Mathematics-1
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Lecture - 11
Taylor's Theorem

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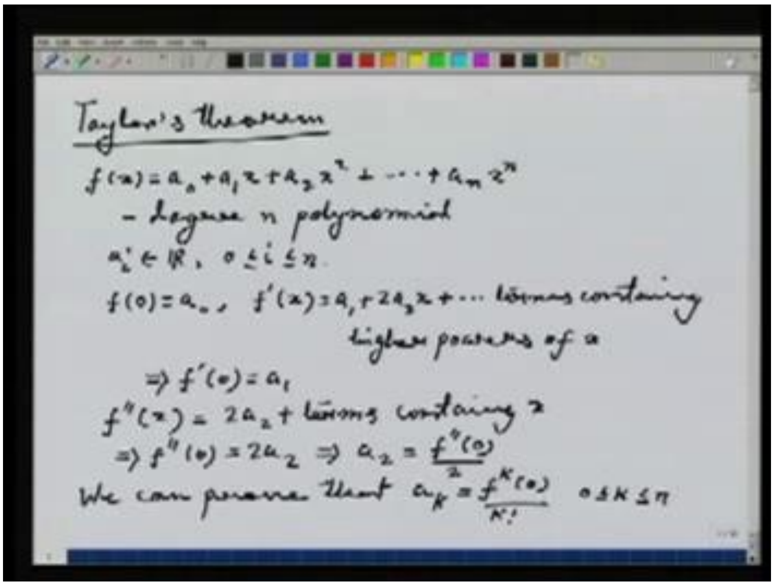

Today we are going to start with Taylor's Theorem. Before going to the exact statement of the theorem, let us try to motivate you. Suppose I have given a polynomial, I will write it as f x. Since it is a polynomial, we can write it as a 0 plus a 1 x plus a 2 x square and so on up to a n x to the power n. So, this is a degree n polynomial. I also assume that all the coefficient of this polynomial, that is a i is, they are real numbers; so a i belongs to r for 0 lesser equal to i lesser equal to n.

Now, since we know that any polynomial is differentiable as many times as you want, the function f is differentiable as many times as I want. And I can see clearly, that is if I look at the derivative of the function all power bigger than, then the derivative certainly become 0. But, now notice something, what is f of 0? I can see very clearly f of 0 is a 0. Then what is f prime at 0? For that I need to calculate f prime of x and I get this is a 1 plus twice a 2 x plus. That is, terms containing higher powers of x this, then implies that f prime at 0 is equal to a 1.

Similarly, I can see that f double prime x which certainly exist, this is twice a 2 plus terms containing x, this then implies, that f double prime of 0 is twice a 2. Which, then implies, that a 2 is equal to f double prime 0 divided by 2. If, I go on like wise, I can actually produce the following formula. We can prove that a n are I would say a k, that is F K 0 divided by factorial k for all k satisfying 0 lesser equal to k lesser equal to n. Here the convention is that the 0 at the derivative means just the function. That means, I am not differentiating.

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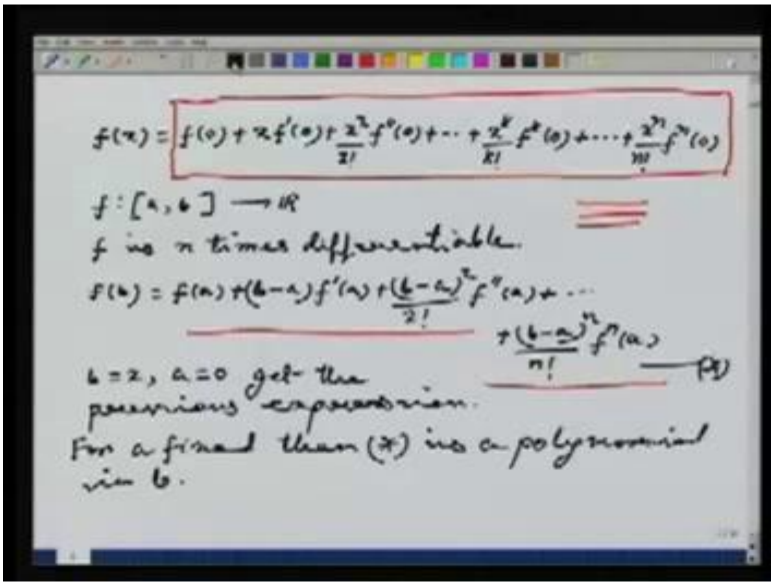

So in like of this, I can actually write down f x as this, f x is equal to f 0 plus x f prime 0 plus x square by factorial 2, f double prime 0 plus x to the power k divided by factorial k times F K 0. And then, we go towards the last term. Since, it is a degree n polynomial, I have to stop after the nth derivative. Because, after that all the derivatives are 0, I get x to the power n by factorial n times f n 0.

Now, here what exactly using about the polynomials, it seems from expression, all I am using is that the function is differentiable as many times as I want. So, given any function, let us say f is a function on a closed interval a b to R. Assume that f is n times differentiable and then I want to see, whether f can be written as a expression, like the above of which I have written here. I am looking the generalization of this, for arbitrary function f, which as many derivatives as I want.

Now, in terms of b and a it look like, that f of b is equal to f of a plus b minus a f prime a plus b minus a whole square divided by factorial 2. F double prime a plus I get the last term. That is b minus a whole to the power n by factorial n, f n. A question is this true, notice that this is related to the previous equation. Just put b is equal to x and a equal to 0 and get the previous expression.

