Mathematics-I
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Lecture - 12
Curve Sketching

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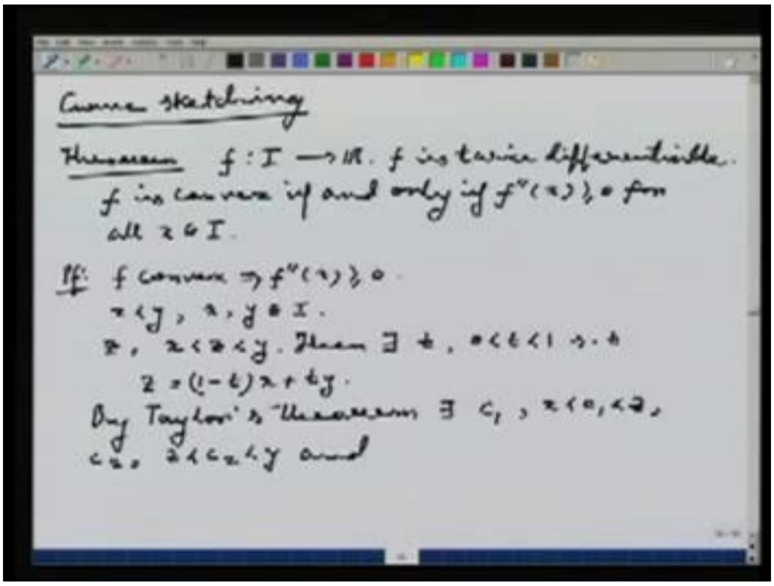

We come to Curve Sketching. We will start with convexity of functions again, there was theorem which I stated and only one part of that theorem has proved. The theorem was this, let suppose f it is from an interval I to R also assume that f is twice differentiable. Then, the theorem is f is convex if and only if f double, everywhere negative. Now, one part of this result, we have already proved that f convex implies f double prime x is bigger than or equal to 0. Now, we want to prove the convex, as I said earlier that this reduce Taylor's theorem. That is I will assume that f double prime is nonnegative everywhere from that I will like to prove that f is convex.

Well I take some x less than y and x and y are in I, let us z satisfies the following that x less z less y, then there exist t which lies strictly between 0 and 1, such that z is equal to 1 minus t x plus t y. Now, what I do is I use Taylor's theorem that is I want to write f x and f y around z. So, by Taylor's theorem, there exist c 1 such that x less c 1 less z and c 2 which satisfies z less c 2 less y.

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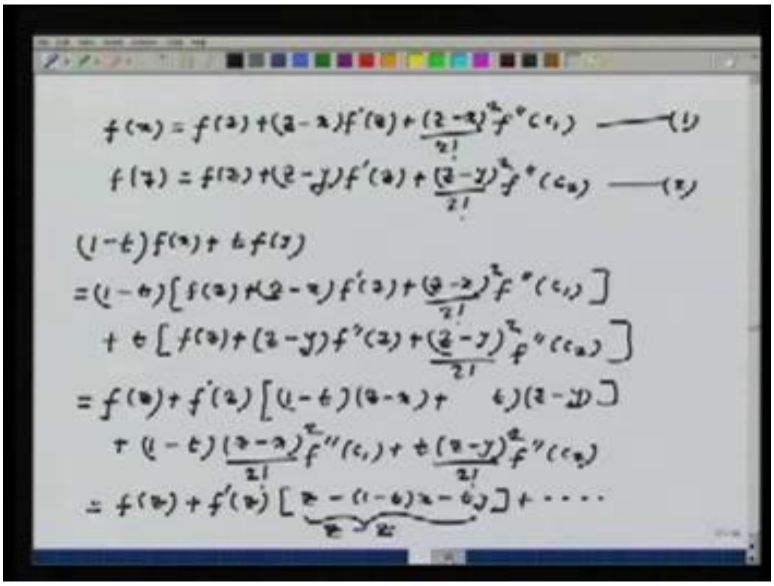

And f of x is equal to f of z plus z minus x times into f prime at z plus z minus x whole square divided by factorial 2 into f double prime at c 1. So, I am using the Taylor's theorem of order 2. Then I write down f of y, which is f of z plus z minus y into f prime z plus z minus y whole square by factorial 2 f double prime c 2, this is my first equation, this is my second equation.

Now, I look at 1 minus t times f x plus t into f y, notice that it if I want to show f is convex. I have to show that this quantity is bigger than or equals to f of 1 minus t x plus t y. So, let us proceed using 1 and 2 I get that this is equals to 1 minus t into f z plus z minus x into f prime z plus z minus x whole square divided by factorial 2 f double prime c 1 plus t into f z plus z minus y f double prime z plus z minus y whole square by factorial 2 f double prime c 2.

