Mathematics-I
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Lecture - 13
Infinite Series I

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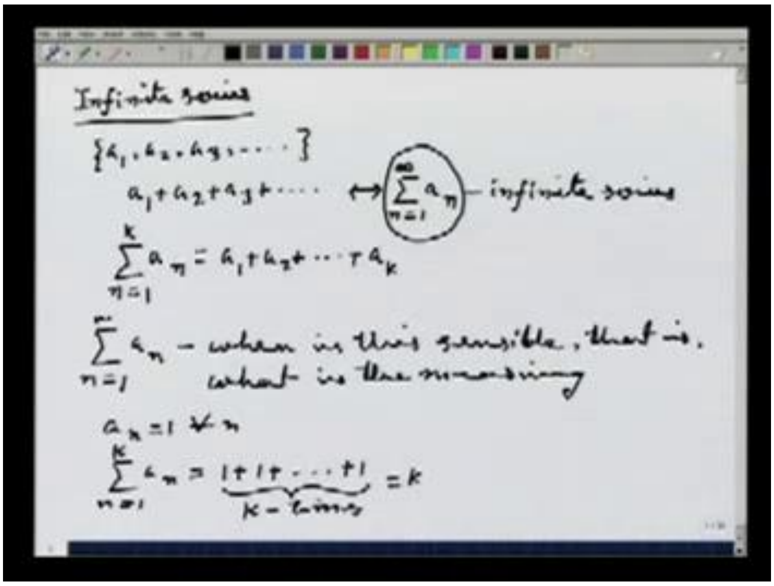

In today’s lecture, we will talk about something called Infinite Series. So, what is the infinite series, suppose I have real numbers a 1, a 2, a 3 and so on, it is an infinite collection of numbers. And I want to talk about expression of the form a 1 plus a 2 plus a 3 and so on; that means, I am not stopping, this all these I want to sum up. So, this gives the expression of this form summation n from 1 to infinity a n this is called an infinite series.

Since there something called the infinite series there should be something called finite series also, but usually we do not use the term series for finite things. So, I can also look at something like summation n from 1 to k a n, the meaning of this very clear. What you mean, that is this is a 1 plus a 2 plus so on up to a k, this has a well defined meaning I have k mean a real number and summing them with each other.

But, when you go to the analogues thing for infinitely many real numbers, then the question makes sense that where do exactly mean by this expression written here. Because this is going to be an infinite process, because I am not stopping, you see this sum is going on and on and on. But then finally, do you land up with something, when you go on summing it up.

So, naturally there is a question of well defined (( )) of this kind of sum, so what do you mean, by that the sum summation n from 1 to infinity a n makes sense. So, the question is summation n from 1 to infinity a n, when is this sensible, that is what is the meaning, for example, I can start looking at an is equal to 1 for all n. Notice that if I look at summation n from 1 to k a n, what I get is, 1 plus 1 plus 1 plus 1 this is k times and what I get is k, so this has a well defined meaning.

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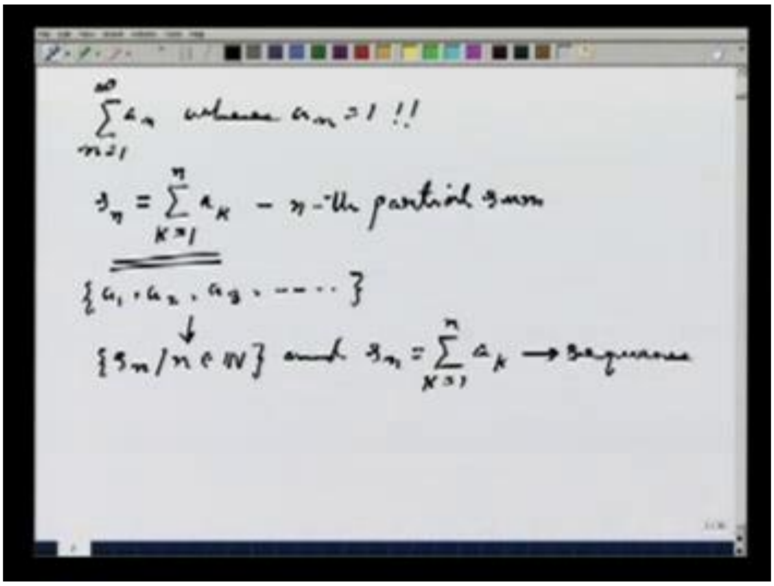

But then, what is the meaning of summation n from 1 to infinity a n, where a n is equal to 1, what should this mean, now intuitively you can see that what is going to happen is 1 is added to getting 1, 1 producing 2. Then again 2 is being added to 1 producing 3 and so on, it goes to produce 4 5 6 7 8 9 and it goes on. So, you can intuitively feel that perhaps this sum, if at all makes sense will produce something like infinity which you do not want to really to deal with. So, you want to know that is there any sensible way of defining summation n is equal to 1 infinity a n. So, that certain choice of a n it ultimately give me a real number, that we will call the infinite sum of this numbers a n. So, we start with definition, but for that I need some notation first.

So, we define s of n to be equal to summation k from 1 to n a k. So, I look at the numbers a 1 a 2 a n which is given to me, which is suppose to sum up infinitely many times i fix any n. Then, I look at the sum of first n terms that is called s n which I have written here, so this is called n th partial sum.

Notice that finally then from the numbers a n which I have started with a 1, a 2, a 3 and so on, this was given from this. I have produced s n where n is natural number and s n is equal to summation k from 1 to n a k, which is a sequence, this is a sequence. Then what I can talk about is the limit of the sequence if at all it exists, s n is just an ordinary sequence of real numbers. I will say that the sum makes sense the sequence s n converges to some number s.

