Mathematics-I
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Lecture - 14
Infinite Series II

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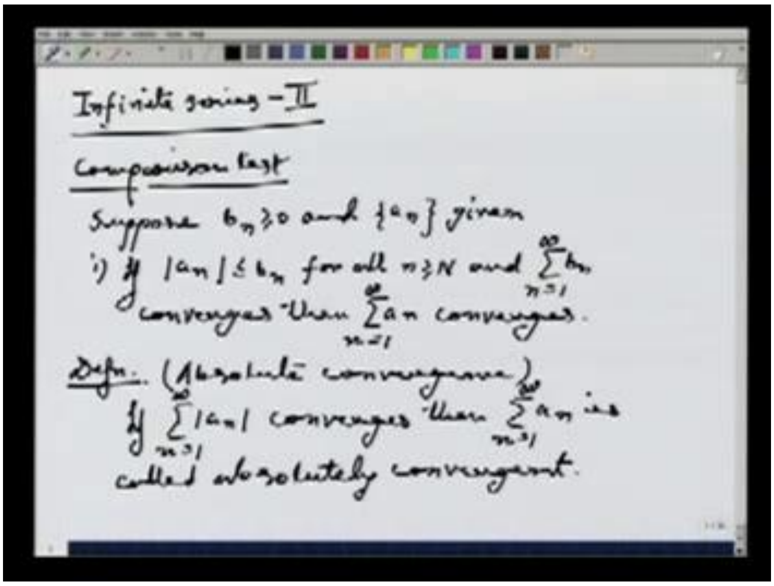

Today, we are going to continue with Infinite Series. The intention is to see a large class of example of convergent infinite series. So for that we need something called comparison test. So, we will start first today's lecture with comparison test. The idea is very simple, the idea is the given the series, which converges how to find another convergent series using this information that given series is convergent. That means, we like to compare, so the precise statement is this.

Let suppose b n is bigger than or equal to 0 and a n is given. So, I have just given two sequences one is b n and every term of the sequence b n is non-negative and a n is a given sequence. Then if mod a n is lesser equal to b n for all bigger than or equal to some capital N. and summation n from 1 to infinity b n converges, then summation n from 1 to infinity a n is also converges.

Now, the proof of this is very simple, but before I come to the proof of this let me introduce one more notation called absolute convergence. So, first let me define absolute convergence this is very simple to understand that if summation mod a n, n from 1 to infinity converges, then summation n from 1 to infinity a n is called absolutely convergent. Now, the question is, what does absolutely convergent has to do with convergent well that is very simple.

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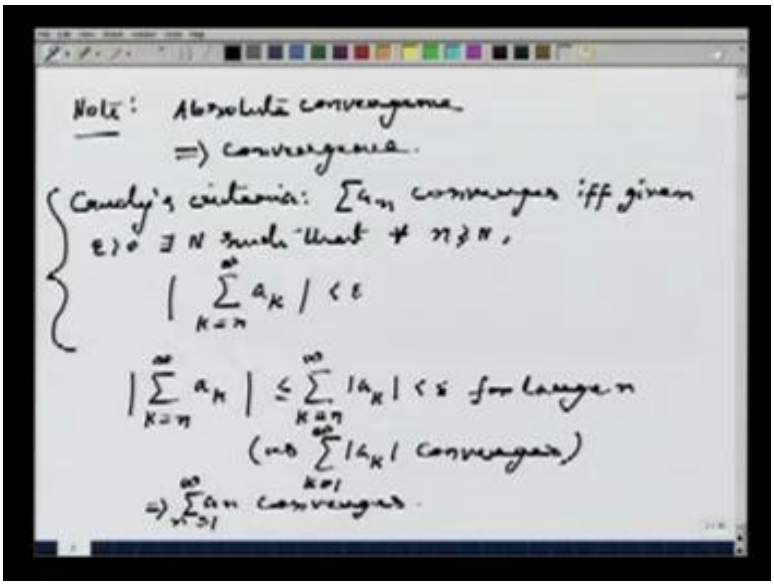

So, I will just put it as a note it says that absolute convergent implies converges, how to prove this that if I have a series, which converges absolutely then it is converges. Well first note one thing that if I have a n infinite series a n such that a n is always non-negative for all n, then the absolute convergence same as convergence there is no difference, because then summation mod a n is same as summation a n.