Now, we are asking for given a, if I take any b, then is it true that f b is given as the previous expression. Now, if you look at this expression which I have written here. if you fix a, if I call the above expression as star, then star is a polynomial in b. That is obvious, if I fix my a as the base point and keep on varying b and I look at the right hand side star that means, it is a polynomial in b.

But given a function, which is differentiable as many times, as I want, why it should be polynomial. For example, you know examples of such functions, e to the power x, sin x, cosine x, these are not polynomial functions. That means, this star cannot be true in general. But for polynomial, it might be true, but for general functions, it cannot expect that it will be true. So, what exactly true, it given by Taylor's theorem. Now, we go the precise statement of the theorem. We will see it very close to the expression, which I have written in the box that is this one. Let us go the precise statement of the Taylor's theorem.

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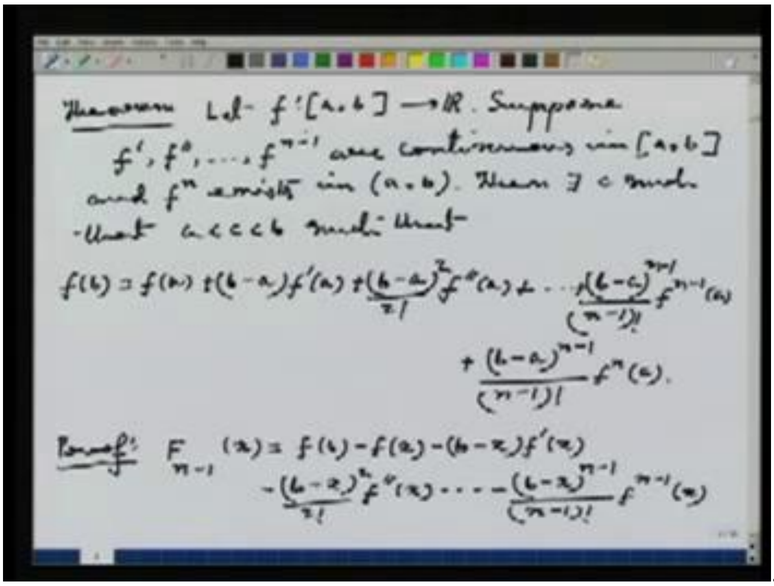

Let f be a function defined on the closed interval a b to R. Suppose f prime, f double prime and so on up to n minus 1, which means n minus 1th derivative of f and are continuous in the closed interval a b and f n exists in the open interval a b. Then, Taylor's theorem says there exist c, such that a less c less b. Such that f of b is equal to f of a plus b minus a f prime a plus b minus a whole square divided by factorial 2, f double prime a plus so on.

Up to b minus a to the power n minus 1, by n minus 1 factorial times f n minus 1 into a plus the last term which is b minus a whole to the power n minus 1, by n minus 1 factorial, but now f n at c. So, everywhere it is the derivatives of a appearing, but the last term, where you get c and c. Now, depends on b, if this c is independent of b. Then, you can see that, f will actually be a polynomial, but that may not be the case. That is why the c, which I have written in the last term, it depends on the point b.

It is exactly, like the foramina in the mean value theorem, the same thing happens in the mean value theorem also. The point c you get lies between a and b, that point depends on the end points. Here also the point c, also depends on the end points. That means, that f not actually a polynomial, it just as this expression. Now, let us try to prove this, well the proof goes like this, first I am going to define a function of capital F, but the function F will depend on n.

So, I define capital F at n minus 1 at x, the idea is you keep b put instead of a in the above expression. That is capital F of n minus 1 x is f of b minus f of x minus b minus x f prime at x minus b minus x whole square divided by factorial 2 f double prime x and so on, up to b minus x to the power of n minus 1 by n minus 1 factorial into f n minus 1 of x.

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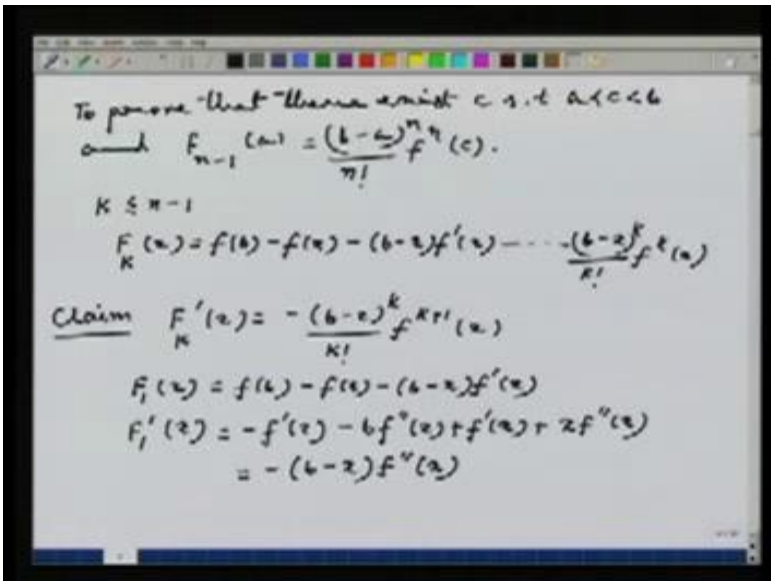

That means what is our aim, I essentially have to prove that, there exist c such that, a less c less b and n minus 1 into a is equal to b minus a to the power n by factorial n. F n of c, this is what I need to prove, what I do is, I vary the parameter n. So, for k less is equal n minus 1, what I do is I define a function F K analogously as the previous 1. I define F K at x is equal to f b minus f x minus b minus x, f prime x and so on, up to my last term is b minus x to the power k, by factorial k, f k x.

