Mathematics-I
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Lecture – 1
Real Numbers

In these lectures, we are going to study a branch of mathematics called calculus.

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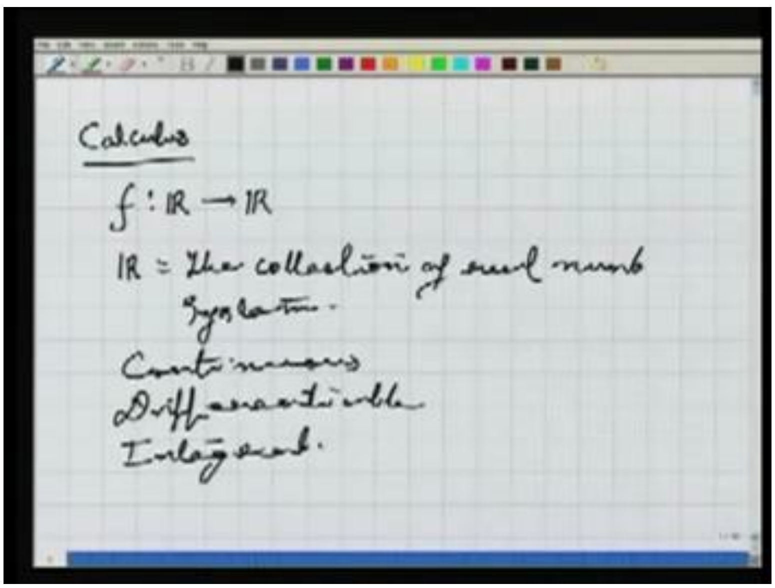

What does this subject exactly deal with? It deals with essentially functions defined on the set of real number systems. In symbol, we write it as function f from R to R, where this R denotes the collection of real number system. But, what kind of functions we do exactly study? We study functions which has certain analytical properties. For example, we study functions which are continuous, which are differentiable and which are integrable. But the problem is at the very beginning. If you want to study all these properties of the functions defined on the set of real numbers, we exactly need to know what these real numbers are. That means we need to study certain properties of real numbers and the problem is, we do not exactly know what are real numbers?

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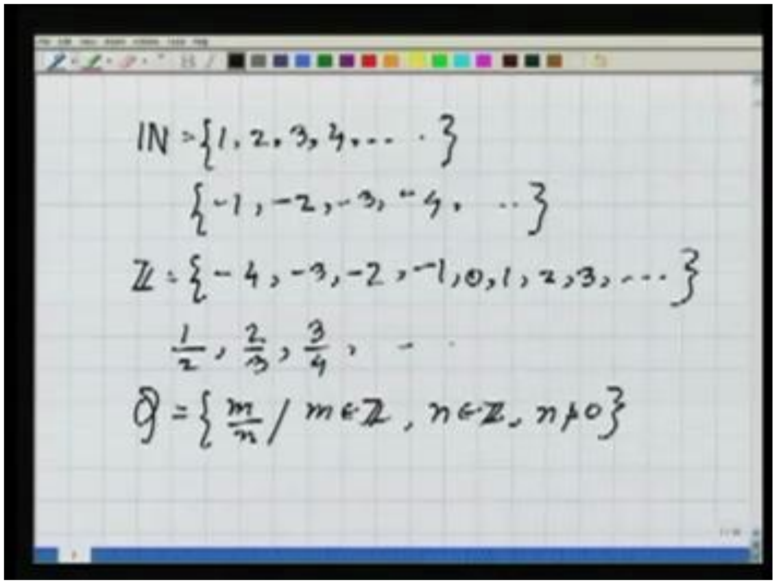

There are certain numbers we anyway know. For example, the set of natural numbers which is denoted by N. The numbers are 1, 2, 3, 4 and so on. We also know examples of negative natural numbers, that is, minus 1, minus 2, minus 3, minus 4 and so on. We also have the number 0 and that comprises the set of integers, denoted by Z. They are denoted by these numbers. We also have example of a bigger set of numbers than the set of integers, for example, the set of rational numbers. What are rational numbers? They are just collection of fractions. For example, if I look at numbers like half, two third, three fourth, these are rational numbers. So exactly, what is the definition of the class of rational numbers?

They are denoted by Q and the elements here are numbers of the form m by n, where m is an integer, n is an integer but n is not equal to 0. We understand why we need this condition, n not equal to 0, because, I want to divide by a non-zero number. Question is, are there any other numbers than this set of rational numbers? We will see that, that is the case. What we do is, let us take a square.

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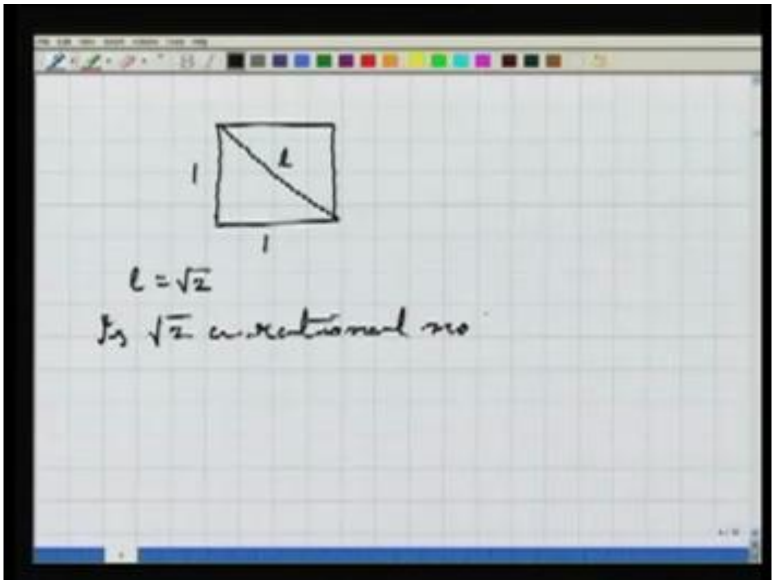

So this is a square with sides equal to 1 and then, let us look at this diagonal. What is the length of this diagonal? If I apply Pythagoras theorem, what comes out, that this diagonal if I call it l, then, l is actually root 2. This is very well known to us. Now the question we ask is, is root 2 a rational number? Let us see whether root 2 is a rational number or not. So, to prove that, what we do is, we assume the opposite of what we want to prove. That is, I will assume that root 2 is a rational number and I will try to get a contradiction out of it.

So if I assume root 2 is a rational number, then I can write root 2 equals to m by n, where m and n are natural numbers and I also assume that m and n has got no common factor, because, if there is a common factor between m and n, I can certainly cancel those factors. So, let us assume, that m and n has no common factor, of course except 1.