Now, this implies if I collect the co efficient of f z I get 1 minus t plus t that means, I get just f z. Then let us look at f prime z, the co-efficient there I get 1 minus t into z minus x plus 1 minus t into z minus y plus the remaining term. That is 1 minus t into z minus x whole square by factorial 2 times f double prime c 1 plus t into z minus y whole square by factorial 2 f double prime c 2, which then turns out to be f z plus then f prime z into. If I calculate 1 minus t z plus I have t y 1 minus t z plus t z is z, then I have minus 1 minus t x minus t y plus the other terms. Now, if I look at this 1 minus t x plus t y is z, so this quantity is actually z minus z, which then is 0.

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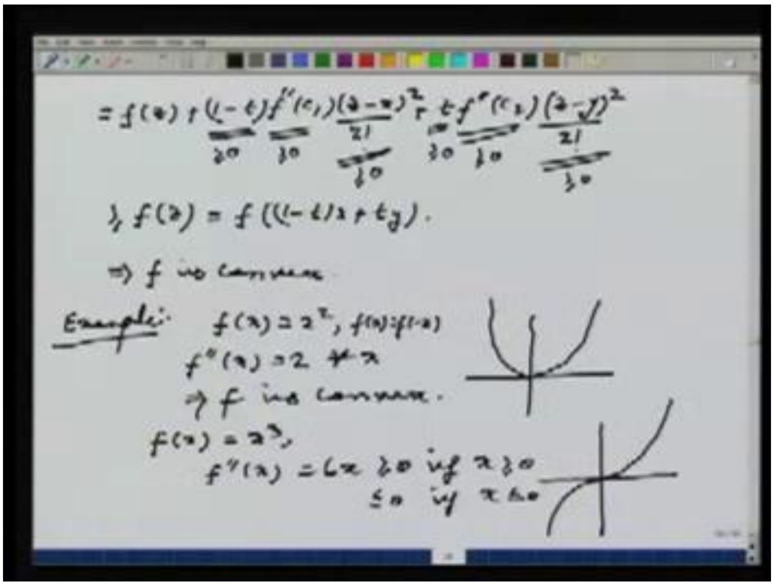

So, what I am left with this actually f of z plus 1 minus t into f prime c 1 into z minus x whole square by factorial 2 plus t f double prime c 2 times z minus y whole square divided by factorial 2. Now, look at all these terms 1 minus t is bigger than or equal to 0, because t lies between 0 and 1 f double prime c 1 by my assumption is bigger than or equal to 0, this quantity being whole square is bigger than or equal to 0.

Similarly, this term is bigger than or equal to 0, this term here is bigger than or equal to 0, so is this turn. That means, this quantity is certainly bigger than or equal to f of z, but then this is equal to f of 1 minus t x plus t of y, this is precisely what we wanted to prove. Because, the left hand side was 1 minus t times f x plus t times f y which here is standing out to be is bigger than or equal to f of 1 minus t x plus t y. So, this satisfies of convexity imply that f is convex, this is the easy criteria to check convexity of functions, if the function is twice differentiable.

For example, if I look at the function f x is equal to x square, then if I look at the double derivative of this function f double prime x this is 2 for all x implies f is convex. But, if you look at f x is equal to x cube the situation is not, so because if I look at f double prime x, this is 6 x which is bigger than or equal to 0. If x is bigger than or equal to 0, and it is less than or equal to 0, if x is lesser equal to 0; that means the function f x is equal to x cube is convex, if x is bigger than 0, but if x is less than 0 the function is concave, and then it, because it is very easy to draw the graph of this function. Suppose this is my axis, then f x is equal to x square also satisfies that f x is equal to f minus x. That means on the right hand side of y axis the function looks exactly as in the left hand of y axis, but on the right hand of the y axis is any way convex and f of 0 is 0. So, f x is equal to x square look like this. And hence on the negative side also it will look like this. But, if I look at f x is equal to x cube it will look like this on the right hand side.

But it will look like this on left hand side, because it is concave on the left hand side this is how this result help us. It becomes the most fundamental result for convex functions to determine convexity. You just look at the double derivative of the function and try to see where it is non negative, where ever it is, it is convex. The same way one can say, the same thing is that a function f is convex. If it is derivative is increasing because, derivative increasing would mean the double derivative non negative. So, we are going to use it at some point of time that a function f is convex, if f prime is increasing or the double derivative is non-negative.