Among this class of functions f n minus 1, I have already told you what it is. Now, the first thing I claim is, this is my claim F K prime at x is equal to minus b minus x to the power k by factorial k times f k plus 1 x. To prove this, we can actually proceed. Let us first to take k to be equal to 1. So what is then, f 1 x, f 1 x is just f b minus f x minus b minus x into f prime x. And then f 1 prime at x, trans out to be minus f prime x minus b times f double prime x plus f prime x plus x f double prime x. If I cancel terms, what I get is minus b minus x times f double prime x, this precisely matches with the formula with the claim, which I have written for k to be equal 1. Now, what I am going to do is assume this result and try to prove it for F K plus 1.

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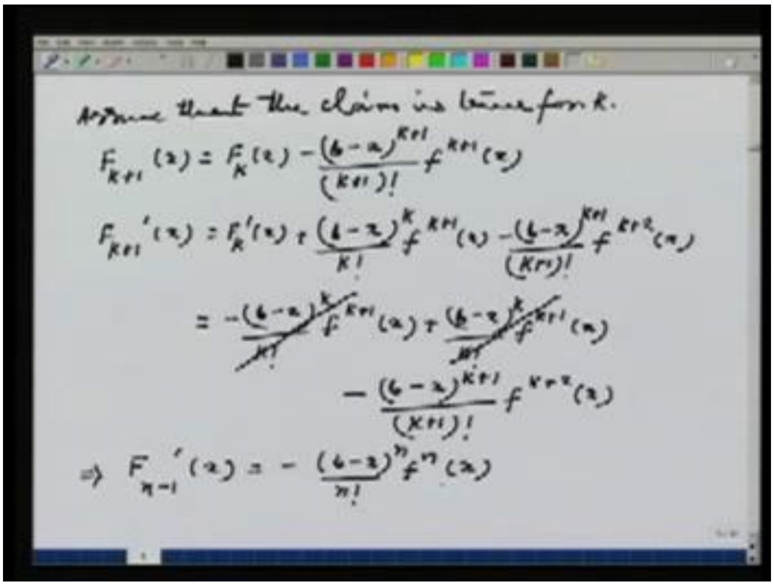

So, assume that this result is true for K ((Refer Time: 16:37)). Let us see the formula for F k s, it is b minus x to the power k, then factorial k, then derivative have one power extra that is k plus 1. So, when you write down F K plus 1 x, it should go to up to k plus 2. From the given formula, it is very clear that F K plus 1 x is actually F K x minus b minus x to the power k plus 1 divided by k plus 1 factorial into F K plus 1 x.

Now, I look at the derivative of this, I look at F K plus 1 prime at x, what I get is F K prime x. Now, the derivative of the next term, which I can calculate surely, which is plus b minus x to the power k by factorial k, F K plus 1 x minus b minus x to the power k plus 1 divided k plus 1 factorial into F K plus 2 x. Now, I am flagging the formula for F K prime x, which I have assumed this.

Then, is minus b minus x to the power k, by factorial k, F K plus 1 x, that is my assumption. Then, I just write down the other terms, which is b minus x power k by factorial k m F K plus 1 x, then minus the other term. That is b minus x to the power k plus 1 by k plus 1 factorial F K plus 2 x. Now, I can cancel the first two terms that is this one and this one.

So, the answer is the third term that is precisely same with my claim. So, we have proved the claim for all k s, in particular this implies this result is true, for all f n minus 1 also. So, f n minus 1 prime x trans out to be minus b minus x whole to the power n by factorial n, f n x.

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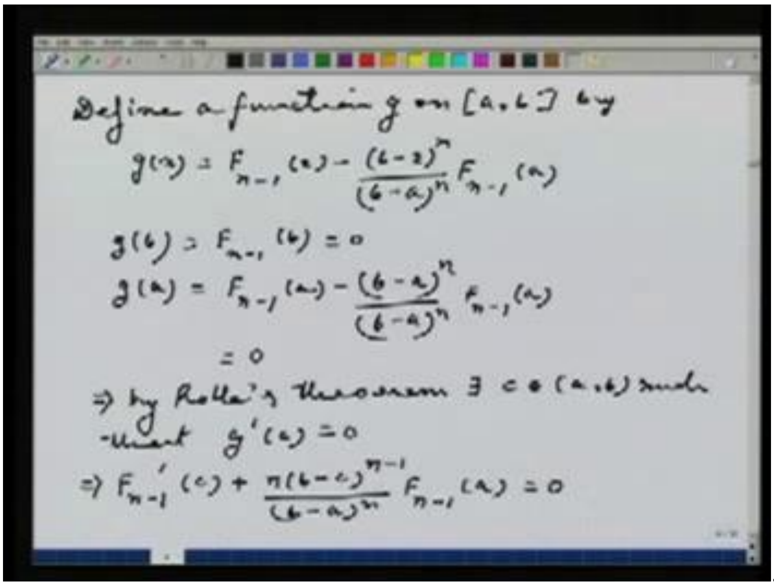

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Now, I am going to use this in the following form. What I do is, I define a function, g on the closed interval a b by g x is equal to f n minus 1 x minus b minus x to the power n, divided by b minus a to the power n into f n minus 1 into a. Now, clearly g is a differentiable function. And then, we check certain things here, what is g of b. That is, f n minus 1 b, because the other term is 0, b minus b which is 0.