So, the absolute convergence is really a different kind of convergence if the terms of the series are not necessarily non-negative. Nevertheless it turns out if the series converges absolutely that is absolutely convergence, then the series converges, well the proof is very simple, if you use Cauchy's criteria of convergence.

So, let us note again, what is Cauchy criteria of convergence, so Cauchy's criteria in the last lecture we have described we just says that summation a n converges, if and only if given epsilon bigger than 0 there exist capital N, such that for all n bigger than or equal to capital N k from n to infinity a k this mode is less than epsilon; that means, the tale after some stage is small.

Now, if I use these criteria, then it is easy to see absolute convergence implies convergence. So, this is, what we are going to use to prove convergence, what I have to do is given epsilon I have to find capital N such that the tale of the infinite series, which starts after capital N is less than epsilon. But, notice that modulus summation k from n to infinity mod a k this quantity is always less than or equal to summation k from n to infinity mod a k. Now, I know that this can be made less than epsilon for large n as this converges.

And hence, the first quantity is also less than epsilon, then by Cauchy's criteria it follows that summation a n converges, so this is simple. So, once again, what is absolutely convergence means it means summation mod a n converges then using Cauchy's criteria we could prove that if the series is absolutely convergent, then it is convergent and to prove that what we have used it just Cauchy's criteria.

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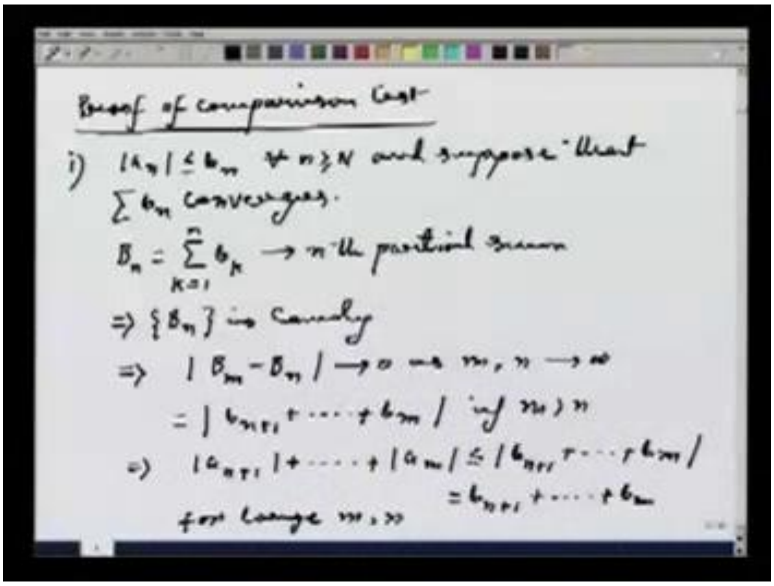

Now, let us go to the proof of comparison test, so the first one it is given that mod a n is lesser is equal to b n for all n bigger than or equal to some capital N and suppose that summation b n converges. So, they we are going to prove is very simple let us say capital B n is equal to summation k from 1 to n b k, so this is nth partial sum.

Now, this would then imply this sequence b n is Cauchy as my assumption is summation b n converges; that means, the sequence of partial sums is a Cauchy sequence this would, then imply that modulus of b m minus b n goes to 0 as m n goes to infinity. Now, notice that this quantity is nothing but modulus b n plus 1 up to b n if m bigger than n if m is less than n, then the role of m and n are getting interchanged.