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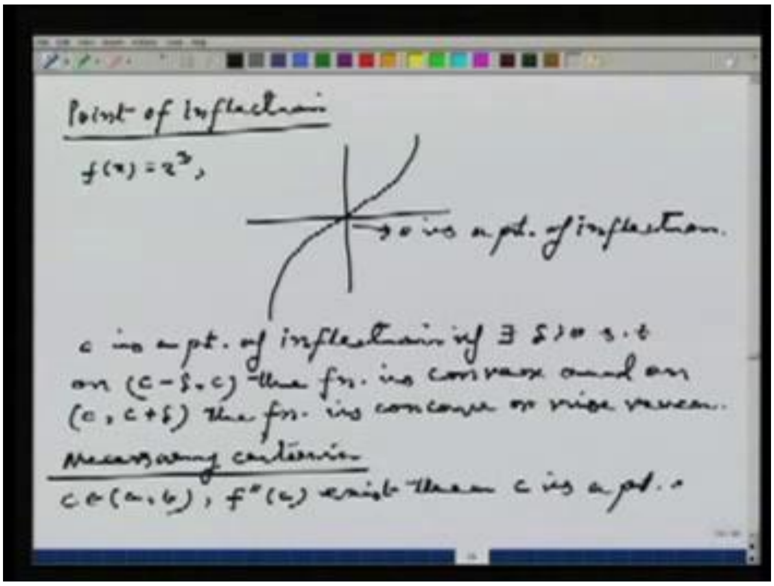

Now, let us go to the second thing we are going to define something called point of inflection, point of inflection actually determines points. Where the convexity of the function, just after that point changes to concave it or the other way. The function was concave, but just after this point it become convex, remember the example f x is equal to x cube. We have seen the graph of the function looks like this kind of. So, 0 here 0 is looking like point of inflection. Because just after 0 the function become convex and before 0 that is on the negative side, it is concave.

So, what is the precise definition, so the definition is this c is a point of inflection, if there exist delta bigger than 0. Such that on c minus delta c, the function is convex and on c, c plus delta the function is concave of vise versa. Now, how do you determine from the just given the function that which one easier point of inflection, for this we have a necessary criteria.

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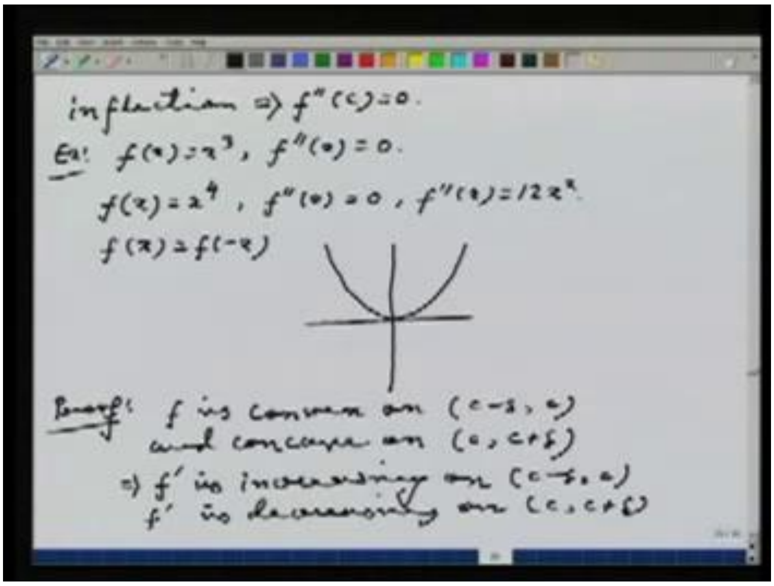

The necessary criteria say is this, that c belongs to a, b and f double prime c exists, then c is a point of inflection. Implies f double prime c is 0 again as a example, I will sight f x equal to x cube, notice that f double prime at 0 is 0. So, 0 is a candidate for being point of inflection, but by this criteria I am not saying that c is a point of inflection, if I already know that c is a point of inflection, then I know that f double prime at c is 0. But, f double prime at 0 does not imply that c is a point of inflection, because now I can look at f x is equal to x to the power 4, then what is f double prime at 0? This is 0.

Because, f double prime at x is equal to 12 x square, the question is, is a 0 point of inflection for f x is equal to x to the power 4, the answer is no. Because, f is an even function because, f of x is same as f of minus x. So, the behavior of f x on the right hand side of the y axis is same as the behavior of left hand side of y axis. But, I already calculated what is f double prime x, f double prime x is 12 x square. So, it is always non negative, that means the function is convex for all x and the graph will look like this and on the negative side also it will look like this.

See 0 is not a point of inflection on the left hand side of 0 the function is convex, and on the right hand also the function is convex. So, all we are saying is, if you know that some point is a point of inflection. Then the double derivative is 0 there, but the double derivative 0 at certain point does not guarantee, that point is a point of inflection. Now, let us see the proof of this how do we prove it assume that f is convex on c minus delta c and concave on c, c plus delta I can say this. Because, my assumption is that c is a point of inflection, so on one side of c it has to be convex or concave on the other side it has to be the other one.

