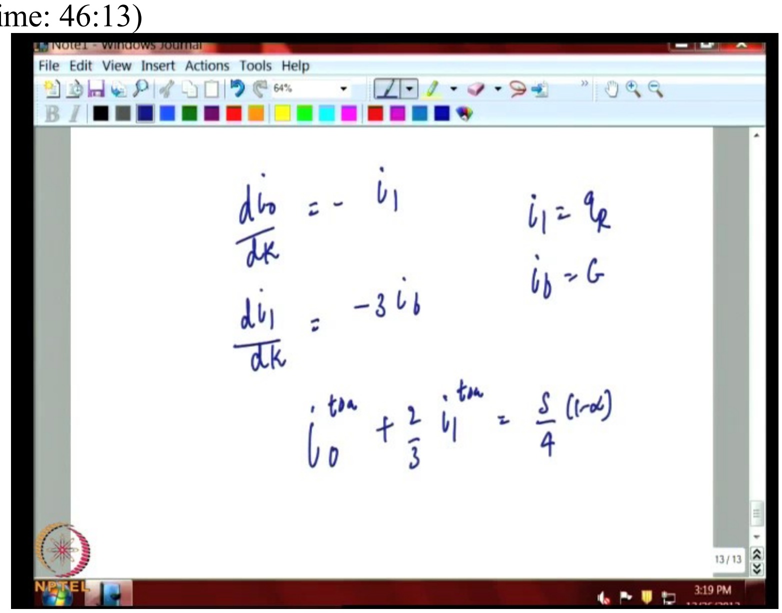

The i one is same as q R. We can again invoke the rate transfer equation, which we used earlier and show that d i zero by d kappa is minus i one and d i one by d kappa is minus three i zero. Now, we will have additional term here, coming in; which invokes the flux at the flux that is going through the system. The d i one d kappa will be related to the outgoing. This we have already invoked earlier.

The only thing now we have to do is that in terms of the radiative flux; which is, if you recall this case, i one is nothing but q R and i zero nothing but G. That we have already used. We can actually then solve the equation. The equation really says that, at the top of atmosphere the total radiation absorbed is, whatever is incoming radiation minus deteriorating radiation.

The intensity at the top of the atmosphere, which is not measured by satellites above this one and this one; top of the atmosphere has to be equal to the absorbed solar radiation. With the inclusion of conduction, the problem thus gets a little complicated because of the fact that the atmosphere is infinitely thick. To simplify this problem, we look at somewhat similar simple exercise, wherein we assume that the atmosphere is strongly absorbing over all layer; q zero, z one. Above that layer we assume that there is no absorption.

This simple logic in this case would be that we are taking into account only infra radiation absorption in the atmosphere in the troposphere, but are neglecting the phenomena going on in the stratosphere. This is essentially a simplification, which will make the problem analytically tractable. That is the aim of this exercise; is to give you a solution which is analytically tractable. We do that by the following approach.

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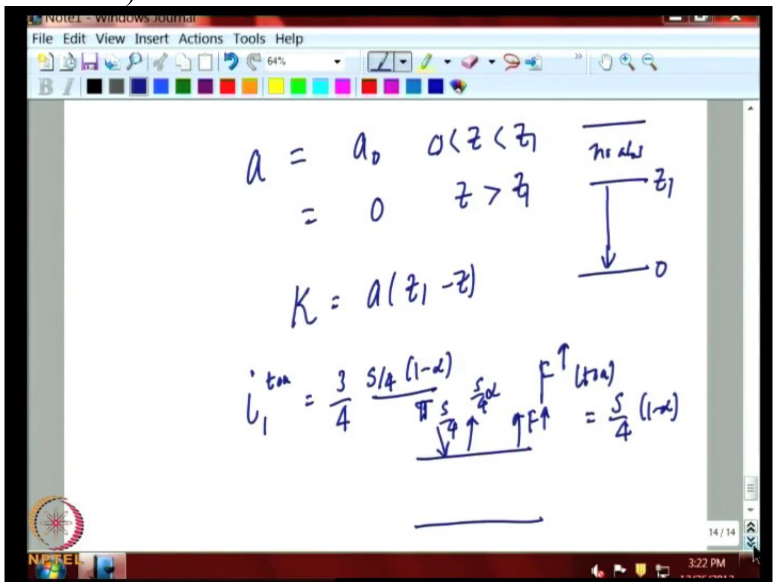

We are saying that the absorption coefficient of the gas a is equal to a zero for the height between zero and certain height; where, typically where the water vapor in atmosphere, and above that height we are assuming that there is no absorption. In this simple model what will happen is because there is no while, since there is no absorption above that layer, our optical depth kappa now will be scaled somewhat differently from what we did earlier. It will be z one minus z because when z equals z one, we are from the top of atmosphere; when z equal to z one, optical depth is zero. If it goes to zero, your optical depth is going up. Here there is no absorption above that. We will get the intensity at the top atmosphere will become nothing but three by four into S by four into one minus alpha, that is, the absorption variation by pi. This is the result we are going to get.