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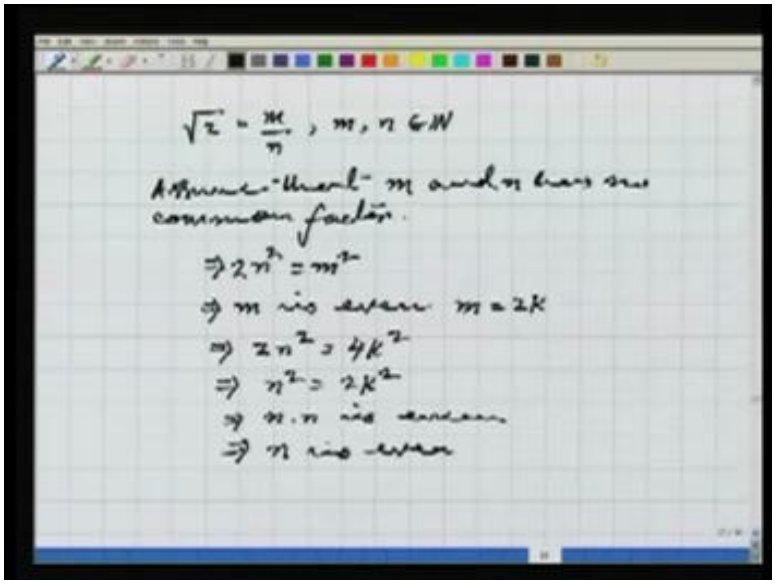

If that is the case, this would imply that n square times 2 is m square. I am just squaring both sides of the equation, right? If that is the case, this implies, since product of odd natural numbers has to be odd and here I see that m into m is even, it implies that m must be an even number. So, I have, m is even. That means m must be looking like 2 k, for some natural number k. This then implies that 2 n square is 4 k square. If I cancel 2 from both sides, it would mean that, n square is 2 k square. That means, n into n is an even natural number. But again, I know that product of two natural numbers which are odd has to be odd. But this is even. That means, n itself is even, implies n must be some 2 p and then can you see I already got the contradiction?

Because 2 is a common factor between m and n and that contradicts my assumption, I said I am taking m and n which has got no common factor. Why is this contradiction arising? It is arising simply because of the fact that I have assumed, root 2 is a rational number, which it is not. So, it implies that root 2 is not rational. That means, we can conceive about real numbers which exist according to the pictures which we have drawn, which are not really contained in the set of rational numbers. That means, there is a need to enlarge the system to a bigger system, where we have all the conceivable numbers which we can have. Well, that introduces us to the set of irrational numbers.

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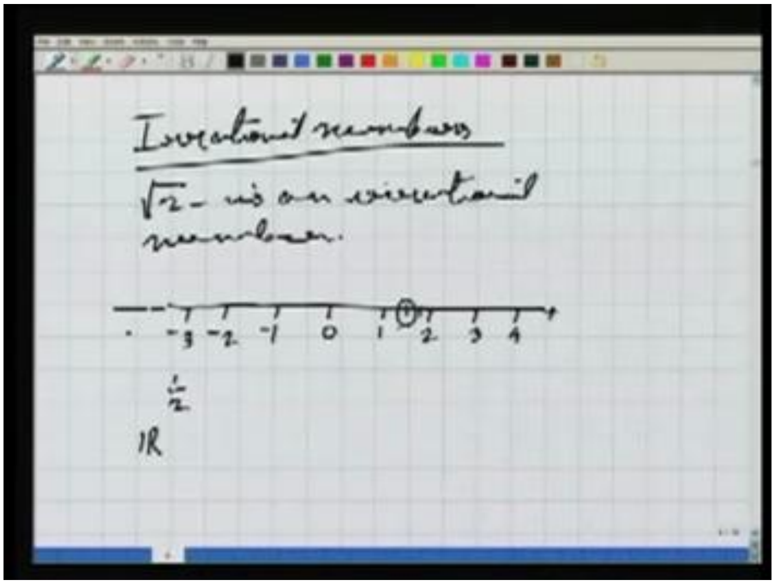

You already have got an example that root 2 is an irrational number. Now, to define exactly what is a real number, it is essentially demanding to know that, we know exactly what are the set of real numbers and that is a complicated thing to go for. So, what we will do is, we will start with a geometric intuition, which will help us doing analysis of functions. So, here we go. Let us look at the line, an infinite line, where somewhere I have 0. Then, at some point, I locate 1. Then, exactly with the same length I locate 2, then, I locate 3, I locate 4 and I go on. Then, on the left hand side of 0, I locate minus 1, I locate minus 2, I locate minus 3 and I go on. Then, it is highly conceivable that I can represent the point, let us say, half. Where it should be? It should be somewhere here. I can certainly look at the point 3 by 2. Let us say, it should be somewhere here and so on.

So probably, I can represent all the fractions. That means the set of rationals inside here, and I will like to assign each and every point of this line with some system, the set of real numbers, which I am going to define, which I am going to have. I want to have points on this line to correspond numbers in R and every element of R should correspond to points on the line. That means, all the irrational numbers, they should correspond to certain points on these lines. So intuitively we will assume that the whole real line, the whole set of real numbers is actually given by this line, every point which are not fractions on this line corresponding to irrational numbers.

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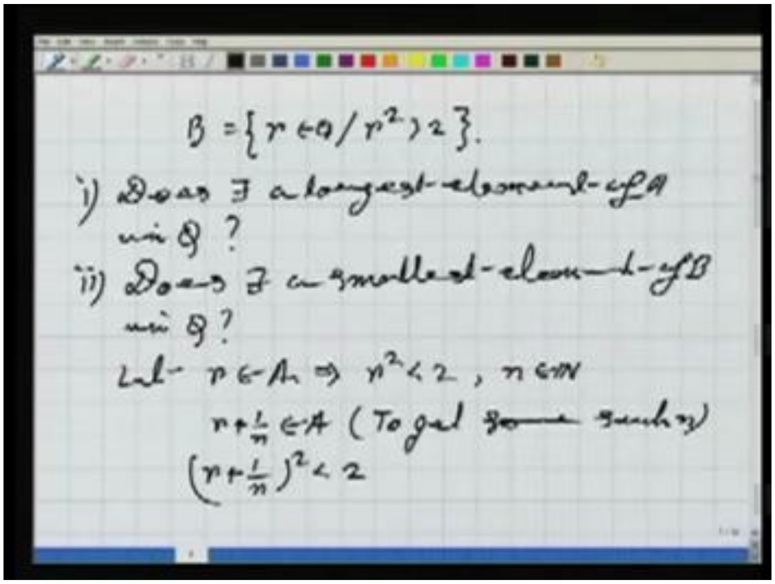

Let us see some more necessity. Why should we have irrational numbers in the set of real number systems at all? Let us look at this problem. I define a set A; it is the collection of all real numbers. I am going to have all rational numbers, which satisfies the property that r square is less than 2. I also define a set B which is given by all rational numbers such that r square is strictly bigger than 2. Well, the question I ask is, does there exist a largest element of A in Q? I can also ask, does there exist a smallest element of B in Q? If both these answers turned out to be negative, then what happens? You can intuitively feel that the largest number of the set A and of the set B, the largest number of the set A and the smallest number of the set B should actually be root 2. But since I am demanding that I am searching for the number which lies inside Q, I will land up with the problem, if root 2 is not a number at all. That necessitates the fact that, if I want to avoid gaps in real line, I must have numbers like root 2 in my system.

So let us see whether the answers to these questions are positive or negative. Let us start with A. So let r is an element of A. This implies, by the defining property of the set A that r square strictly less than 2. Then, I choose some natural number n in N and I look at r plus 1 by n. I am trying to say that I can always manufacture a natural number n, such that r plus 1 by n still is in A. So I want to get some such n. Let us see whether we can do it.