So, we are assuming the left hand side the function is convex and on the right hand side it is concave. Now, this would then imply that f prime is increasing on c minus delta c and f prime is decreasing on c, c plus delta, this I can say because I already know my criteria that is f is convex, if and only if the double derivative is nonnegative.

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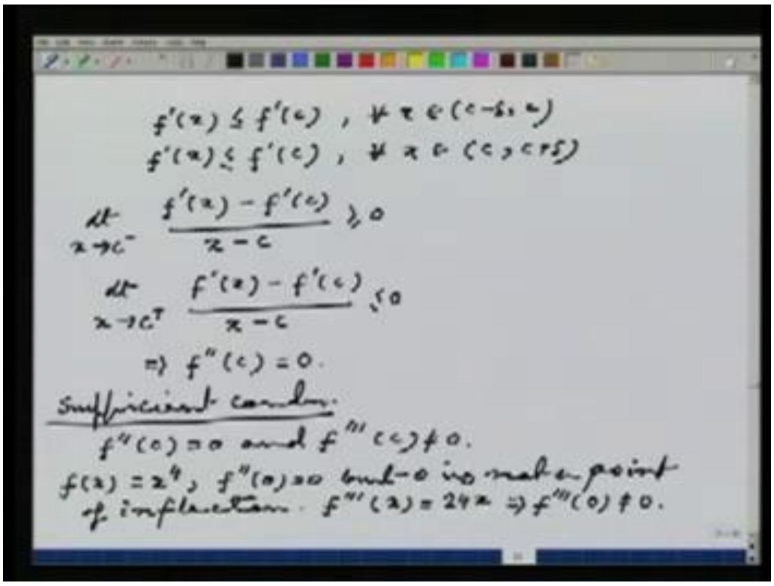

Now, this means f prime of x is lesser equal to f prime of c for all x in c minus delta c and f prime of x. Since f prime is decreasing on the right hand side of see, I have that this is still true, this is lesser equal to f prime c for all x in c, c plus delta. Now, I look at the quotient f prime x minus f prime c divided by x minus c and I take the limit as x going to c minus. That means, I am approaching c from the left hand side of c, this would then imply what because, f prime x is any way lesser equal to f prime c.

So, the numerator is negative, but x is also lesser equal to c; that means, the denominator is also negative. That means, this is bigger than or equal to 0 and then I look at limit x going to c plus and write the quotient f prime x minus f prime c divided by x minus c. Here, notice that f prime x is again lesser equal to f prime c; that means, the numerator is negative, but x is bigger than c that means, the denominator is nonnegative.

That means the whole quantity is lesser equal to 0, and now I use the assumption that f double prime c exists. Since, f double prime c exists, it means it has to be same as this limit, then implies that f double prime c which is same as both the limits and both the limits can have the only common value which is 0, so f double prime c is equal to 0. But, as said this the necessary condition, but not a sufficient condition. Well we state the sufficient condition for you, without the proof the sufficient condition is that c is a point of inflection. If f double prime c is 0 that has to be there and f triple prime c, if it exists of course, this is not equal to 0.

Now, notice again the example of f x is equal to x to the power 4, we have notice that f double prime 0 is 0, but 0 is not a point of inflection. And now we know this has to be happen because, if I look at f triple prime c, what is f triple prime x that is 24 x. So, this implies then that f triple prime x is 24 x; that means if triple prime 0 is not equal to 0 and that is why 0 is not a point of inflection for f x is equals to x to the power 4.

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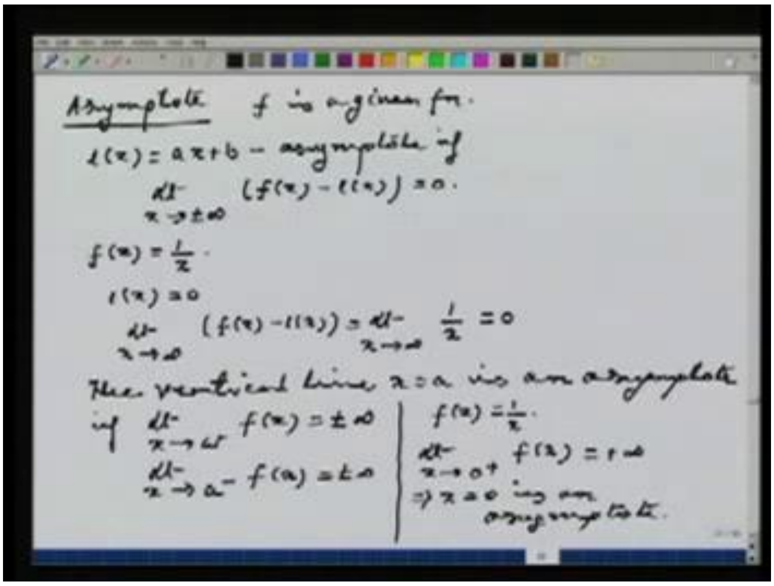

Now, I move towards next topic of ours, which is important for drawing curves of functions. That is asymptotes, intuitively asymptotes means that they are tangential to the curves, but at infinity if you go towards infinity the distance between the curve and the function decreases to 0 you know something like this. The precise definition is this, that I look at a line l x, f is a given function, I look at a line l x, which is of the form a x plus b, this is an asymptotes if limit x going to plus or minus infinity f x minus l x goes to 0.