But, then notice this also implies that mod a n plus 1 plus mod am that is less than or equal to this actually happening, because I am assuming all the b m are non-negative the modulus is not actually require this quantity actually is b n plus 1 up to b n. Now, I know that mod a n is lesser is equal to b n all large possible n after a stage capital N.

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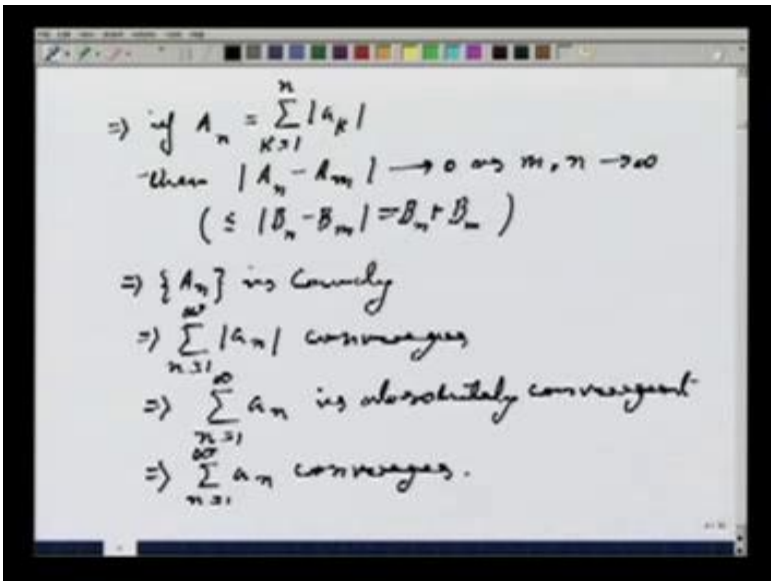

So, I can say that this is true for large m n; that means, if m n goes to infinity, then modules of this would imply if capital A n is equal to summation k from 1 to n mod a k, then modulus of a n minus am goes to 0 as m n goes to infinity. Because, this quantity is dominated by modulus of b n minus b m, which actually b n minus b m, if I assume m to be bigger than 0, then it is plus b m minus b n.

Now, that quantity goes to 0, because summation b n converges; that means, the nth partial sum forms a Cauchy sequence. So, modulus of b m minus b n goes to 0 and since modulus of am minus a n is dominated by those things they also go to 0, so this implies that the sequence a n is Cauchy this implies summation n from 1 to infinity modulus a n converges this implies summation n from 1 to infinity a n is absolutely convergent by our definition of our absolutely convergent, but this then implies then one of our previous observation says that summation n from 1 to infinity a n converges, what it says is very simple that if you have two infinite series.

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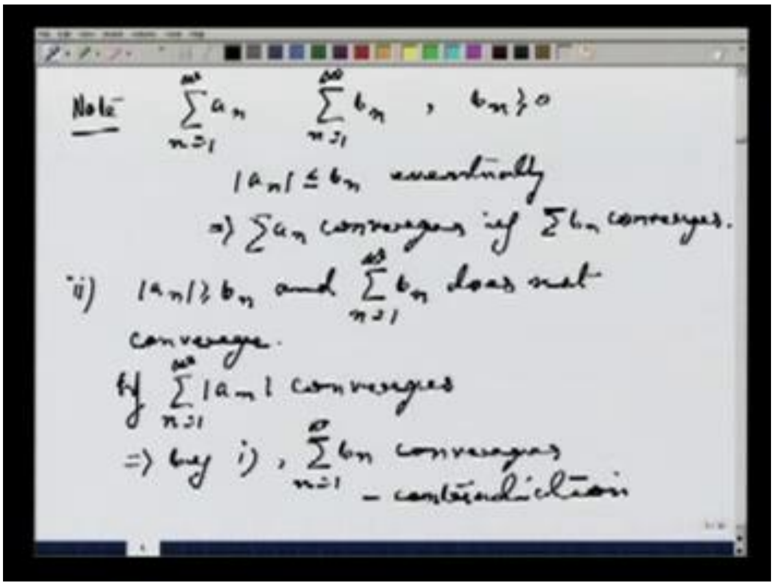

So, again I will put it as a note you have two infinite series one is summation a n n from 1 to infinity another is summation b n assume b n is non negative. If compare the terms now that is, why it is called comparison test look at modulus a n, if it is lesser equal to b n for all n or after some stage or you can say eventually this implies summation a n converges if summation b n converges. It is very simple to remember that if you have a infinite series whose terms or after some stage less than term of the convergent infinite series, then your series is also converges.