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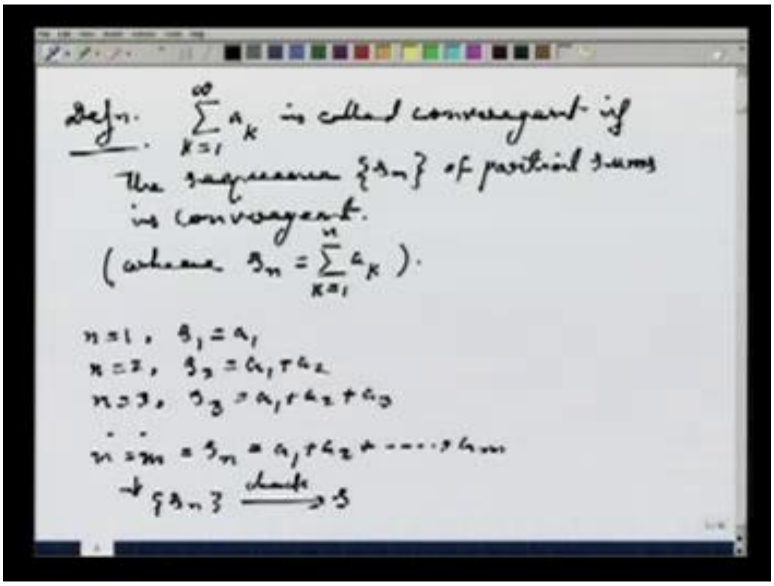

So, now I come to the definition, summation k from 1 to infinity a k is called convergent, if the sequence s n of partial sums is convergent. In other words, the definition demands that the summation n k from 1 to infinity a k is a sensible infinite sum. If the sequence or partial sums of that infinite series which we are denoting by s n is convergent. So here, I will not where s n is summation k from 1 to n a k, so how do we check. When, the infinite series makes sense very simple i fix n, then I look at the first n term of the infinite series i sum them up, I call it s n.

Now, I am varying n right, so I get a sequence now i just state whether the sequence s n converges or not. Because, given a sequence we know how to check the convergence of that sequence, whatever method we have sandwich theorem. Or whatever we will use that method or even if somehow show that the sequence of partial sum is Cauchy.

Then one of previous theorem, we know that the sequence is converges, and hence the infinite series will makes sense. So, the process is very simple I repeat once, again that given an infinite series summation k from 1 to infinity, what you do is, first you look at for n is equal to 1, what is s 1? s 1 is equal to a 1. Now, you fix n is equal to 2, then what is s 2, it a 1 plus a 2 fix n is equal to 3, then what is s 3, it is a 1 plus a 2 plus a 3. I go on doing this, if n is equal to m, then s m is a 1 plus a 2 and so on up to a m. So, finally this produces the sequence s n. I just want to check whether this converges to some limit s, if it does then we say that the infinite series converges. Now let us see by this definition does any infinite series at all converges.

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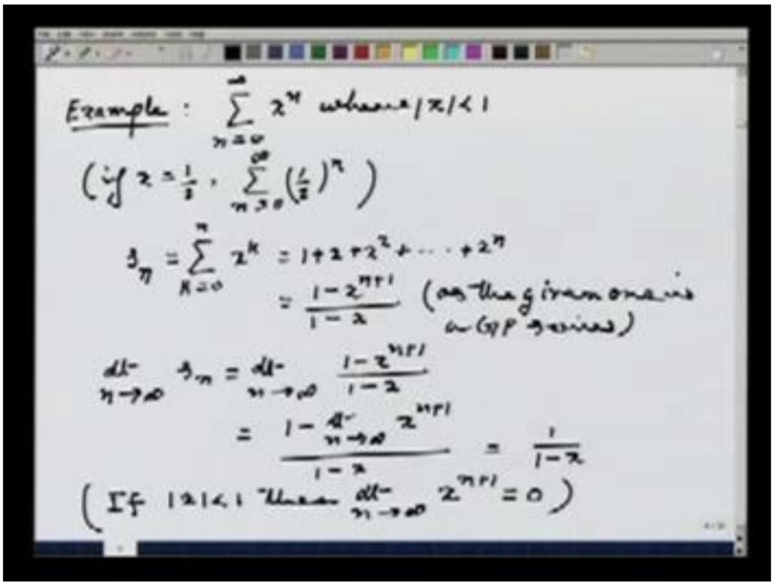

We start with this infinite series, this is our first example. Let us look at the summation n from 0 infinity, x to the power n where modulus of x is strictly less than 1.Now, this is an infinite series x is a fixed number, x is not varying in particular, if you like I can choose x to be equal to half, I can choose x to be equal to one-third.

So, in particular if x is equal to half I am looking at the series summation n from 0 to infinity, half to the power n. So, that makes sense I want to see whether this infinite series converges at all by the way you have defined the convergence of the infinite series. So, what we have to do for that well we have to look at the sequence of partial sums and try to see whether they makes sense.

So, I define s n is equal to summation k from 0 to n, x to the power k, so this is actually 1 plus x plus x square up to x to the power n. Now, this being a GP series, I know how to calculate this sum, I know that this is simply 1 minus x to the power n plus 1 divided by 1 minus x, as the given one is a GP series.

Now, I have to check, what happens, if I take the limits as n going to infinity of s n; that means, I am looking at limit n going to infinity 1 minus x to the power n plus 1 divided by 1 minus x. Now, x as nothing to do with n, so I get this is 1 minus limit n going to infinity x to the power n plus 1 divided by 1 minus x.