That is the distance between the line and the curve is actually goes to 0, as an example let me look at the function f x equals to 1 by x. And then I look at the function x axis, that is y is equal to 0. So, then I should look at, so l x is equal to 0 this is my line. And then I look at limit x going to infinity f x minus l x, which is limit x going infinity just 1 by x and I know this is equal to 0. That means, the x axis is asymptotic to the curve 1 by x, now there is another kind of asymptotes a vertical line.

The vertical line x is equal to a is an asymptote, if limit x going to a plus or a minus or limit x going to a minus f x is plus or minus infinity. Again as an example, I look at f x is equal to 1 by x and I look at the y axis, that is x is equal to 0. And then I check that limit x going to 0 plus f x equal to plus infinity, this implies x equal to 0 is an asymptote.

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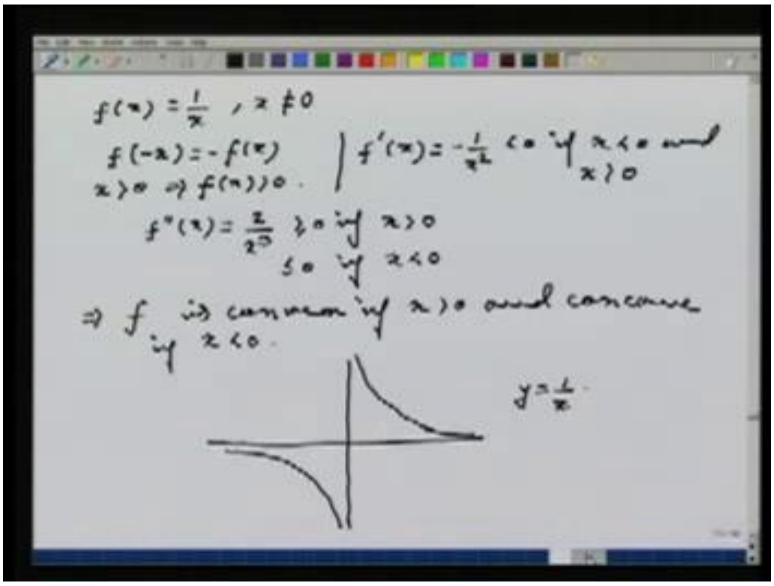

That means, with the curve y is equal to 1 by x I have got two asymptote, one is the x axis, other is the y axis. Let us see whether we can draw the curve of this f x is equal to 1 by x. Notice this is define only when x is not equal to 0, for x equal to 0. I do not know, this has the property that f of minus x is equal to minus f x and x bigger than 0 implies f x is also bigger than 0.

That means, if x is on the right hand side, the curve will lie above the x axis, if x is on the left hand side the curve will lie below the x axis, and not only that they will very symmetric towards each other. Now, how the curves wends to understand that, I look at f double prime x. That is f prime x I know which is minus 1 by x square and then f double prime x is 2 by x cube, which is bigger than or equal to 0, if x is bigger than 0, it is lesser than or equal to 0, if x is less than 0.

So, f prime x is minus 1 by x square which is less than 0, if x is less than 0 and x is bigger than 0. That means, on both sides of y axis I will have my function to be a decreasing function, then I look at the double derivative. I see that at f double prime x is 2 by x cube, which is bigger than or equal to 0, if x is bigger than 0 and less than or equal to 0 if x is less than 0. That means, f is convex if x is bigger than 0 and concave if x is less than 0, I already know that x axis and y axis both are asymptotes to the function.

And now I can draw the graph of the function, it looks like this, this is the y axis, this is x axis from the convexity of the asymptotes I see that the function look like this is on the right side. And similarly it will look like this on the left side, so this is the graph of y is equal to 1 by x, I have used asymptotes increasing decreasing and convexity. Now, using all these let me try to draw another curve for you, which is slightly more complicated than this, but not very complicated.