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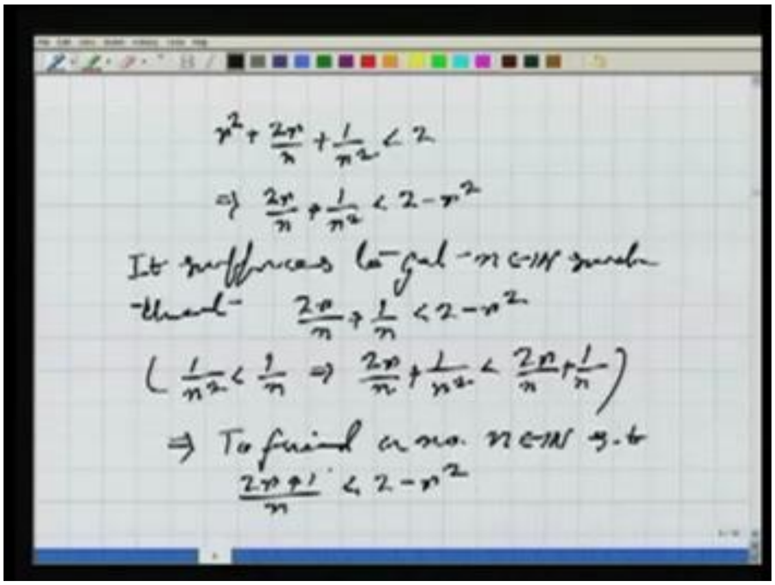

If r plus 1 by n has to be in A, that means, it has to satisfy the defining property of the set A. That means, r plus 1 by n whole square must be strictly less than 2. Well, this implies r square plus twice r by n plus 1 by n square must be strictly less than 2. This implies twice r by n plus 1 by n square strictly less than 2 minus r square and this is anyway positive. This right hand side, 2 minus r square because I assumed that r square is strictly less than two. Now, if I want to find some such n, I say that it suffices to get a natural number n such that twice r by n plus 1 by n strictly less than 2 minus r square. This is simply because of the fact that 1 by n square strictly less than 1 by n and this implies twice r by n plus 1 by n square is strictly less than twice r by n square twice r by n plus 1 by n. Now, if I can find an n such that twice r by n plus 1 by n is strictly less than 2 minus r square because of this inequality, it will trivially follow the twice r by n plus 1 by n square is strictly less than 2 minus r square. So that is good enough. That means I have to find a number, a natural number in the set of natural numbers, such that twice r plus 1 divided by n is strictly less than 2 minus r square. That is good enough.

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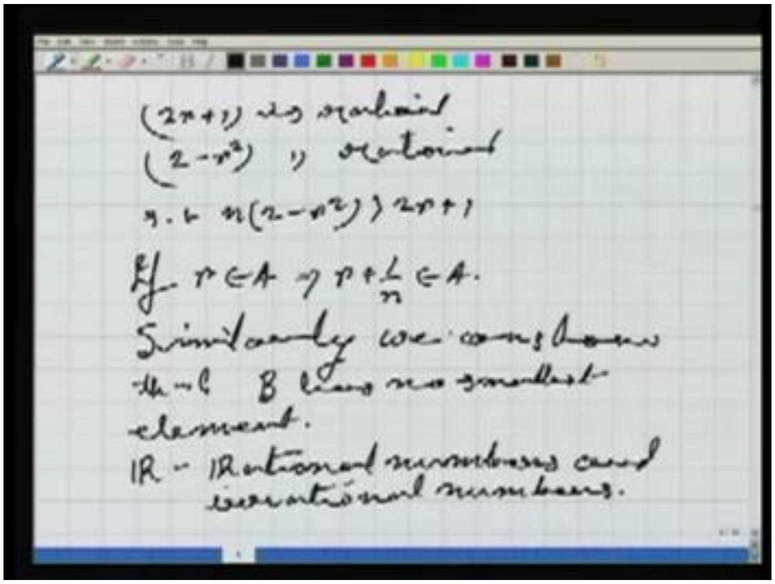

Then I say the proof is intuitively clear, because twice r plus 1 is rational. Similarly, 2 minus r square; this is also rational. Then, you can always arrange for an n such that n into 2 minus r square is strictly bigger than twice r plus 1. That is not very difficult. That shows that, if r is in A, this would imply, r plus 1 by n is also in A but what does it show? If I take any r, then r plus 1 by n, again a rational number, which is strictly bigger than r and still it is in A and there is no end to this process.

Similarly, we can show that B has no smallest element and this essentially shows that the set of real number has got certain gaps. That is, if you do not assume that the set of irrational numbers are needed, just to fill in those gaps you throw in the set of irrational numbers. That is, now the whole collection of real numbers stands for rational numbers and irrational numbers. Although mathematically difficult, but actually one can construct the set of real numbers starting from the set of rational numbers and not only that: the usual operations of mathematics, which I had on the set of rational numbers can actually be extended to the full set of real numbers.

For example, I have plus because I can add two rational numbers certainly, that will obviously include the case that I can subtract two rational numbers. I can multiply two rational numbers and another fundamental thing that, given two rational numbers I can determine which one is larger, which one is smaller. That means I have an ordered relation like this among the set of rational numbers.

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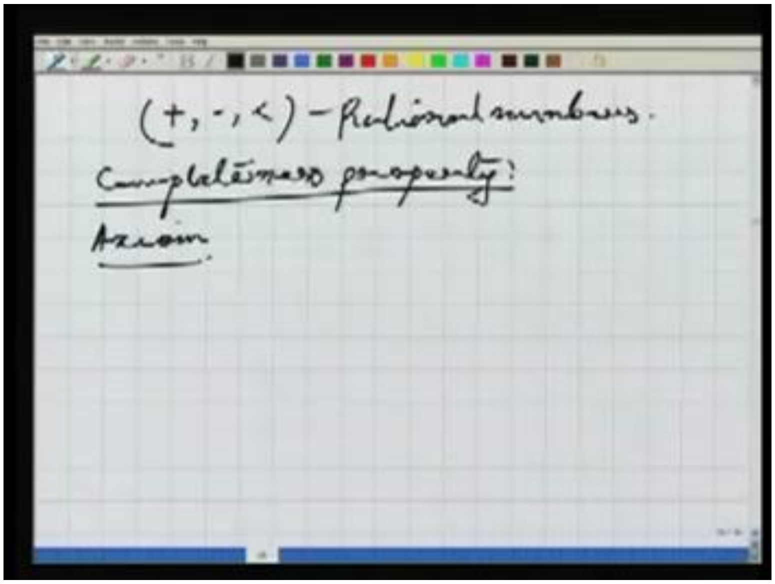

Well, it is possible to construct a set of real numbers throwing in the irrational numbers with the set of rational numbers, such that all these operations actually can be extended to the whole set of real numbers. What does extension mean? It means, among real numbers, if I can somehow choose rational numbers, let us have two rational numbers, then the addition of the real numbers is same as the usual addition of rational numbers. The order relation of two real numbers, if I restrict my real numbers to the set of rational numbers, it coincides with the order relation of rational numbers, which I had earlier aimed with these things.

Now, I will say that if I want to fill in the gaps with the rational numbers, with the irrational numbers. Then the most important property of real line, which I am going to have is, the completeness property. Well, what is completeness property? Notice that we have not really constructed the set of real numbers. We have just assumed some such thing exists. If that is the case, I cannot really deduce what completeness property is. So I will like to throw in as an axiom. That means, you have to just, you know, you have to just assume it, I mean.

So axiom, it just means, that it is a statement with a mathematical statement which you will not like to prove and you are neither supposed to question that statement. You just have to assume the statement. What the completeness axiom is? It turns out that, in whole calculus, this plays a very fundamental rule. Now question is, what is completeness property? Well, to tell you what exactly completeness property is, we need to define certain things.