Now notice, from our knowledge of sequence that we know that this fact is true if mod x is less than 1, then limit n going to infinity x to the power n plus 1 is actually is equal to 0. So by this I get that this is 1 by 1 minus x; that means, the infinite series summation n from 1 to infinity x to the power n which we are going to look at this infinite series actually converges by using the definition of convergence this infinite series, there exist a particular definition. So, I am just checking that this infinite series satisfies the criteria convergence which we have defined.

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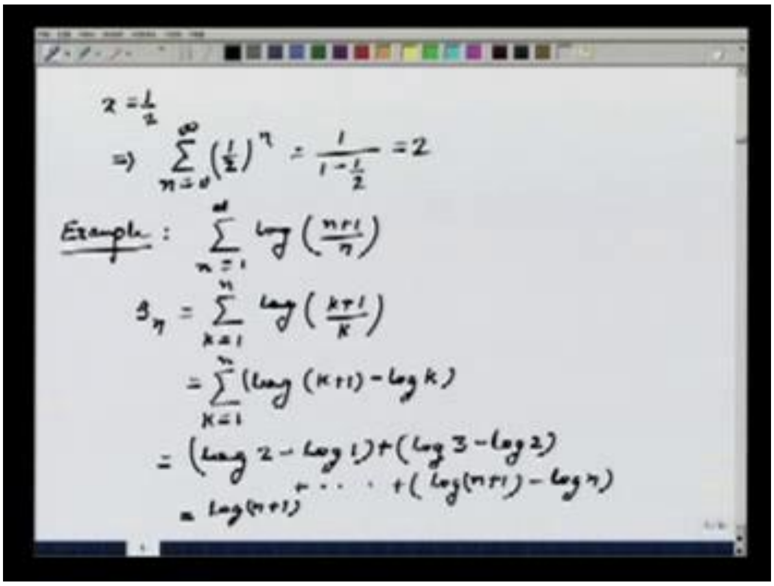

So, in particular if I choose x is equal to half it then implies that summation n from 0 to infinity, half to the power n that is equal to 1 by 1 minus half that is equal to 2. Now, let us see some more examples, let us look at this infinite series summation n from 1 to infinity log of n plus 1 divided by n I want to check whether this infinite series converges or not.

So, I define s n summation k from 1 to n log of k plus 1 by k this I can certainly write as summation k from 1 to n log of k plus 1 minus log of k. If I open it up write down, we immediately see some cancelation happening, that is if I put k is equal to 1 plus k is equal to 2 log 3 minus log 2 plus and so on, then the last term is log n plus 1 minus log n. Now, if I look at the cancellation, what I am going to get is log n plus 1 minus log 1 which is 0.

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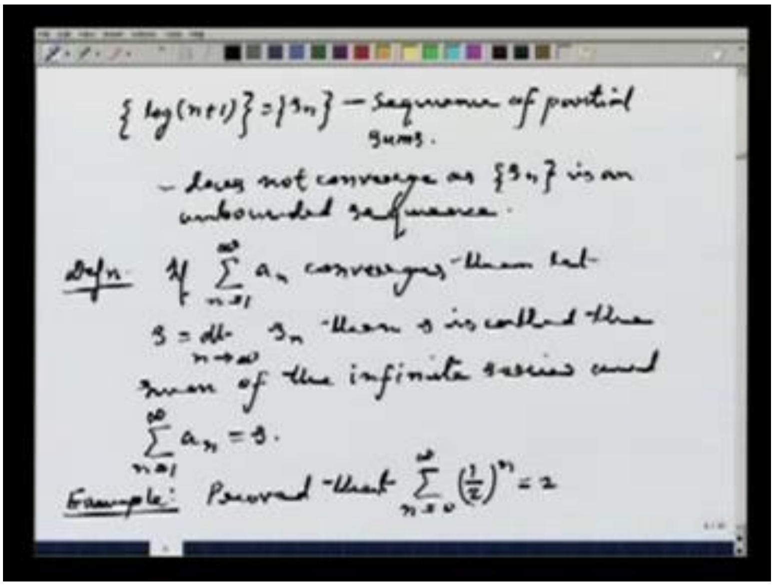

Now if I want the infinite series to converge, then I must have my sequence of partial sum s n to converge. But here, what is the sequence of partial sums, s n. So finally, the sequence of partial sums which I got the sequence log n plus 1, this is my sequence s n, this is the sequence of partial sums. Question is does the sequence converge, well we again know from the knowledge of sequences that if a sequence converges that it is certainly bounded. No unbounded sequence can converge, but look at the sequence log n plus 1, this is certainly a unbounded sequence and hence it does not converge. So, this does not converge as s n is an unbounded sequence, this infinite series does not make sense.

Now, we are going to define again, so it is definition if summation n from 1 to infinity a n converges. Then let s be the limit of the partial sums, if the infinite series converges;

that means, the sequence of partial sums converges and since the sequence of partial sums converges it converge to the unique real number which called s, then s is called the sum of the infinite series.

And just, we write it this form that n from 1 to infinity a n just we define to be s, so in this light we have already proved that summation n from 0 to infinity 1 by 2 whole to the power n is actually is equal to 2, because we found out that the partial sums which we have constructed use GP series explicitly evaluate it took limit and found that limit is actually 2. So, by the previous definition the sum of the infinite series is 2.