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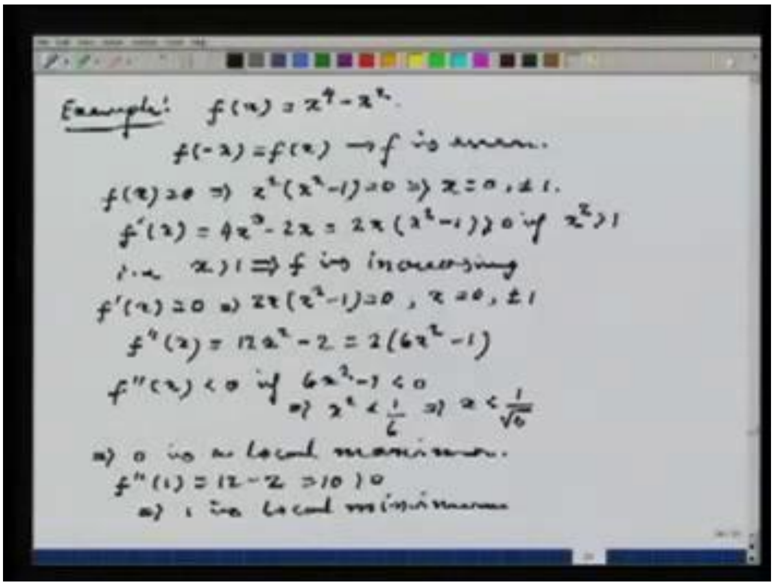

The curve is this f x is equal to x 4 minus x square, again notice the first thing that f of minus x is equal to f x. That means, it is an even function. So, it is enough to draw the graph of the function on the right hand side of the y axis, this implies f is even it is enough to draw the graph of the function on the right side of the y axis. Now, let us first try to see, what are the 0's of the functions? Where does it cut the x axis. So, f x is equal to 0 implies x square into x square minus 1 equal to 0 implies x is equal to 0 or plus minus 1.

So, there are two points on the right hand side where the function cuts the x axis, the minus 1 has to appear because, the function is even function, because the same behavior will be shown on the left hand side of the y axis. Now, let us calculate to understand the increasing, decreasing property of the function. Let me calculate what is f prime at x, that trans out to be 4 x cube minus twice x, that is twice x into x square minus 1 since, I am looking for positive x, I can say that this is bigger than 0, if x square is bigger than 1 that is x is bigger than plus 1.

Now, what are the extreme points the maximum minimum of the function that also I have to understand. So, I have to look at the critical point that is f prime x equal to 0, this implies twice x into x square minus 1 equal to 0; that means, x equal to 0 and plus minus 1. I have already noticed for x bigger than 1 with implies x is increasing. Now, I have to find out, what is the nature of the critical points, where do we have maximum, where do we have minimum for which now I have to look at the double derivative, that is f double prime x, what is f double prime x well that trans out to be 12 x square minus 2, that is 2 into 6 x square minus 1. Notice that this is f double prime x is less than 0, if 6 x square minus 1 is less than 0, this implies x square is less than 1 by 6, this implies x less than 1 by root 6. In particular it follows that at 0 f double prime is negative, this would then certainly imply that 0 is the local maximum.

What about the other point plus minus 1's are also critical points. Well, what happens at 1 f double prime at 1 is 12 minus 2, that is 10 which is bigger than 0 this implies 1 is local minimum, but in the process I have also obtain some more information that is when x lies between 0 and 1 by root 6 then the double derivative is negative.

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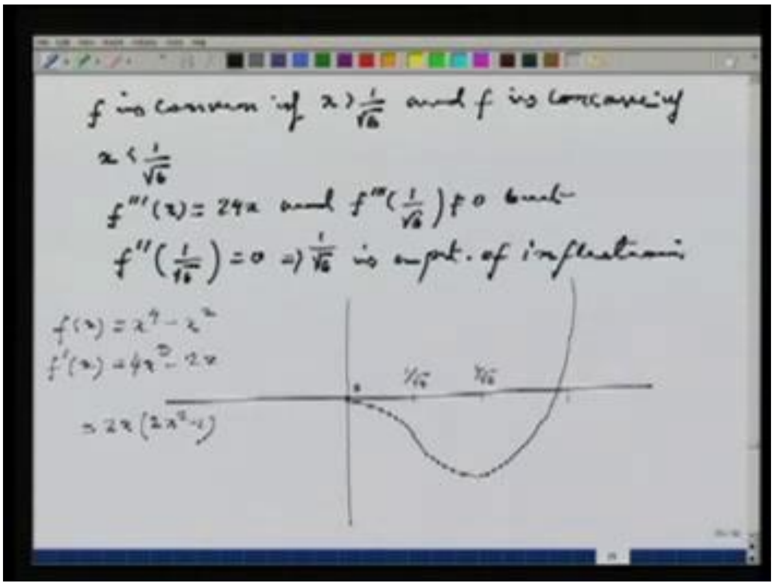

That means the function is concave there, and if x is bigger than 1 by root 6, then f is convex. Let us write down that also that f is convex, if x is bigger than 1 by root 6, and f is concave, if x is less than 1 by root 6 with this information. That means, actually the 1 by root 6 is a point of inflection, we can actually check that. Because, what is f triple prime x I know the expression of f double prime x it is turning out to be 12 x square minus 2. So, f triple x is 24 x and f triple prime at 1 by root 6 is non zero, but f double prime at 1 by root 6 is equal to 0 this implies 1 by root 6 is a point of inflection.