So, I just get f 1 minus 1 b, but now let us look at the formula for f n minus 1 b. The greeting formula, if I put x to be b, I can certainly see from this formula. That is, f n minus 1 b is 0. So, this is equal to 0, what is g of a f n minus 1 a minus b minus a power n divided by b minus a power n into f n minus 1, a which is same as 0. So, g is a differentiable function which vanishes at the end points.

So, this implies by Rolle's theorem there exist c in the open interval a b, such that g prime of c is equal to 0. Now, g prime of c is equal to 0 implies f n minus 1 prime at c minus the derivative of the other function at c. That means n times b minus c to the power n minus 1, the sign would be plus divided by b minus a to the power n, f n minus 1 a, that is equal to 0.

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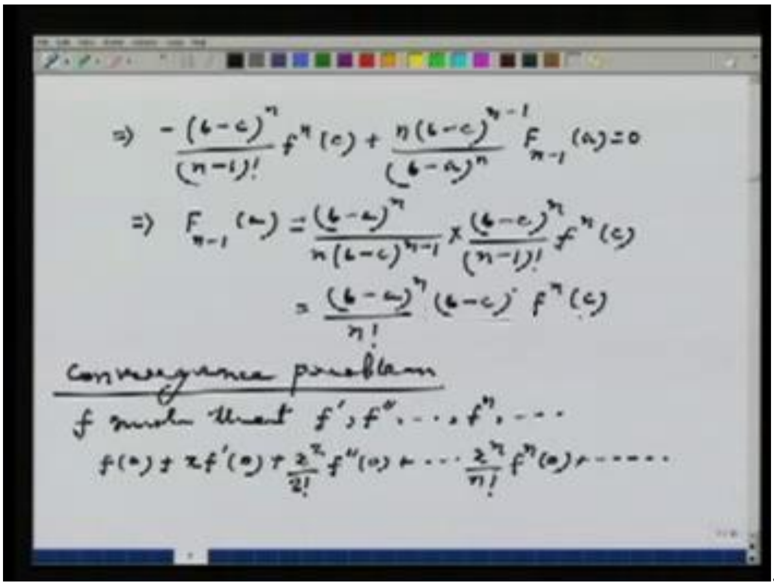

But remember, I have an expression for the function capital F n minus 1 prime, which I am going to use now. This would, then imply that minus b minus c whole to the power n by n minus 1 factorial, into f n c, this follows from my claim, which I have proved. Then plus the term which I have calculated b minus c to the power n minus 1 divided by b minus a, whole to the power n times f n minus 1 a, which is equal to 0.

This would then imply, that f n minus 1 a, remember the I have defined the function capital F n minus 1. My job was to show the capital F n minus 1 at a is actually the last term in the Taylor’s series. That is the term involving c, which we will see, how it comes out, this is then equal to b minus a power n divided by n b minus c, whole power n minus 1 times b minus c power n divided by n minus 1 factorial times f n c.

If, I calculate, what comes out is b minus a power n, divided by factorial n b minus c f n c, which is precisely, I wanted to prove. Now, I just write down the definition of f n minus 1 at a, which is equal to the last term of the Taylor series. That proves Taylor’s theorem, now we address another assume, it is the convergence problem.

Suppose I have the function f such that, all the derivatives of f exists, of all possible order, that is f prime, f double prime and so on f n and it goes on. So for example, I can take f x is equal to the 2 to the power x, it is a function. Whose derivatives of all order exists I could have also taken f x is equal to sin x for which also the derivative of all order exists.

So, I look at some such functions and then I can certainly write down this f 0 plus x f prime 0 plus x square by factorial 2, f double prime 0. And I go on writing it, notice I can go on writing it, because derivative of all possible order exists and hence the series makes sense.

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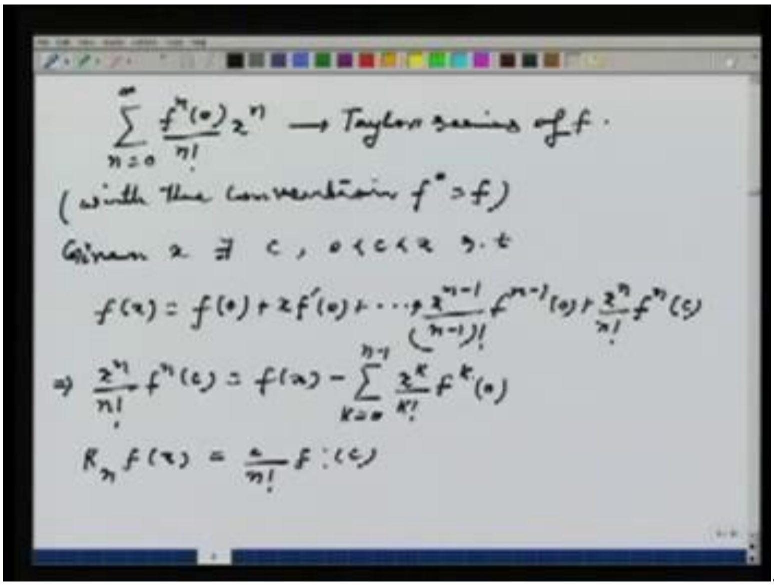

So, I actually have a infinite series at my hand, I look this summation n from 0 to infinity f n 0 by factorial n times x to the power n with the convention of course, with the convention f 0 is f. So, I get an infinite series or if you recall actually get a power series, this is called the Taylor's series of f. But the question now is, whether this series makes sense, does this series converge, because it is an infinite series and if it does it converge to the function f.