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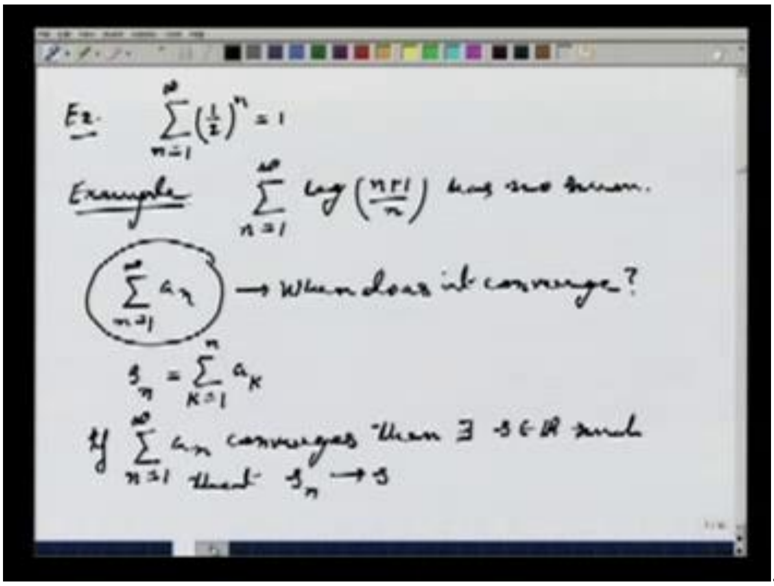

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Then, suggest the following easy exercise you should check it yourself that summation n from 1 to infinity half to the power n is equal to 1, check it yourself which should be true. But, I want to check this in the following way, find first the partial sums calculate the partial sums if you can and take the limit, so that the limit is 1.

So, the powers of half if you go on adding up itself, what it produces, is 1 in the same light if I look at this example, this is a bad infinite series summation n from 1 to infinity log n plus 1 by n has no sum, because it does not converge. So, we can talk about sum of a infinite series if and only if it converges.

So, the next question is, which infinite series, actually converges, how do you understand that, looking at the coefficient appearing the infinite series; that means just I will look at this expression summation n from 1 to infinity a n, I will just look at the this expression. Then, from this a ns I like to conclude whether it converge or it does not, when does it converge? Can we find it out from the behavior of the a ns that is the question the element a n which appear is there property of those a ns which tell me the infinite series converge. So, for that again we look at the partial that is s n that is summation k from 1 to n a k. If summation n from 1 to infinity a n converges, then there exists s a real number such that s n converges to s now comes the following trick.

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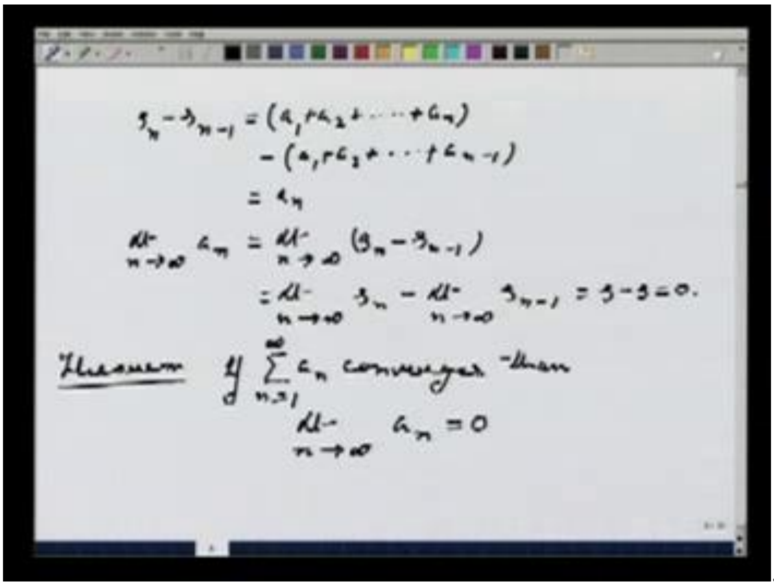

Notice that s n minus s n minus 1, what is this, if I write down the quantity a 1 plus a 2 it goes up to a n. From this I subtract a 1 plus a 2 plus a n minus 1, and just I have written down the definition of the partial sums, then everything gets cancel only what remains is, a n. Then what is limit n going to infinity a n, this is same as limit n going to infinity s n minus s n minus 1 which I can certainly write as limit n going to infinity s n minus limit n going to infinity s n minus 1, because both the limits exists which is s minus s that is 0.

So, we get the following theorem as an interesting criteria, if summation n from 1 to infinity a n converges, then limit n going to infinity a n is 0, so you take an arbitrary sequence a n and try to sum them up it will not out. At least if you want to make sense of it you have take sequences which converge to 0, otherwise there is no way that the infinite series makes sense.

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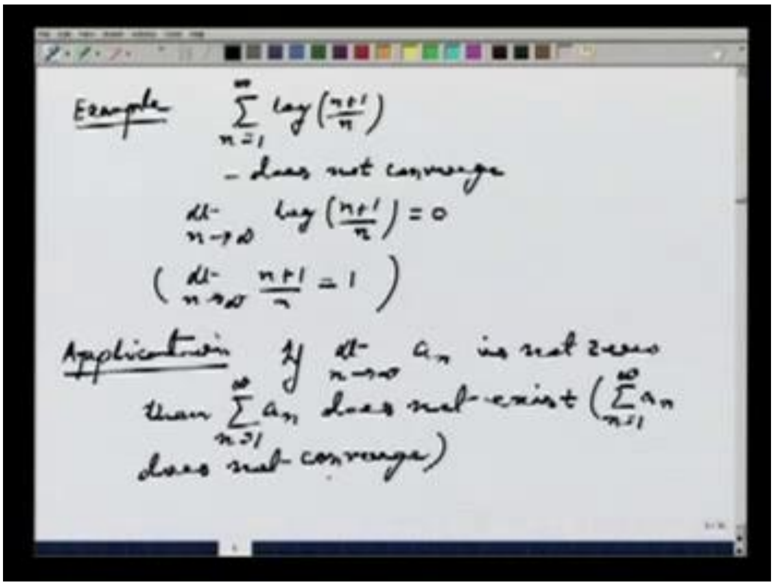