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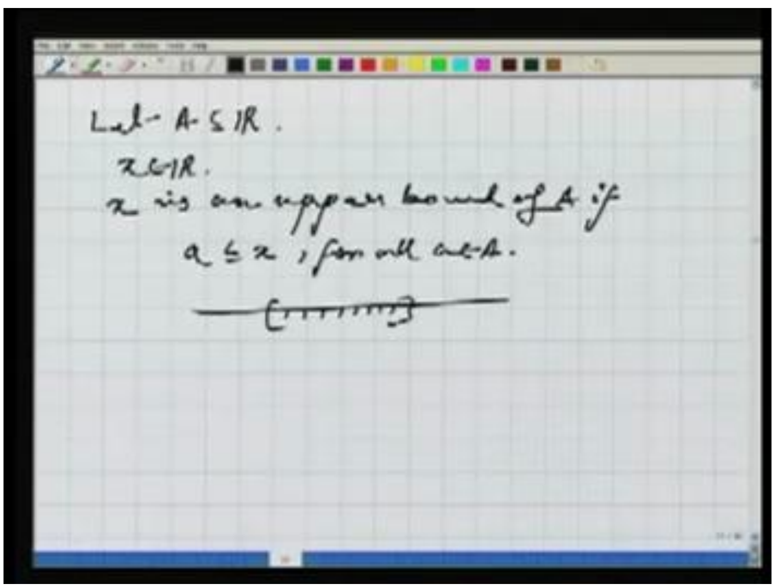

Let A be a sub set of R. Then, let us say, some x. Let x belong to R. We say x is an upper bound of A. If every element of A is smaller than this x, you know, that is, a is less or equals to x for all a in A. If you look at the picture, that means, if I represent real numbers in a line, let us say A is certain collection of points. Here, then x actually has to be here onwards. It can be this end point, but any x after this end point will work as an upper bound. Pictorially, it means just this.

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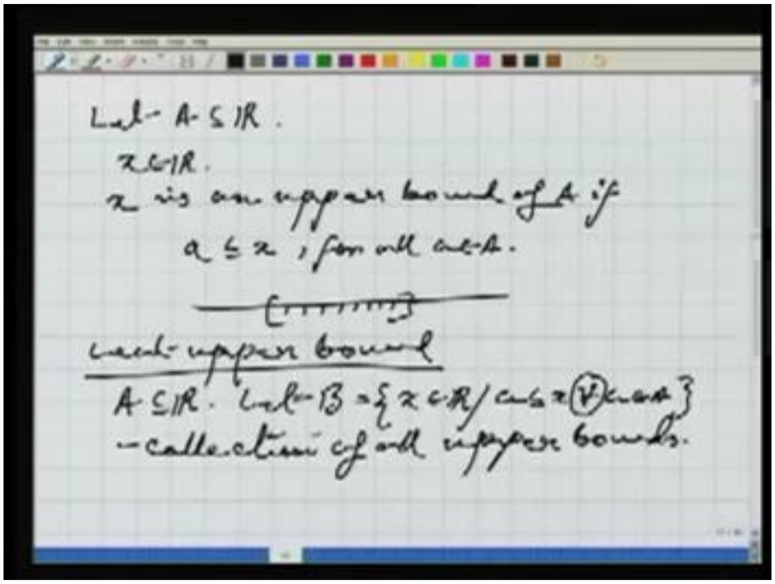

Now let us talk about least upper bound. What does this mean? Well, it means exactly what it says. It is the least among all the upper bounds. Let A be a set again and let B be the collection of all real numbers x in R such that a is less or equals to x for all a in A. By this symbol, I mean for all. Well, then what is b actually? This B is just the collection of all upper bounds and then what is least upper bound? Least upper bound is the least member in the set B. So, least upper bound, in short, we call it l.u.b. It is the least member. In other words, it means the smallest number in the set B. That means what?

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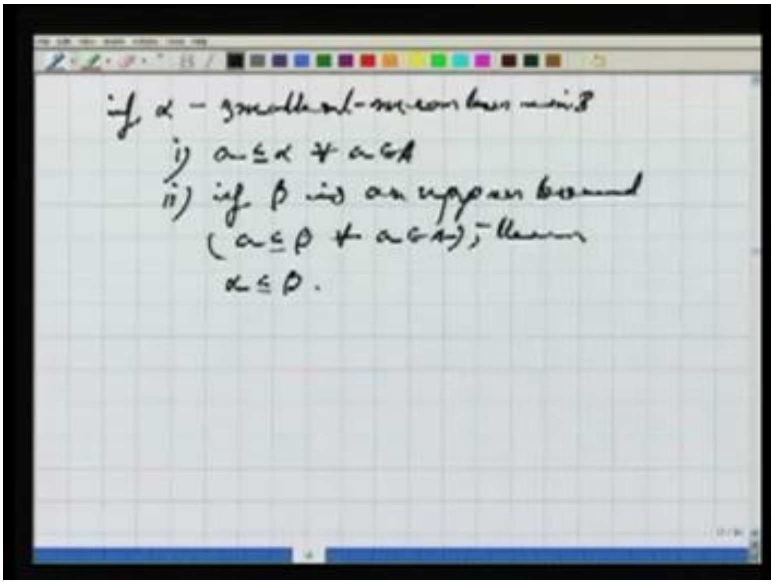

Well, it means by definition, that if alpha is the smallest number in B, as it is a member of B, it must be an upper bound of the set A. That is, the first condition is, that a is less or equal to alpha, for all a in A. Second is, it is the least. That means, if beta is an upper bound, which means a is less or equal to beta for all a in A. Then, this beta must be bigger than or equal to alpha. That is, alpha is less than or equal to beta right?

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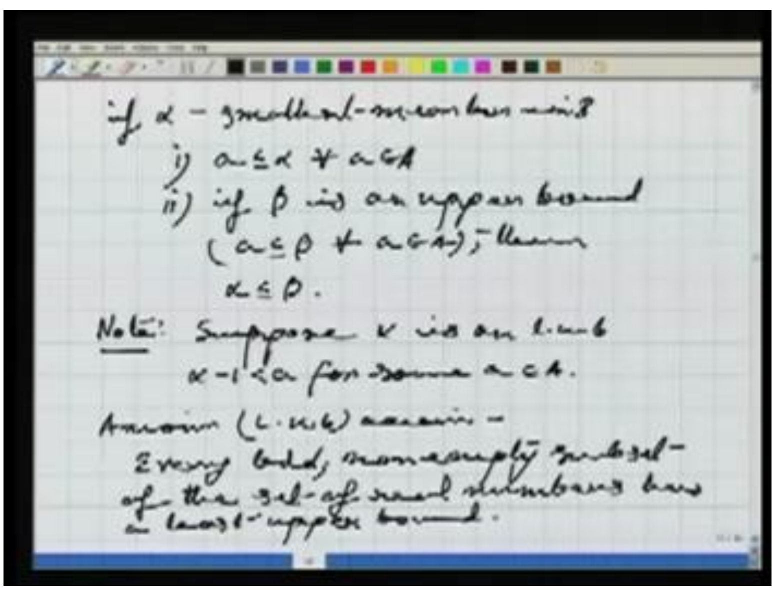

Now, let us just make an observation. I will say it is a note. Suppose, alpha is an l.u.b and I look at some alpha minus 1, the question is, can this be an upper bound? Well, you know least upper bound means it is the least among the upper bounds. So, l minus 1 can no longer be an upper bound, means what? Something is not an upper bound; it means there must be some element in the set A, which is bigger than this number alpha minus 1. That is, it must be strictly less than a, for some a in A.

Now, if you go back to the previous example of the sets A and B which I defined in terms of r square less than 2, then I had a problem that the set B, they had no smallest element, among the set of rational numbers. I want to remove that difficulty. That means, I want to have every subset the least element and that is actually what the completeness axiom says. So to be precise the axiom is, we call it the l.u.b axiom. It just says that every bounded, non-empty subset of the set of real numbers has the least upper bound. This l.u.b or least upper bound, which we have defined, is also known as supremum and the l.u.b axiom is also known as the supremum axiom or the completeness axiom or the completeness property.