That is on the left hand side of 1 by root 6 the function is concave, but on the right hand side of 1 by root 6 f is convex, as we have calculated. Now, with this information it is very easy to draw the graph of the function that we will see. Now, let us have the y axis here, we have the x axis here. The curve has 0 at 0 that we have seen, let us mark the points this is 0, then I have another point, this is 1 by root 6 and I have another point 1 by root 2 that I know, the local minimum arises.

And then I have the point 1, where the function cuts the x axis I know at 0 the function has a local maximum and after that from the derivative exists follows that up to 1 by root 6 the function is concave. So, the graph goes like this in a concave fashion up to 1 by root 6, but after 1 by root 6 I know that the function is convex and 1 by root 2 is a local minimum. So, it goes now like a concave fashion again from the calculation of derivatives I have seen that if x is bigger than 1 by root 2, the derivative is positive right.

We can actually do the calculation once again because, f x was x 4 minus x square, so f prime x is 4 x cube minus twice x, which is twice x into times twice x square minus 1 you see that means, after minus 1 by root 2 the function increases and reaches 0 at 1 and then goes on increasing in a convex fashion. So, it goes now like this it goes in a convex fashion, similarly now I can since the function is an even function on the left hand side of the y axis I can do the same proceed here to produce the graph of the function.

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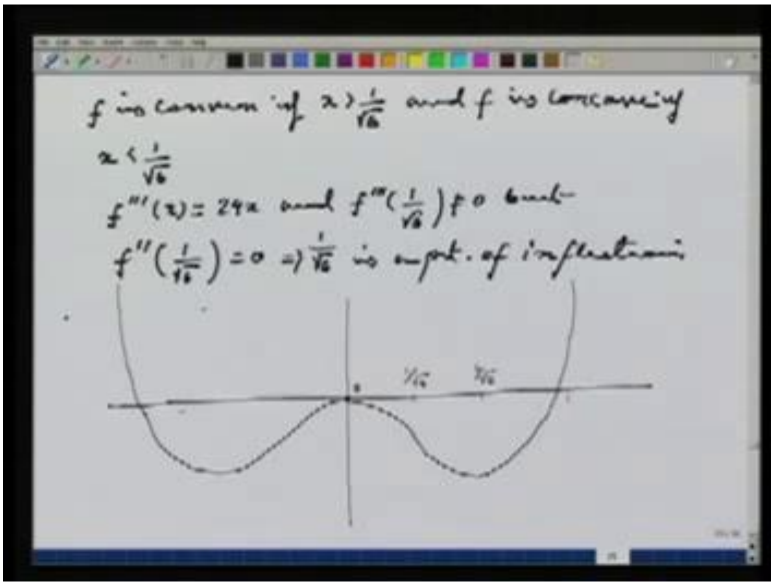

It will then look like, it is exactly the raplic of the right hand side it will go exactly like this, this what now the graph of the function looks like. Now, let us try to sum up what we have done in these set of lectures, we started with derivatives and then after mean value theorem is started connecting the behavior of the function with derivatives.

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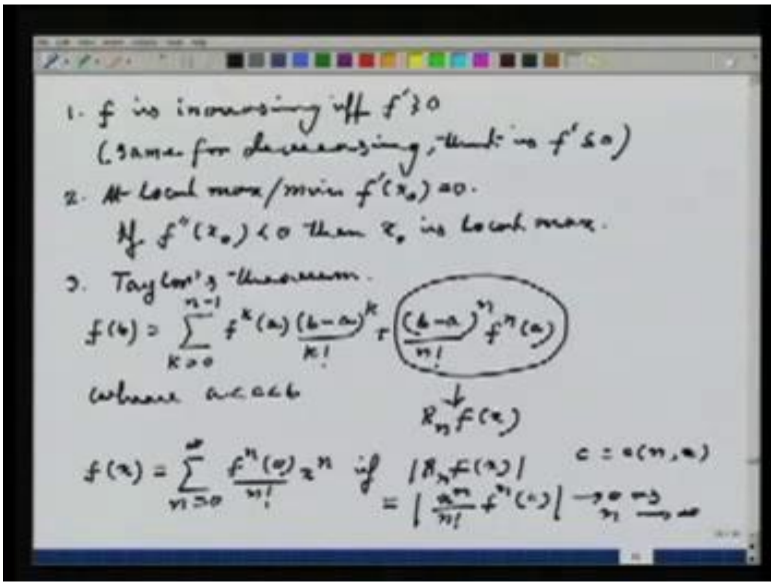

The first one is f is increasing, if and only if f prime is bigger than or equal to 0 same for decreasing that is f prime is less than or equal to 0. Second point was the extreme points at local maximum or local minimum f prime has to be equal to 0, but then to guarantee whether the point is maximum or minimum, we have to go to the double derivatives. That is if f double prime at x naught trans out to be less than 0 then x naught is a local maximum and I will mention again this conditions work, if the extremum point in the interior of the interval where you are looking at, if it is a boundary point then this theorem they not be true we have seen counted examples of that kind.