Let us look the bad example summation n from 1 to infinity log of n plus 1 by n look at this, I know this infinite series does not converge that we have proved. But, what is limit n going to infinity, log of n plus 1 by n. Notice that limit n going to infinity n plus 1 by n that is 1; that means, if I take log, then this limit is 0, what does it say, that the infinite series summation 1 to infinity log n plus 1 by n, this series has the property that its terms converges to 0 as n goes to infinity. But still, the sum does not make sense. Does this contradicts, the previous theorem which we have proved let us look back at the previous theorem, what does this say, it says if the infinite series summation a n converges. Then, the term converges to 0 as n goes to infinity it does not say if the terms converge to 0.

Then, the sequence the infinite series converges, so this theorem which we have proved that convergence of infinite series implies an converges to 0, in which way this result can actually we used this theorem actually be used in proving the fact that certain sequence certain infinite series does not converge. You just show that the term of the infinite series does not converge to 0, you just show that that will immediately imply by this theorem that the infinite series does not converge.

So, how to use this result is this, so the typical application of the result is the following. If limit n going to infinity a n is not 0, then summation n from n is equal to 1 to infinity a n does not exist. In other words, summation n from 1 to infinity a n does not converge, so in the technical language that the terms of the infinite series is going to 0 is a necessary criteria for the convergence of the infinite series, but it is no way sufficient.

Now, we come to some important infinite series which will encounter time and again and then try to device the producer by which we can test whether the infinite series converge or not. Because, we want to produce example of many infinite series and there should be some anti rule by which one should able to check whether converge. For this, we need to study some infinite series which converge and then deduce information about other infinite series from this infinite series.

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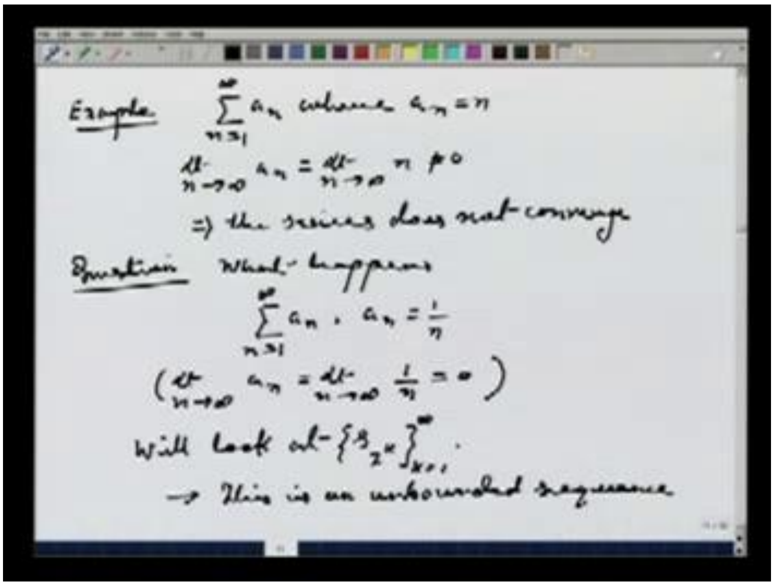

Now, we are going to look at some fundamental infinite series which will keep appearing time and again. First let us, look at this infinite series summation n from 1 to infinity a n where a n is equal to n. Now you go by the direct method of finding the partial sums or by the previous criteria you will get the same result that this infinite series does not make sense it does not converge. Simply because limit n going to infinity a n that is limit n going to infinity n this is certainly not equal to 0, so this implies the does not converge. But question is, what happens, to this sum summation n from 1 to infinity a n where a n is equal to 1 by n, notice that limit n going to infinity a n that is equal to limit n going to infinity 1 by n which is equal to 0.

But we know that this condition does not guarantee that the infinite series convergent, all we can say is well there is some hope this infinite series might converge to the at least the terms are converging to 0. But, the question still remains does the infinite series converge, we will see the answer is in the negative that this infinite series does not converge.

So, that is the thing we are going to show now, so what we do is, we start looking at the partial sums. Now, if I can somehow show that the partial sums that forms an unbounded sequence, then we can certainly show conclude the infinite series does not converge. Because, the infinite series converge the partial sums the sequence of partial sums has to converge; that means, at the least that the sequence of partial sums must be bounded sequence.

Now to show, that the sequences of partial sums that is the a n unbounded sequence, what I will do is, I will concentrate on a sub-sequence of the sequence of partial sums and show that sub-sequence unbounded that will certainly imply that the whole sequence is also an unbounded.

So, the trick involved here is to look at which sub-sequence, we will consider that will work for us. Will, we look at s 2 the power k that is for each k. I will look at the sum of 2 to the power k many times, that is certainly the sub-sequence of the whole sequence of partial sums and I am trying to prove that this sequence will prove that this is an unbounded sequence.