Well, it turns out many cases, it does converge to the function, but there are examples of functions for which the Taylor's series, even if it converges, it may not converge to the function. So, we will see this entire example. But before that let us try to understand, how exactly we determine, whether the Taylor's series converge to the function or not for that.

I am going to use Taylor's theorem, what I have proved is that the given x there exist c, which lies between 0 and x. Such that f x is equals to f of 0 plus x f prime 0 plus x to the power n minus 1 by n minus 1 factorial, f n minus 1 0 plus x to the power n by factorial n times f n c. Now, writing in different fashion, I will say that this imply, that x to the power x by factorial n times f n c is equal to f x minus summation k is equal to 0 to n minus 1 x to the power k by factorial k, F K 0.

Notice that, the term returning the sum precisely the nth partial sum of the Taylor's series and if I want to know whether the Taylor's series convergent to the function or not. All I have to check is that, the modules of f x minus the partial sum go to 0 as n increases, which is same as saying the left hand side should go to 0 as n go to infinity. We call it remain, R n f x to equal to x to the power n by factorial n times f n c, notice here the c do depend on x.

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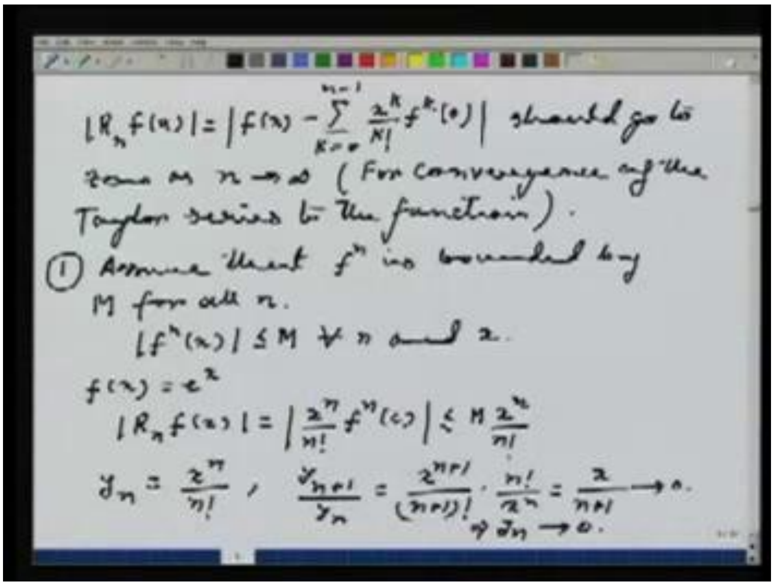

So, for the convergence of the Taylor's series, what we have to look at is mod R n x, which is same as f x minus summation k is equal to 0 to n minus 1, x to the power k by factorial k times F K 0 should go to 0 as n goes to infinity. This is required for convergence of the Taylor's series to the function. Well, there are certain easy cases, where this happens.

So, I will write it here, assume that f n, that is the nth derivative of f is bounded by a number M, for all n, that is I am saying that modulus of f n x is lesser equal to M, for all n and x, if this happen. Well as example, first let us see the function f x is equal e to the power x, but first we have to be sure, that if the bounded of the derivative is there. Then, I will get the convergence, why do these happens, because then modulus of R n f x. I look at, what is it is definition, I know modulus of x to the power n by factorial n times f n c.

Since, f n c is bounded by M, this would mean that M times x to the power n by factorial n. Now, just I will guarantee the x to the power n by factorial n converges to 0 as n goes to infinity for that what we do is, I define y n is equal to x to the power n by factorial n. And then I note that y n plus 1 divided by y n trans out to be x to the power n plus 1 by n plus 1 factorial into n divided by x to the power n, which is x by n plus 1.

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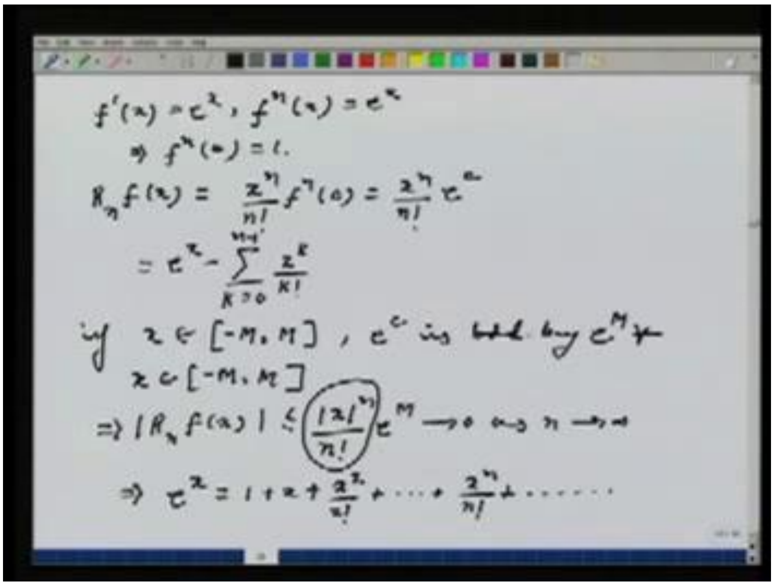

But my x is fixed that converges to 0. And then by previous exercise, which we are dealt with sequences, this implies the actual sequence y n converges to 0. That means, R n f x converges to 0, that means, the nth partial sum converges to f x for a given x. Now, I am going to use this for the function f x is equal to e to the power x. Let us what happens, l have then f prime at x is e to the power x, not only that for any arbitrary n, f n x is e to the power x.