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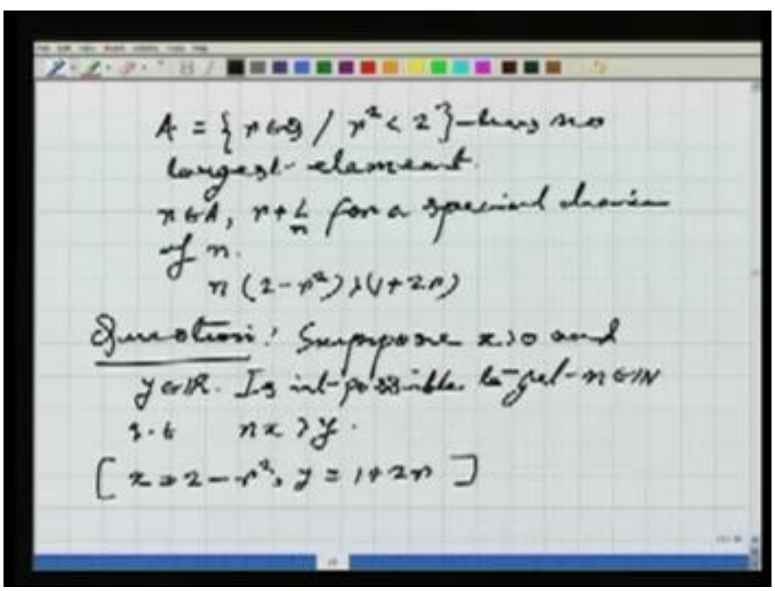

Now, I want to show you the power of this axiom in some familiar situation. Let us go back to the proof, that the set A, that is all rational numbers such that r square is less than 2, has no largest element. We have seen the proof of this. How did we go about this proof? Well, I had taken some real number, some rational number r in A and I just looked at r plus 1 by n, for a special choice of n and what kind of n actually I have chosen? I had chosen an n which has the property that n of 2 minus r square is strictly bigger than 1 plus twice r. Since everything is rational here, 2 minus r square and 1 plus 2 r, I could manage some such n. But now I want to generalize it. For example, let me ask this obvious looking question.

Suppose, x is a positive number and y is just another real number. Is it possible, to get a natural number n, such that n times x is strictly bigger than y? You can see why I am asking the question. For example, I have already shown that if x equal to 2 minus r square and y equals to 1 plus 2r, I can manufacture some such n. But now I am generalizing. I am telling you, suppose x is an arbitrary real number that is no longer a rational number, so is y, is it possible to get a natural number n such that this happens? You will see that, yes, that is the case. So let us try to prove it. There are two cases.

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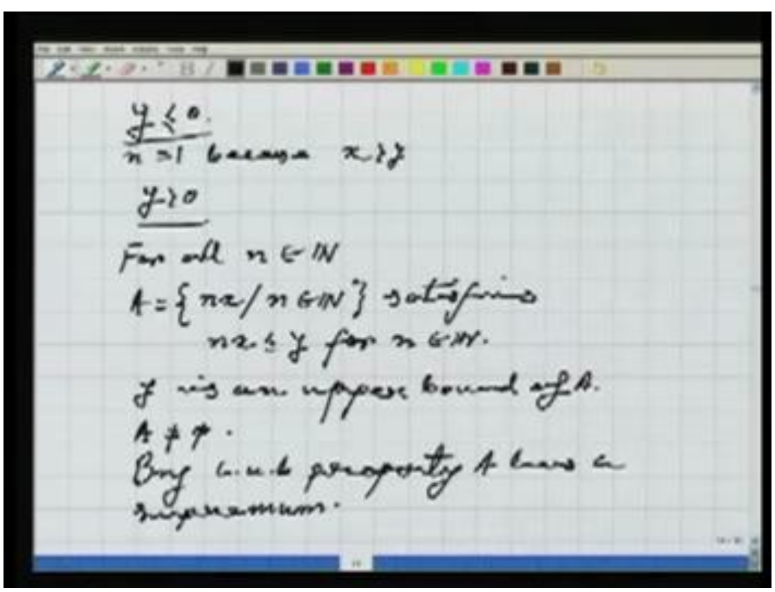

First of all, I assume that y is less than or equal to 0. In that case, I would say it is obvious. You can take n to be equals to 1, because x is certainly strictly bigger than y, because I have taken x to be positive. But the real non trivial task is, if I assume y strictly bigger than 0, what happens in this case? Well, we use something called contra positive argument. That means, you assume that the result is false and then you try to contradict which you already know to be true. It means that your assumption must be false. So what is the contradiction of my statement? The contradictory statement is, for all n in the set of natural numbers, the set nx satisfies for all n.

Now, look at this set. If I call this set A, then it turns out that, y by definition is an upper bound of this set A. So, we have y is an upper bound. Well, I can easily check that the set A is non-empty. Because, certainly the element x is there. Then, by the least upper bound property, A must have a supremum. By l.u.b property, equivalently, the completeness axiom A has a supremum. I denote the supremum by alpha, let us say.

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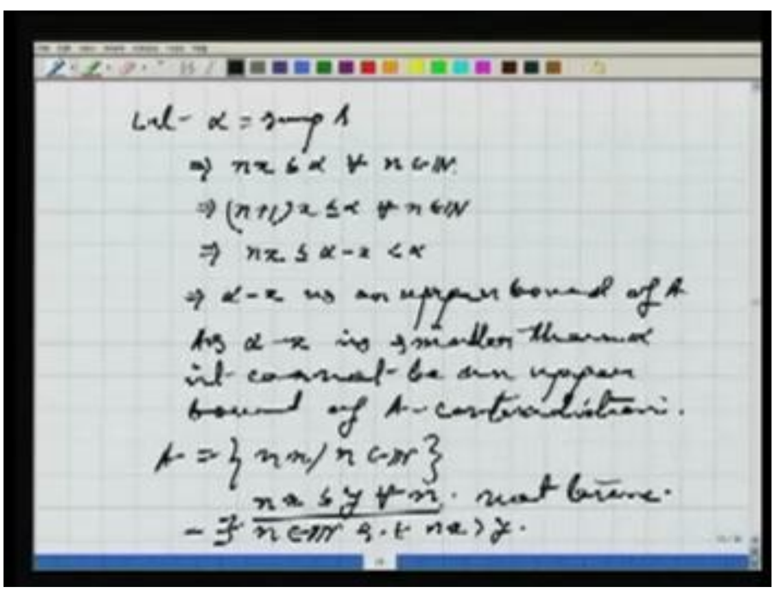

Let alpha be the supremum of A. In symbol, I write alpha is equal to supremum A. This implies nx, certainly, less or equal to alpha, for all n in N. Well, that also means that, n plus 1 x is also less or equal to alpha, for all n in N because if n is a natural number, so is n plus 1. This implies that nx is less or equal to alpha minus x. Notice that this alpha minus x is positive. What does it mean? As, alpha minus x is less than alpha, I see that it implies alpha minus x is an upper bound of the set A.