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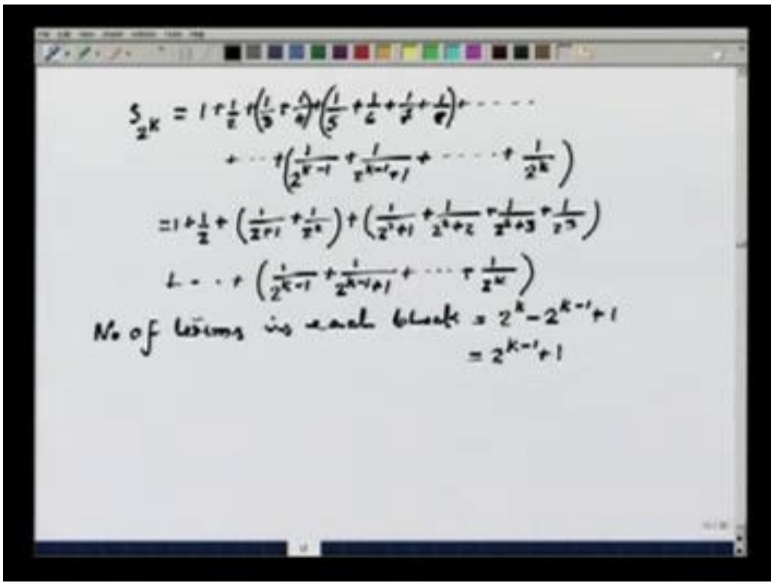

So, let us see how to do that, so I first start writing down, what is s 2 to the power k, this is 1 plus 1 by 2 plus one-third plus one-fourth plus one-fifth plus one-sixth plus one-seventh plus one-eighth plus goes on plus 1 by 2 to the power k minus 1 plus 1 by 2 to the power k plus 1 plus so on, and up to 1 by 2 to the power k.

Here, what I do is, I regroup the terms that is this is my first regrouping I take one-third and one-fourth, then I take one-fifth and up one-eighth. So, what is the grouping, from one-third to one-fourth, what I am doing is, I am collecting two terms, then from one-fifth , I go up to 1 by 2 to the power 3, if I write down in this language it is 1 plus half plus, here what I have is 1 by 2 plus 1 that is one-third plus 1 by 2 square, then what is one-fifth, that is 1 by 2 square plus 1 plus 1 by 2 square plus 2 plus 1 by 2 square plus 3 plus 1 by 2 to the power 3, it goes on like this plus. Then, the last term you see the pattern here this 2 the power k minus 1 plus 1 by 2 to the power k minus plus 1 and 1 by 2 to the power k. Now, I can easily calculate in each block, how many terms are there, so the number of terms each block that is 2 to the power k minus 2 to the power k minus 1 plus 1. So, how many terms are here, this is 2 to the power k minus 1 plus 1.

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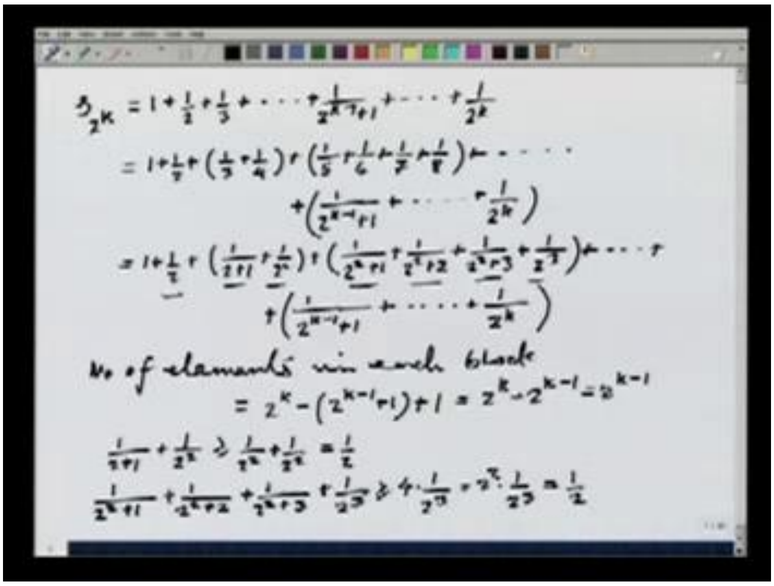

Let us write down, what is s 2 to the power k, that is 1 plus half plus one-third and so on, 1 by 2 to the power k minus 1 plus 1 and so on up to 1 by 2 to the power k, what I am going to do now is, I am going to group terms in the following way. I first write 1 plus half, then plus one-third plus one-fourth this is one block, then comes the next block which is one-fifth plus 1 by 6 plus 1 by 7 up to 1 by 8. Then, starts the next block it goes on like that then finally, I have 1 by 2 to the power k minus 1 plus 1 plus up to 1 by 2 to the power k, I can also write in the following form that is 1 plus half plus 1 by 2 plus 1 that is one-third plus 1 by 2 square. Then the next block with 1 fifth which is 1 by 2 square plus 1 plus 1 by 2 square plus 2 plus 1 by 2 square plus 3 plus 1 by 2 to the power 4 which is one-eighth, it goes on like this then comes the last block which is 1 by 2 to the power k minus 1 plus 1 plus up to 1 by 2 to the power k.

Now, the question is in each block, how many terms are there? Well here instead of 1 by 2 to the power 4 I have actually 2 to the power 3 which is 8. Now in each block, how many terms are there, if I want to count, what I need to look at is, so number of elements in each block is equal to 2 the power k minus 2 to the power k minus 1 plus 1 then plus 1 that is the total number of elements that is it is 2 to the power k minus 2 to the power k minus 1 which is 2 to the power k minus 1.

So, if I take the case for example, k is equal to 3, then I should have four terms which precisely the case here 1 2 3 4, if I take k to be equal to 2, then I should have 2 terms which is the case. Now, the question is how many blocks are here? Well that is very easy to calculate, that my first block is here, this is my second block this is my third block, so there are actually k blocks.