This, then implies that f n 0 is equal to 1, then what is my R n f x, this is then x to the power n by factorial n f n c. That is x to the power n by factorial n times into e to the power c. And this is suddenly equal to f x, which is e to the power x minus summation k from 0 to n minus 1 x to the power k by factorial k, F K at 0, which I have noticed actually is equal to 1.

Now, I have given a fixed x, if x lies between minus M and M let us say, then e to the power c is bounded by e to the power M, for all x in minus M and M. This then implies that mod R n f x is lesser is equal to mod x to the power n divided by factorial n into e to the power M ((Refer Time: 35: 22)). So, I get mod R n f x is lesser is equal to mod x power n divided by factorial n times e to the power n.

Now, in e to the power m there is no n depends, n dependence only on mod x by factorial n. But, I have already established that the first factor, this goes to 0 as n goes to infinity. And e to the power is the constant and this goes to as n goes to infinity. This then implies the familiar formula that e to the power x is actually 1 plus x plus x square by factorial 2 plus x to the power n by factorial n and so on. This series converges to the series e to the power x, for all x and the right hand side is precisely the Taylor's series of the function.

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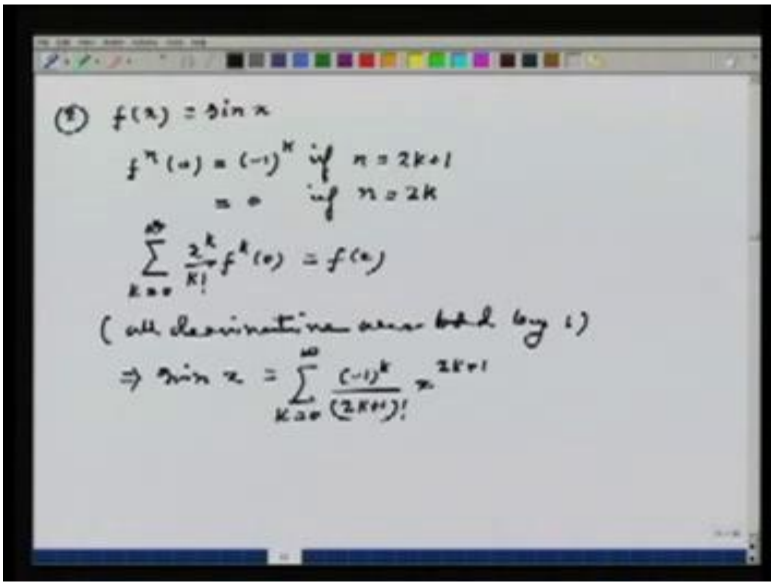

Now, let us look at the examples of another familiar function, I take f x is equal to sin x, this the good example to deal with you will see. Because, whatever derivative of f x I look at, it is either sin or cosine, in any case there are all bounded functions. So, the convergence of the Taylor series will not be much have a problem, and since I want to write down the Taylor's series of the function around 0. It will turn out the derivative precise value of the derivative; I can calculate either sin or cosine.

Well, the calculation is, that f n at 0 is equal to minus 1 to the power k, if n is of the form 2 k plus 1 and it is equal to 0, if n is equal to 2 k, that means the even powers of x, which should appear in the Taylor's series, will disappear because the derivative at 0 will be 0. So, the power of x, which will appear in the Taylor's series of sin x functions are suddenly going to the odd powers.

Well, this actually reflects the facts that sin of minus x is minus sin x, that property taken care of by this odd powers. Well, we will try to write down the Taylor's series of the function, it will then turn out to be summation k from 0 infinity x to the power k by factorial k into F K 0. I can say that this is equal to f x as of all possible derivatives are bounded by a single constant, all derivatives are bounded by 1.

This then implies, that sin of x is summation k from 0 to infinity minus 1 to the power k by 2 k plus 1 factorial times x to the power 2 k plus 1 using the same technique. Now, you can write down the Taylor's series of the function cosine x also, because again I can calculate the entire derivative at 0. Similarly, all possible derivatives are bounded by 1, so the Taylor's series will certainly converge and if you calculate the derivative, you will see at 0, all the even terms are vanishing.

And hence, you get the Taylor's series of the cosine function, which you probably already know as the series for the cosine x, but that series represents sin x or cosine x, actually follows from the Taylor's theorem. And since, the Taylor's series converges the function is given by the same series so far; so good. Now, we will look at the example, the Taylor's series converges, but it does not converges to the function given that can also happen.

Now, to construct the examples of the function for which the Taylor's series does not converge to the function. The basic idea is to construct a function, which has derivative of all orders that has to be there otherwise I cannot write down the Taylor's series. But, manage the functions, such that all it is derivative at the point 0. Let us say is 0, suppose that is happen, then what will happen, this is my example 3.