Now, if I go back to the observation which I just made after the definition of supremum, which essentially says that anything lower than supremum cannot be an upper bound, alpha minus x is smaller than alpha, how can this be an upper bound? This is a contradiction. So, as alpha minus x is smaller than alpha, it cannot be an upper bound of the set A. Well, I say this is a contradiction and that means my proposition is wrong. That means this set A which I have defined, the set A which is collection of nx; this cannot satisfy that nx is less or equals to y for all n. This is not true. This is not true means, there must exist some n for which this is false. That is, nx is strictly bigger than y. That is, what we wanted to prove. This is called the Archimedean property. So, the precise statement of this is, if x is a positive real number and y is an arbitrary real number then there exist a natural number such that nx is strictly bigger than y.

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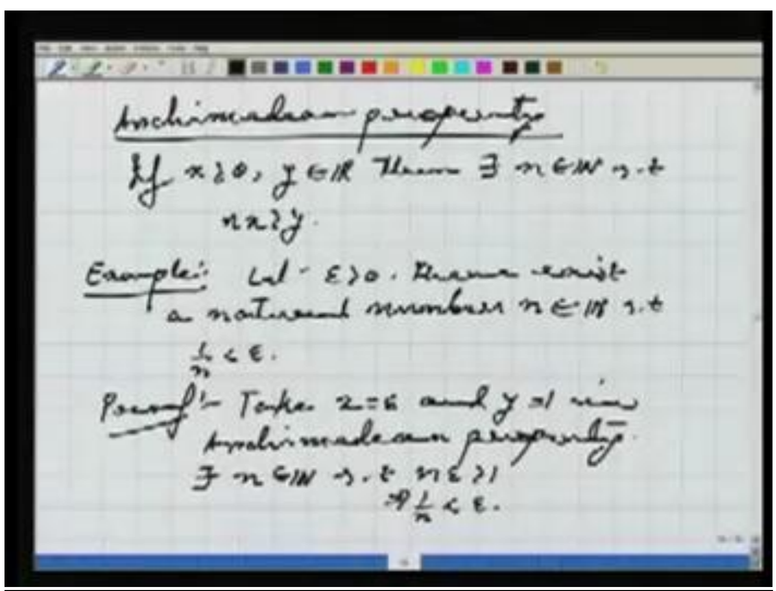

Well, let me show another application of this. Let epsilon be any positive real number. Then I say, there exists a natural number n such that 1 by n is strictly less than epsilon. How do you prove it? Very simple; just in the archimedean property, take x equal to epsilon and y equals to 1, in archimedean property. Then, by the Archimedean property, there exist n in N such that n epsilon is bigger than y. This precisely means 1 by n strictly less than epsilon. Let us look at some more examples of Archimedean property. That is, applications of it. What does it really imply?

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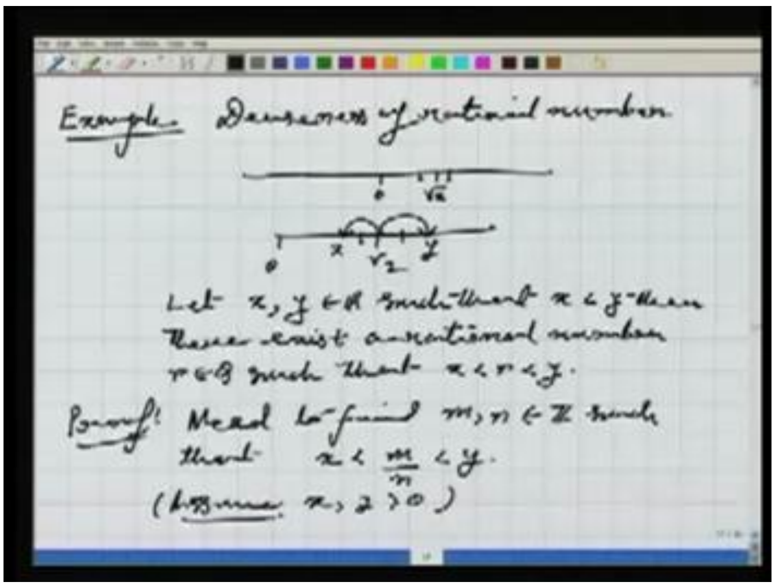

The second example, which I have in mind, is the denseness of rational numbers. So let me write, denseness of rational numbers. What does denseness really mean? To illustrate, let us look at the picture again. This is the real line. Let us say, I am standing at the point root 2, which is an irrational number, now you know. Denseness essentially means this, that suppose, I move little bit right, let us say here or I move little bit left let us say here, then I am actually crossing a rational number. That is, if I magnify this picture a little bit more, let us say this is root 2, this is 0, I am standing here.

Suppose I want to go here. I am moving left or I want to go here, that is right. Then, I am crossing a rational number. That means, there is some irrational number sitting here or there is some rational number sitting here. That is what denseness means. The precise mathematical formulation of this is as follows. Let x and y are 2 real numbers, such that, x is strictly less than y. This x, you can think of, is square I have moved and y is my right movement. I want to say, then, there exist a rational number r, such that r lies between x and y. That is, x is strictly less than r; this is strictly less than y. The proof of this is just a very simple application of Archimedean property. If you people think a little bit, you will also come up with the proof.

So, the proof is this. First thing is, what exactly I want? I want a rational number r, which lies between an x and y. That means I need to find 2 integers, let us say, m and n, such that, x is less than m by n and this is less than y. For simplicity, let me first assume that x and y, both are positive. The other thing you can actually manage yourself. So I assume x, y both are positive. That means, now my job is reduced to finding two positive numbers m and n, that is, no longer integers, natural numbers, such that x is less than m by n and m by n is less than y.

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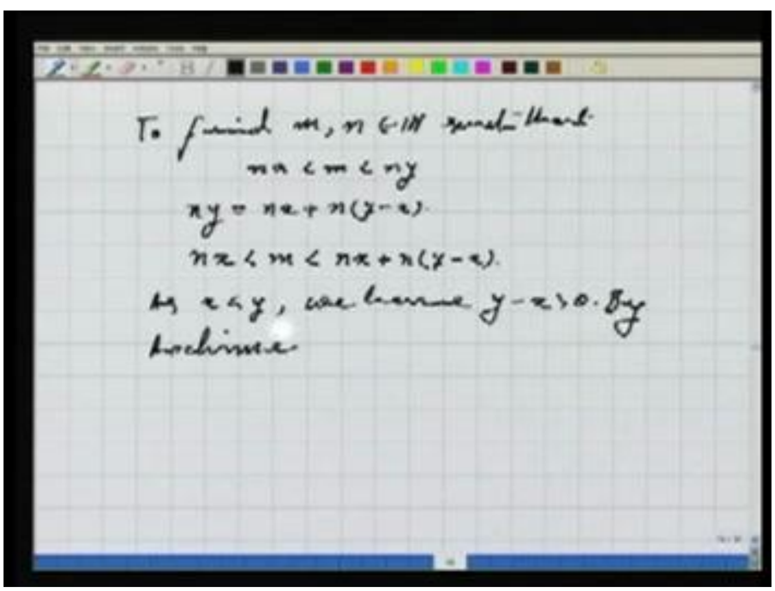

What does it mean to say that, I will find these numbers? It means this, that, I need to find now m and n, such that, nx is less than m and this is less than ny. Now, what I do is, I use a very simple trick. I just write ny as nx plus n of y minus x. So I need to find, m and n, such that this is true. nx is less than m; it is less than n x plus n into y minus x. Now, since x is less than y, I already know that y minus x is strictly bigger than 0. In this case, I say by Archimedean property, I can always manage to find a natural number n, such that n into y minus x, let us say, is bigger than 10 because y minus x is a positive number, 10 is a given real number.