Now, look at the each of the blocks for example, 1 by 2 plus 1 plus 1 by 2 square here, this is bigger than or equal to 1 by 2 square plus 1 by 2 square that is half. Then, what is the next term, that is 1 by 2 square plus 1 by 2 square plus 2 plus 1 by 2 square plus 3 plus 1 by 2 to the power 3, what is the smallest possible term here? The smallest possible term here is 1 by 2 to the power 3. Each of the term are bigger than 1 by 2 to the power 3 and there are total four terms it is bigger than or equal to 4 times 1 by 2 the power 3 that is 2 square into 1 by 2 cube again half.

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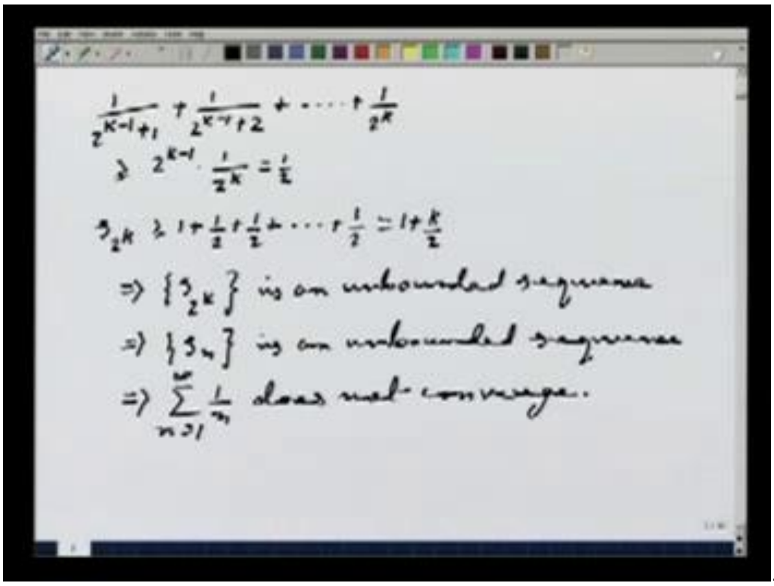

So, if you look at the last term of the example, I have 1 by 2 to the power k minus 1 plus 1 plus 1 by 2 to the power k minus 1 plus 2 it goes up to 1 by 2 to the power k which is the smallest term here. So, it is bigger than or equal to, and since I know that there are 2 to the power of k minus 1 terms which is bigger than 2 to the power k minus 1 times 1 by 2 to the power k which is half.

As a result, I get that s 2 k 2 to the power k bigger than or equal to 1 plus half plus half how many half are going to come; that means, how many blocks are there, but I already counted there are k blocks starting with the first block there is only one element half. So, get half k many times, so what I get is 1 plus k by 2.

So, s 2 to the power k is bigger than or equal to 1 plus k by 2, this is immediately implies, that the sequence 2 to the power k is an unbounded sequence. This certainly implies the sequence of partial sums s n is a n un bounded sequence which certainly implies that summation n from 1 to infinity 1 by n does not converge.

So, this is an example of an infinite series where the terms are converging to 0, but the whole sequence does not converge. Now we are going to try to get some necessary as well as sufficient condition for convergence of an infinite series which is certainly be of theoretical nature. But, to develop practical tool in testing whether an infinite series converges or not this is going to be very important.

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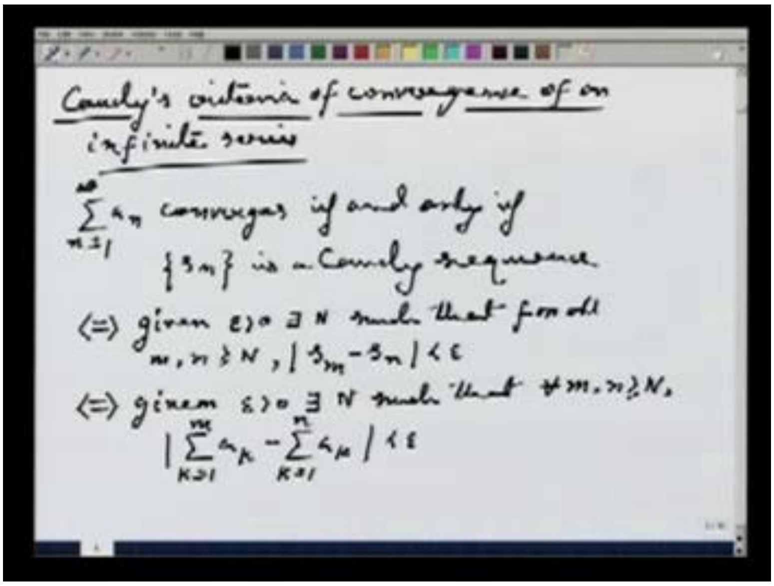

So, this is Cauchy criteria of convergence of an infinite series, this criteria actually rests on Cauchy criteria on convergence of a sequence and hence the name the idea is very simple. You have an infinite series summation a n, I want to find a necessary and sufficient criteria for the convergence of this infinite series. Now, by the definition this infinite series converges if and only if the sequence of partial sums s n converges. Now, the sequence of partial sums will converge if and only if it is a Cauchy sequence this is precisely the idea. So, I will just write that summation from 1 to infinity a n converges if and only if the sequence s n of partial sums is a Cauchy sequence.

Now, this implies and implied by the definition of Cauchy sequence that is given epsilon bigger than 0. There, exists a N certainly depending on epsilon, such that for all m n bigger than or equal to N mod of s n minus s m is less than epsilon. What I have written just is the definition of Cauchy sequence, this just the fact that s n is a Cauchy sequence.