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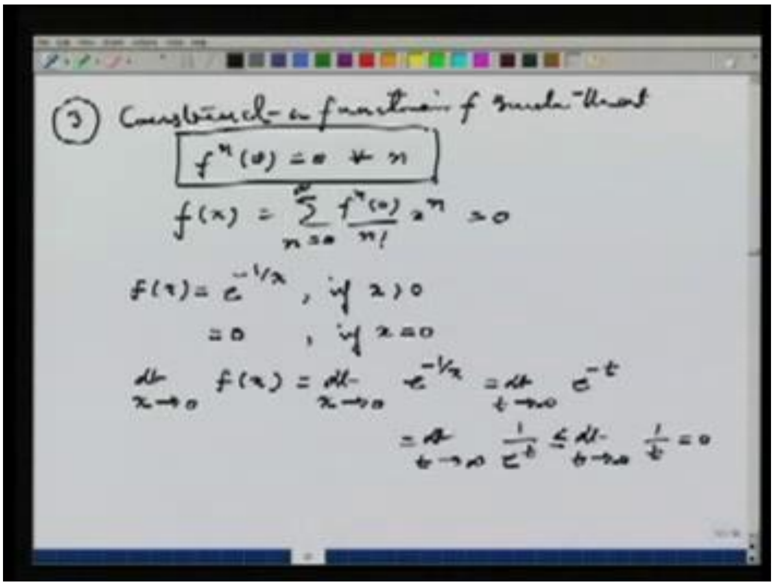

Try to construct the function the function f, such that f n at 0 is 0, for all n, suppose I do this. Then the Taylor's series converge to the function would mean, that f x is equal to summation n from 0 to infinity f n 0 by factorial n times x to the power n. But, I know that f n 0 is 0, that means, the right hand side is actually is equal to 0, so all I have to do is, I have to construct a function, which is nonzero and whose derivatives at 0 is 0.

If that happens, the Taylor's series of the function cannot converge to the function. So, the question is whether some such functions exist. Well they do this is, how we construct them, I define the function f x is equal to e to the power minus 1 by x, if x is bigger than 0. I define it to be equal to 0, if x is equal to 0; notice that one thing is very clear here, that the function is continuous.

Well, just I have to check limit x going to 0 f x is 0, that means, limit x going to 0 e to the power minus 1 by x is equal to 0, if I put 1 by x is equal to t. This would mean limit t going to infinity, e to the power minus t, which is same as limit t going to infinity 1 by e to the power t, which is lesser is equal to limit to going to infinity 1 by t. This is simply because, e power t is bigger than or equal to t that follows from the Taylor's series as t is positive, which is equal to 0, so the function is continuous.

So, the question is the function differentiable at 0, obviously the function differentiable, if x is less than 0 and since the function is 0 on the negative axis of the real line, you can also see the derivative also has to be equal to 0. Now, if x is bigger than 0, e to the power minus 1 by x is certainly a, differentiable function, so 0 is the only point, where we have to check the differentiability, so let us check it at 0 and try to find out what is f prime 0.

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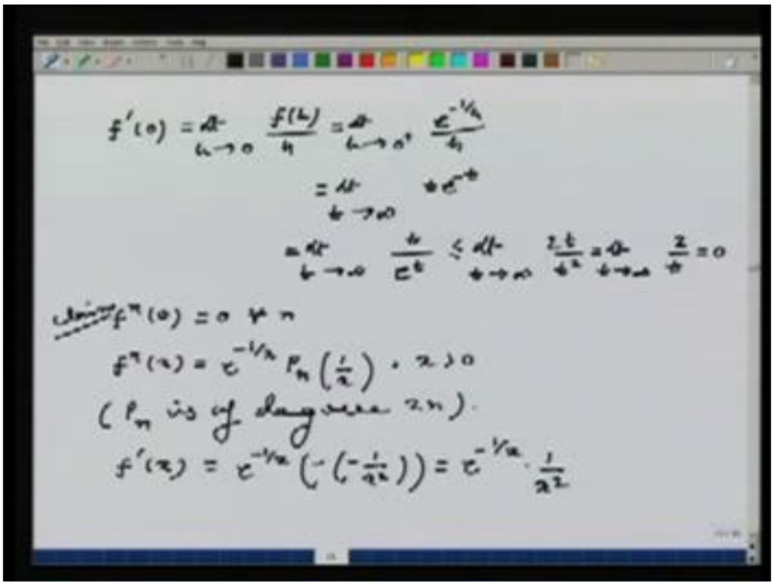

So, f prime 0, by definition is limit h going to 0, f h minus f 0, but f 0 is 0 it is f h by h, that is limit h going to 0, e to the power minus 1 by x divided by h. I again play the same trick of putting 1 by h is equal to t, I will then get limit t going to infinity, well here I am bother about h going to 0 plus, let me say that what I get is t times e to the power minus t. This is then limiting t going to infinity; t divided by e to the power t, which then lesser is equal to limit t going to infinity t by t square.

Because, e to the power t bigger than or equal to t square t by 2, so what I get here is limit t going to infinity 2 by t, which is again is equal to 0. So, the derivative exists and not that, only that at 0 is 0, so I want to prove that f m 0 is equal to 0 for all n this is my claim. I will try to prove by induction and for that obviously, need the expression of x n if x is bigger than 0.

So, the another claim is that f n x is equal to e to the power minus 1 by x times a polynomial p n at 1 by x for x bigger than 0, where it will turn out t is of degree 2 n which is not important for us right now, but this is what we want to prove. So, let us see the thing is true for n is equal to 1, the first one we have seen that f prime 0 is 0, that is f prime at x, if x is positive.

We can simply differentiate, it is e to the power minus 1 by x, times the derivative of minus 1 by x, that is minus of minus of 1 by x square, which is e to the power minus 1 by x times 1 by x square.