At the top of atmosphere, we are saying that there are flux at the top of the atmosphere has to be equal to S by 4 into one minus alpha because we recall that what is the radiation absorbed has to be emitted back. We have here S coming, S by four coming in, S by four alpha going out and the outgoing flux has to be equal to this minus this; because by there is no balance in the that atmosphere.

Once invoked, we will get the solution with conduction. The conduction solution has the advantage that we would not see the discontinuous, continuous that you had absorbed earlier in the pure radiative equilibrium solution. Now, the details of the solution are little bit intricate.

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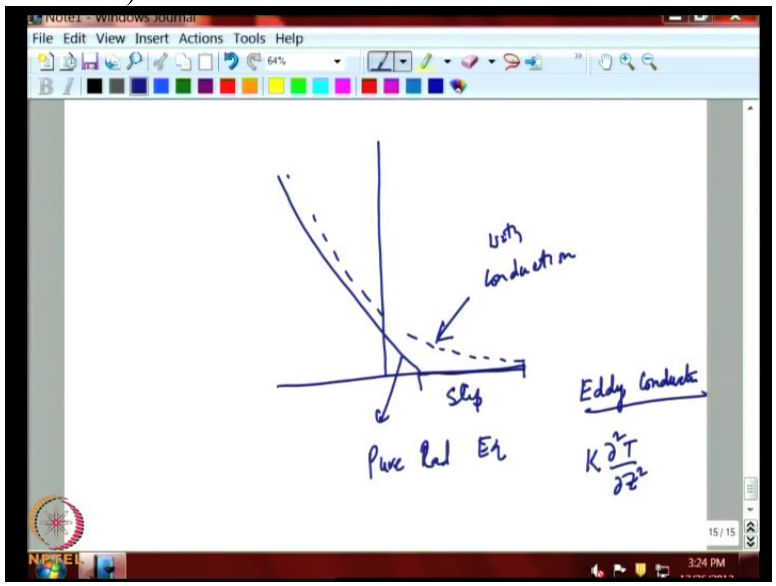

If we just look at the solution, we will see that the radiative equilibrium solution will have a break here; it is slip. Once we allow for conduction, this solution will be absolutely smooth. This is pure radiative equilibrium. Once we include the conduction term, then the solution will be extremely smooth all the way through the wall.

This in a way mimics what happens in the earth atmosphere. Although we have simulated this as a molecular conduction, in reality what is happening is due to term "Eddies". We can talk about this "Eddy Conduction". This term that is coming in there, which normally is used to represent the bulk conduction very close to wall can also be use approximately used as Eddy conduction in the boundary layer of the earth's atmosphere, which is typically about one kilometer thick. There we can assume that Eddies are doing the transfer, and they are doing it similar to the way it is done for the bulk conduction.

We have seen that in the lecture that, we can get the result we obtained earlier for Exponential Kernel approximation by other doings; either by assuming that the intensities in the upper, lower hemisphere are constant and not functional angle or appealing to the moment method and using higher order moments to obtain more equations to eliminate the unknowns in the problem. But, whether we use the Kernel approximation or we use the moment method or the approximation of two stream approximation of Simon and Schuster or it is also called Eddington, the answers are all very similar. There are slight differences in the numerical value of the coefficients, but these are somewhat minor. When we plot these results, they look extremely similar. They provide the same insight that we got from the Kernel approximation.

The advantage of the new method we discussed in this lecture is that, they can be invoked in even more complex problem, wherein there is scattering; wherein the Kernel approximation, now it is so easy to implement. We talked about two stream approximation. Now, this has been extended now to four streams, six streams, and eight streams and so on. With the availability of powerful computers, extending this two stream to multiple streams is a very trivial task. It is only extending the complication. We will discuss other issues of miss scattering in the next lecture.