Then by the statement of Archimedean property, there is a natural number n such that n into y minus x is bigger than 10. But this, in turn, implies that n y, sorry, n x plus n y minus x minus n x is bigger than 10. What does it say? It says, that the distance between this real number, which is actually n y and this real number is 10 and if there are 2 real numbers whose distance is 10, I can always find a natural number, which lies between these 2 numbers. That is, there exist a natural number m, such that n x is less than m and this is less than n x plus n into y minus x. But this righter number is nothing but n y. That means my job is over. I have found a natural number m such that, n x is less than n, which is less than n y. That in turn implies that, x is less than m by n less than y.

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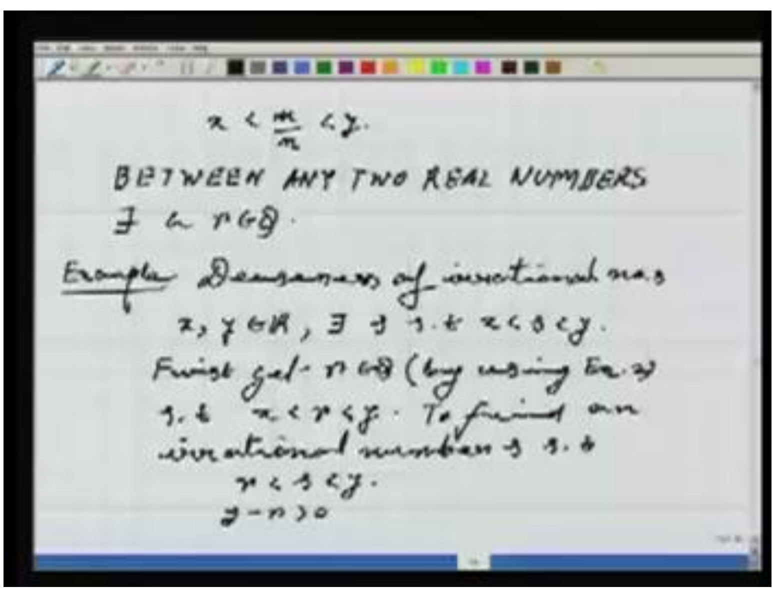

It again means that between any two real numbers, there exist a rational number, and this is precisely what denseness of rational numbers in the real number system means. Now, I show another application of Archimedean property. This is the third example. I want to prove denseness of irrational numbers. Now, you people know already what does this mean.

Exactly analogous to the rational numbers, I will like to prove between any 2 real numbers, there exists an irrational number also. That is, if x and y are 2 real numbers, I want to prove that there exist an irrational number s, such that x is less than s, less than y. How you go about it? What I do is, given x and y, I can always, squeeze in some rational number between x and y. That is, by the previous example, I call that number r. That is, first get a rational number by using example 2, such that x lies between, x is less than r and r is less than y and then I want to find an irrational number s which lies between r and y, that is, to find an irrational number s such that this happens: r is less than s and it is less than y.

You will be surprised to see that I can manage some such s with an explicit knowledge of only one irrational number, which we have seen so far, that is, root 2. What I do is as follows. I look at first y minus r and obviously, this is bigger than 0. In the Archimedean property, if you think this as an x and choose another irrational number, root 2, I say, by Archimedean property, it follows that, there exist a natural number n such that n into y minus r it is strictly bigger than root 2. Let us have a look at it again.

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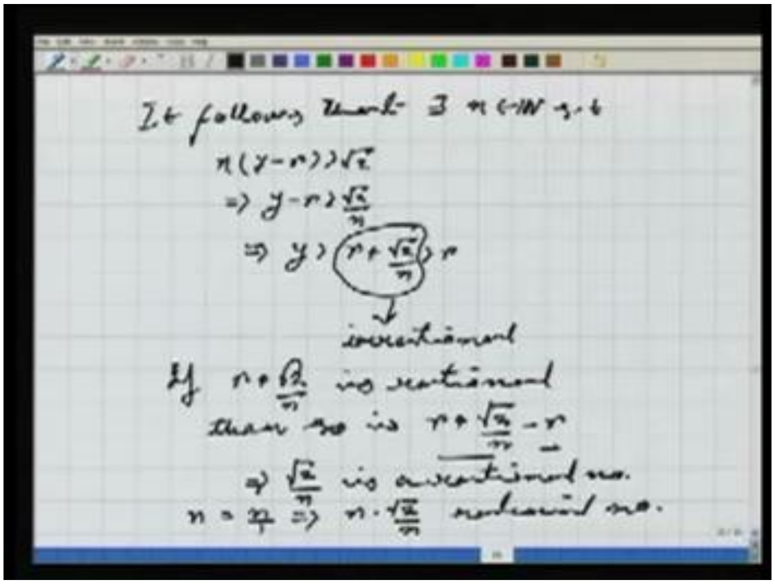

I am saying that y minus r is a positive real number. That is clear because r is less than y. Root 2 is a real number. Then, by the Archimedean property, there exist a natural number n such that n into y minus r is strictly bigger than root 2. That is fine. What does this mean? This implies that y minus r strictly bigger than root 2 divided by n. This also implies, that y is strictly bigger than r plus root 2 by n and notice that root 2 by n is a positive number. That means it is strictly bigger than r.

Now if I can somehow guarantee, that this is irrational, then I will prove. But why it should be rational? Again, we use the contra positive argument. About rational number system, we know that if we add or subtract two rational numbers, I land up within the set of rational numbers. If I multiply two rational numbers, I am still inside the set of rational numbers. So let us assume that this is not irrational. That means, again, I am using the opposite of what I am supposed to prove and then I will try to get a contradiction out of it. That means my assumption is wrong.

So if r plus root 2 is rational, then so is r plus root 2 by n minus r because difference of two rational numbers is a rational number and I have assumed that r is a rational number. But this implies that root 2 by n is rational number. I also know that product of two rational numbers is a rational number, and of course, any natural number is a rational number because n, I can always write as n by 1. This implies n into root 2 by n must be a rational number. Notice that n is a rational number, so is root 2 by n.

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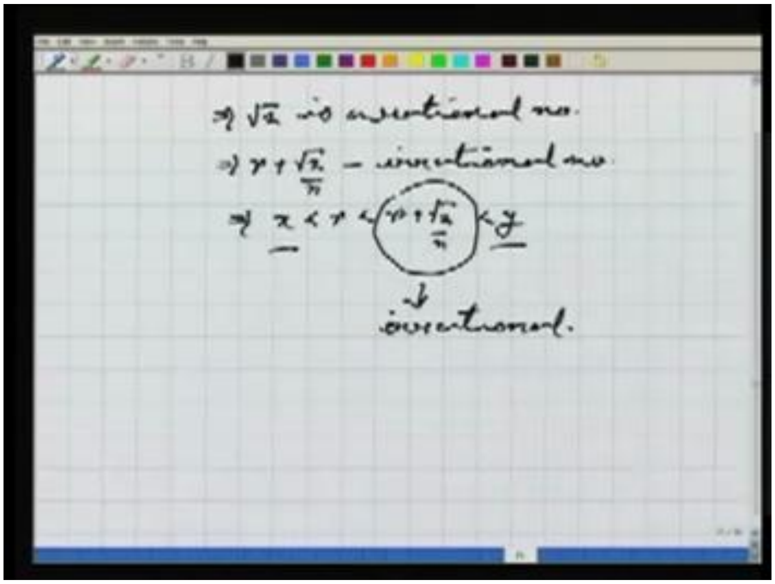

This implies in turn, that root 2 is rational number, but that is a contradiction because this is the first result we started with; that root 2 is an irrational number. This implies that r plus root 2 by n is actually an irrational number, and this implies what I got finally is x less than r, which is less than r plus root 2 by n, which is in turn, less than y. x and y are the two real numbers. I started with and finally I landed up with r plus root 2 n, which is irrational. That is, between any 2 real numbers you can always find rational number as well as an irrational number. That is what is called the denseness of irrational numbers.