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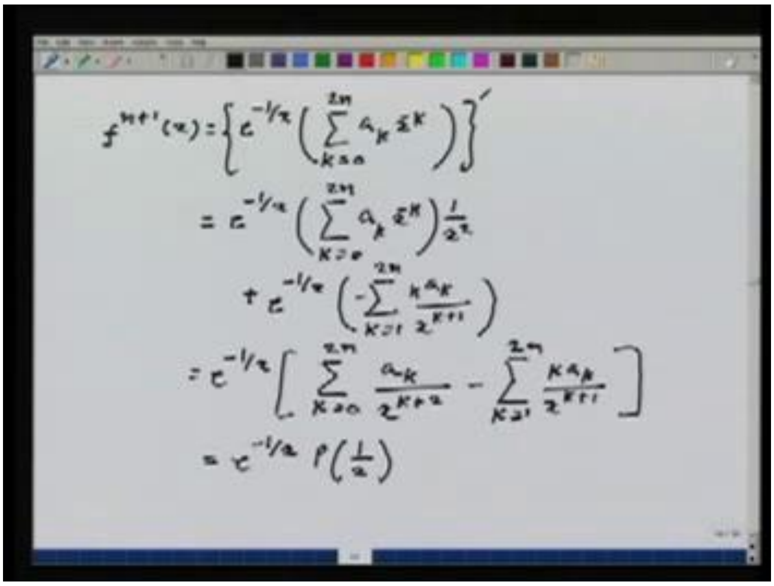

So, I will assume that the induction hypothesis, that the result is for n, I want to prove it for n plus 1, so what is f n plus 1 x, that is the derivative of the function e to the power minus 1 by x times p n x. Now, there is a polynomial here, which I will know, so I write down the polynomial here, it is k from 0 to 2 n a k x to the power k, the prime of this function.

This is my f n plus 1 x, well this then turns out to be e to the power minus 1 by x into summation k from 0 to 2 n, a k x to the power k times 1 by x square plus e to the power minus 1 by x times the derivative k from 1 to 2 n. So, f n plus 1 x is e to the power minus x times the polynomial at 1 by x. So, I have a power minus here and then I differentiate, what I get is k equal to 1 to 2 n with minus sign a k divided by x to the power k plus 1 times k.

This is, what I get now if add the polynomial terms I see that, I get a polynomial of degree 2 times n plus 1. So, this is e to the power minus 1 by x times summation k from 0 to 2 n a k by x to the power k plus 2 minus k is equal to 1 2 n k a k divided by x to the power k plus 1. You see I get a polynomial, whose degree is 2 times n plus 1 and in the variable 1 by x, so I get into the minus 1 by x times the polynomial p, at the point 1 by x, so this verify my second claim.

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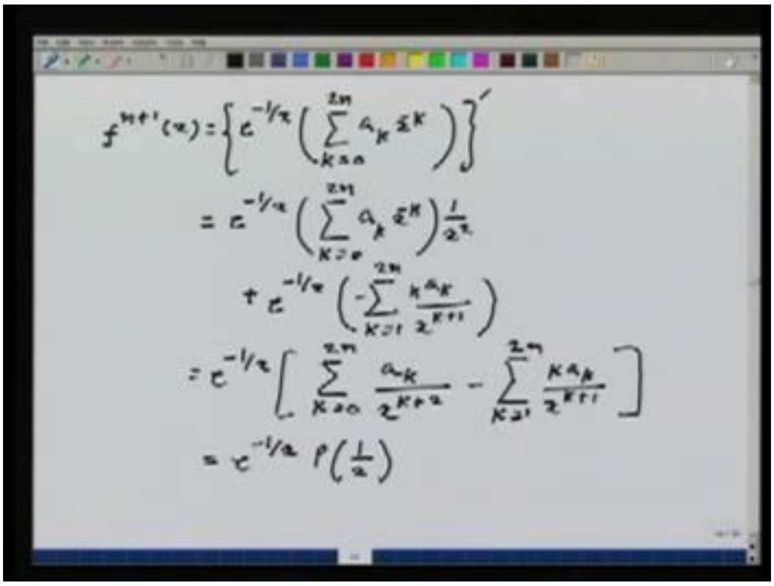

But, what about f n plus 1 0 then, that I have to calculate this, then limit h going to 0, f n h minus f n 0 divided by h minus, since the function is 0 on the negative axis, all these f n h are anyway 0. So, h going to 0 minus m if I take certainly to get 0, what I am bother about is limit h going to 0 plus. Now, I have an expression for f n h, I will put limit h going to 0 plus, I know the expression of f n h, it is e to the power minus 1 h times a polynomial p at 1 by h, then I have another 1 by h.

Now, if I write down the expression of the polynomial and term by term, I can look at the limit. So, I am essentially bother about tending the limit plus e to the power minus 1 by h times 1 by h to the power some m, m is bigger than or equal to 0, certainly. Now, I calculate this limit again use the same trick, which I have used limit t going to 0, infinity t to the power m, e to the power minus t.

Now, this then lesser equal to limit t going to infinity t to the power m times m plus 1 factorial divided by t to the power m plus 1, as e to the power t bigger than or equal to t to the power m plus 1 divided by factorial m plus 1. Now, this certainly goes to 0, as t goes to infinity, this implies all the derivative at 0 is 0, and hence the Taylor's series of the function, if I look at the Taylor's series turns out to be 0.

So, it cannot converge to the function e to the power minus 1 by x is always non 0, if x is positive. So, how it can happen, that the Taylor’s series of the function converge to the function, for all x that is not true, it can converge to the function. If x is negative for positive x, it cannot converge to the function. Here we see examples of a function for which, the Taylor’s series although converges, it is 0 for all x it does not converge to the function.