Then comes the third one Taylor's theorem, it says that if we have a function f in a closed interval a b then f b can be returns as summation k from 0 to n minus 1 f k a b minus a to the power k by k factorial plus b minus a to the power n by factorial n into f n c, where a less c less b.

And the last term which we have written, this is called the remainder after the n'th terms we have noticed it as written it as R n f x term, we have also seen that the Taylor's series of the function converges to the function, if and only if this remainder goes to 0 as n goes to infinity, that is I can write it f x to equal to summation n from 0 to infinity f n 0 by factorial n into x to the power n, if modules of R n f x that is modulus of x to the power n by factorial n into f n c goes to 0 as n goes to infinity.

Notice that the point c which I am writing here, it should actually be written as c should be written as c n x, it depends on the point x as well as n which I am looking at, if the remainder goes to 0. Then the function is actually a power series given by it is Taylor's series and the power series is of the form f n 0 by factorial n into x to the power n.

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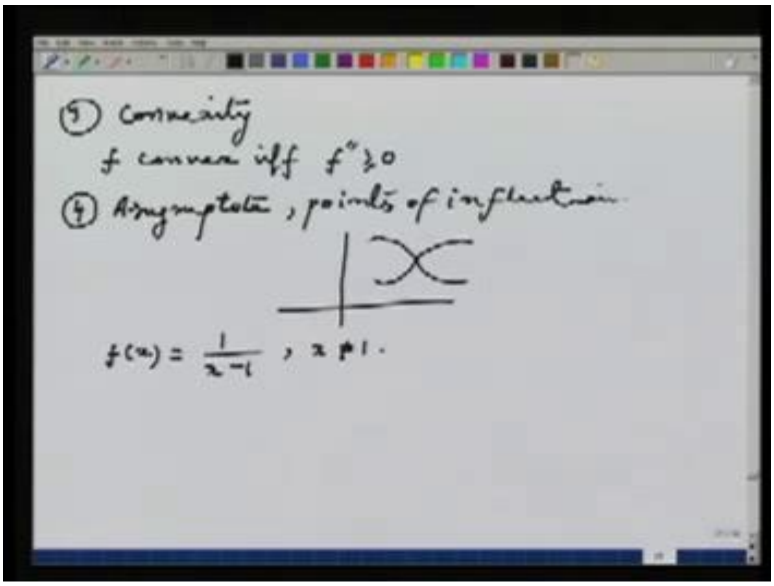

After doing Taylor's series, we have gone to convexity and the most important result about convexity is what we have proved that f is convex if and only if f double prime is bigger than or equal to 0. But, it the same thing as saying that f prime is increasing concave similarly means f double prime is less than or equal to 0 or f prime is decreasing, after this we have defined asymptote and point of inflection, one is tangent at infinity that is asymptote point of inflection as we know what it means, you can always remember some point is a point of inflection means, the curve of the function has to look like this or it has to look like this.

That is on the left hand side you have convexity and on the right hand side you have concavity or the other ((Refer Time: 43:37)). After this we have drawn curves for you I will know give as a exercise for you to draw the graph of this function f x equal to 1 by x minus 1 for x not equal to 1 just try to draw the graph of the function. Now, there are certain tips to draw the graph of the function, for check whether the function is even or odd it may not be anything, but if it is your job is simpler.

If it is even function or an odd function, just draw the graph of the function on the right hand side of the y axis. If it is the even function draw the same graph on the left hand side, you get the graph of the function, after doing this if you look at the first derivative of the function to find out the increasing, decreasing behavior of the function and the extreme point, that is the 0's of the derivatives. Then the next thing you do is, you examine the 0's of the derivatives, whether they are local maximums of local minimums.

Once you do that and where it cuts the x axis that is important, that is what we have used in one of our examples. Once you know that you have some good idea about where the graph of the function cuts the x axis and which way it is increasing or decreasing, that you get from the behavior of the derivative. Now, go towards the double derivative because, once you want to draw the curve you have to know which way bends, that is always given by the convexity and concavity of the function. You calculate the double derivatives, see it behavior where it is positive, where it is negative, where it is 0. If you are some 0 look at the triple derivative of the function, you might get the point of inflection and then using all this information's, I am sure you can draw the graphs of many, many function.