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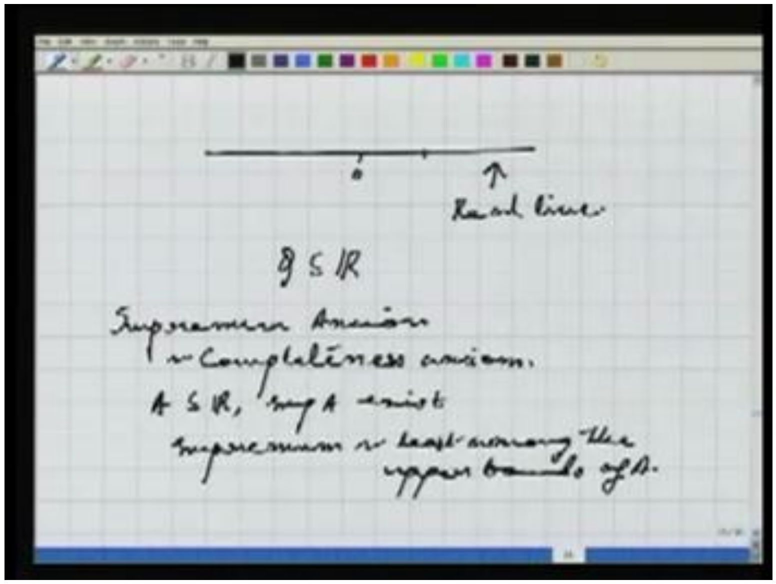

Let us summarize what we have learnt in today’s lecture. Although we have not really defined for you, explicitly, what are real numbers, but we certainly do have an intuitive feeling towards it. Let me explain. Let us look at this line and I have the set of real numbers. Then, I say, that there is a one-to-one correspondence among points in this line and real numbers, that is, elements of the set written as R. What is the correspondence?

Given any point, I can always choose a point here so that the length represents the real number. That is, if I take the rational numbers inside here, then the points having fractional length from 0 corresponds to rational numbers and any other point which does not have this kind of fractional length, they are actually representing the irrational numbers and that is the reason, sometimes, this is called the real line.

Now the second thing, the most fundamental about this real line was the supremum axiom. Sometime, this is also called the completeness axiom. What does it say? As you already know now, that if you take any non-empty subset of real line A, then A has a supremum and supremum means what? Supremum means, the least among the upper bounds of the set A.

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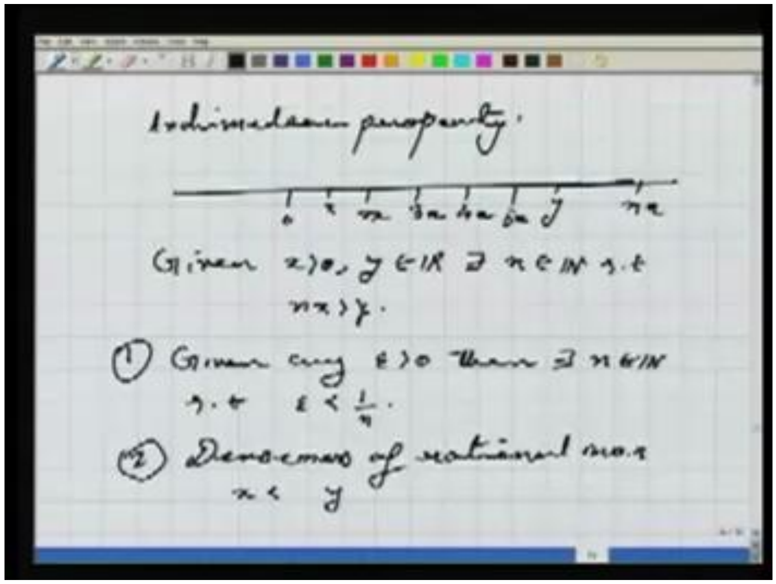

The next thing was the fundamental application of the supremum axiom, which we call the Archimedean property. Pictorially, it means just this. You have this line, you take some y far away from 0 and you take some x here. Then what you do is, you go on adding it to itself for sufficiently many times, you know, that means this is x, 2x, 3x, 4x, 5x and so on. At some stage, for a large n you will have nx sitting here. That is what Archimedean property says. That is, given any x bigger than 0 and y, an arbitrary real number, there exist a natural number n such that nx is strictly bigger than y and as you must have noticed that, if y is actually a negative real number, that is, it is less than 0, then there is no point in trying to prove it because it is obviously true.

Then we have seen some fundamental applications of that Archimedean property. Number one was, that given any epsilon bigger than 0, this epsilon might be very large, it might be very small, also something like 1 by 10,00,00, you know, it does not really matter, then there exist a natural number such that this epsilon is strictly less than 1 by n. This was the simple application of the Archimedean property. Then the second property was denseness of rational numbers. That is, given any 2 real numbers, x and y, suppose, x is strictly less than y, then always, there exist a rational number r, which lies between x and y.

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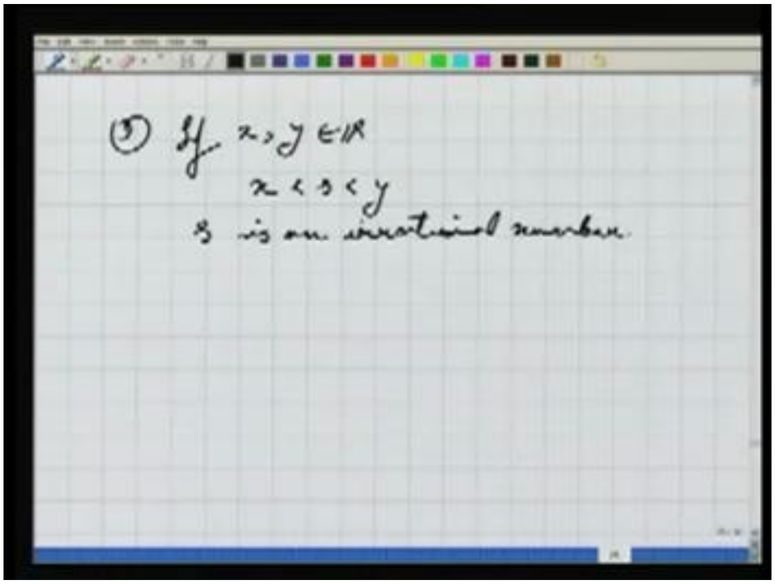

And then, we have analogous property for the irrational numbers that is, if x and y are two real numbers such that x is strictly less than y, then just like rationals, there are irrationals also which lie in between x and y, that is, x is less than s, less than y and s is an irrational number. That is, there are enough rational numbers and irrational numbers in the real line. The most fundamental among those properties, which we have learnt so for is, the supremum axiom. It is also called the completeness axiom, what does it say? It says, that if I take a subset A of R, which is non-empty, as well as, bounded, then it has the supremum and what is the supremum? Supremum is an upper bound of the set A, but it is the least among all the existing upper bounds.

These are the most fundamental properties of real line. In the forthcoming lectures, we will see that, when we start studying functions, which are continuous differentiable or integrable, to study them, actually, we will be frequently using these above mentioned properties and that is why, these properties are the most fundamental properties, directly related to the structure of real line.