But, what does it actually mean, I can assume that m is bigger than n of if you like n can also be bigger than m, let me assume m is bigger than n. Then, this is if and only if given epsilon bigger than 0. There exist N such that for all m n bigger than or equal to N modulus summation k from 1 to m a k minus summation k from 1 to n a k is less than epsilon.

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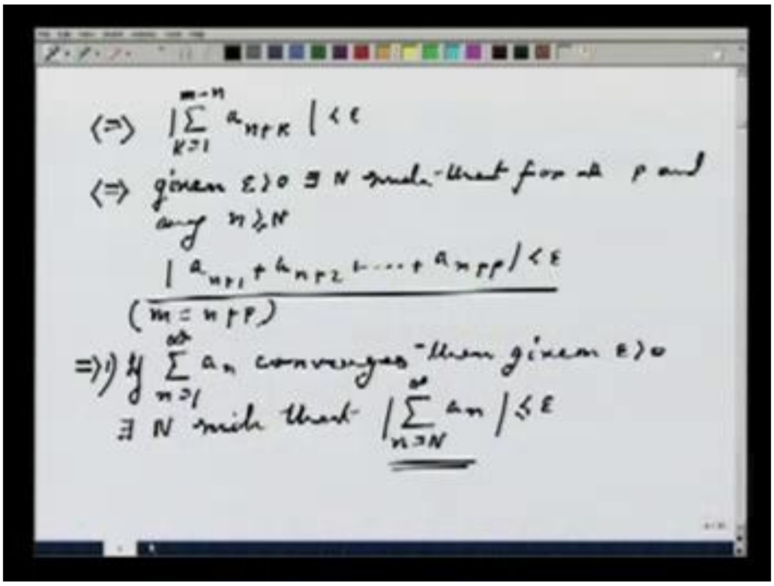

But, if you open up this summation and cancel terms, this is implies and implied by modulus summation from 1 to m minus n a n plus k is less than epsilon. This then, if and only if I can just write in the following form that given epsilon bigger than 0. There, exist N, such that for any p any n bigger than or equal to n modulus of a n plus 1 plus a n plus 2 up to a n plus p is less than epsilon. I would actually like to say for all p.

So, what I have written down here is the case m is equal to n plus p, in the previous step if you put m is equal to n plus p, then what you get is precisely this quantity. So, this then implies actually this gives me some information about convergence, that is if summation n from 1 to infinity a n converges.

Then given epsilon bigger than 0, there exist N, such that modulus of this infinite series, this quantity is less than epsilon. Why this is the case, because if you look at the previous quantity written here, this is nothing but partial sums of the infinite series which I have written here. Since, the partial sums that is less than epsilon then the whole sum since it is already exist the infinite series is convergent. So, this sum is also less than epsilon, so this is called the tale of the infinite series if an infinite series converges then it is tale has to be small.

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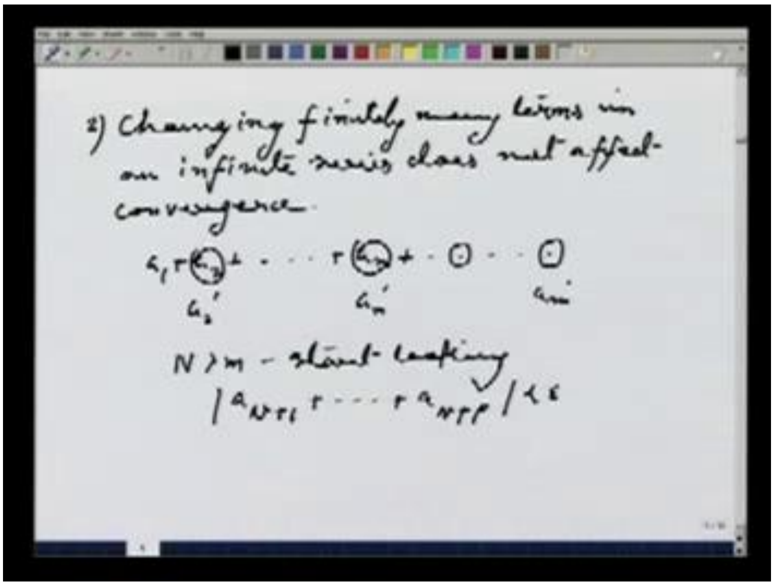

One more thing that comes from this is observation one. The second observation is that the changing finitely many terms in an infinite series does not affect convergence just like sequences that is very simple. Suppose, I have this infinite series a 1 a 2 a n and so on. Now, at finitely many stages, let us say this a 2 changed a n is changed, some terms here is changed and here is changed to some a 2 prime a n prime.

Suppose, I change this term it does not affect the behavior of convergence at all, it is not going to show up in the tale. Because, what did I say? I said given epsilon bigger than 0 there exist N such that after that term if I look at sum of finitely many terms that is less than epsilon that is the necessary and sufficient criteria for convergence of an infinite series, there exist a stage is the key here.

I will choose my stage or at least I will search for that stage, after this finitely many points or finitely many numbers which I have checked. For example, this m is the maximum 1 I will start looking for N which is strictly bigger than m, I will start looking for that. And then check whether summation a N plus 1 plus a N plus p whether that is less than epsilon you see.

So, the previous terms which I have changed does not matter at all, so that is the important thing, which we have learnt that if I have an infinite series and I change only finitely, many terms of the that infinite series it does not affect behavior of the convergence of the infinite series at all. So in the next lecture, we well elaborate on the Cauchy criteria, and try to find more examples of infinite series which converges.