Now, let us come to the proof of part two I know that mod a n is bigger than or equal to b n and summation does not converge I want to prove that summation a n is also does not converge. Suppose on the contrary if summation n from 1 to infinity mod a n converges, then I already know that b n is lesser equal to modulus of a n and summation mod a n converges try to apply part one. If you have a n infinite series whose terms are less than or equal to term of the convergent infinite series, then the infinite series converges.

So, b n constitute a n infinite series summation b n converges and each terms lesser is equal to modulus a n and summation mod a n converges that would imply. Then by one summation b n, n from 1 to infinity converges I am applying one, but already assumed summation 1 from 1 to infinity b n does not converge that is the given fact; that means, then summation mod a n cannot converge, because if that converge that would imply that summation b n converges, but it given to me that summation b n does not converge, so this is a contradiction.

So, this contradiction tells me that it cannot happen that summation mod a n converges. So, we learnt two things that if I have two infinite series a n and b n given to me b n is a n infinite series consisting of non-negative terms, if summation b n converges and mod a n is lesser than or equal to b n, then summation mod a n is also converges. In fact, it converges absolutely on the other hand, if I know that mod a n is bigger than or equal to b n and summation b n does not converge, then summation also cannot converge. Now, we are going to use this states to produce some more examples of convergent infinite series.

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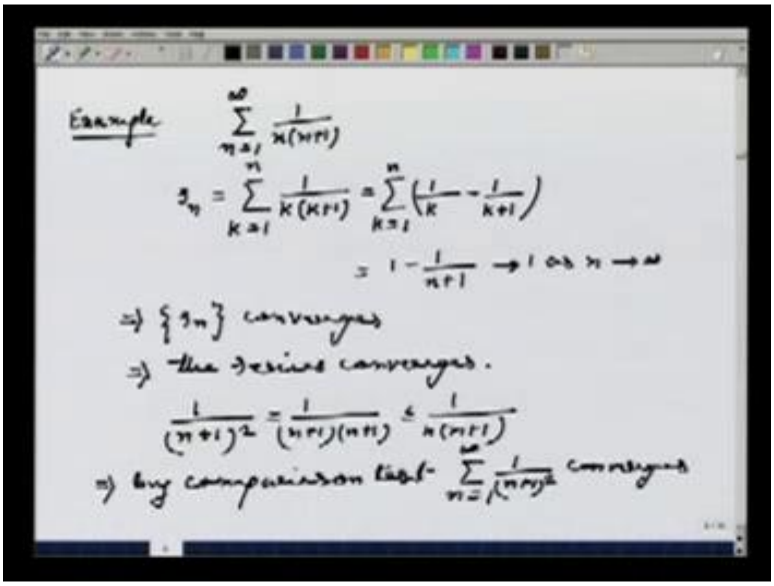

So, first look at this series n from 1 to infinity 1 by n times n plus 1 question is whether infinite series is a n convergent infinite series or it is not. So, what we do is we try to calculate the partial sum and see what happens, so what is s n s n is summation k from 1 to n 1 by k into k plus 1 this I can write as summation k from 1 to n 1 by k minus 1 by k plus 1 this is, so called telescopic sum.

So, if I open up the summation and write down each and every term then cancellation takes place, what I have left with this 1 minus 1 by n plus 1. So, this goes to 1 as n goes to infinity, this implies that the sequence s n converges this implies then, by the definition that the series converges. Now, notice that 1 by n plus 1 whole square, if I look at that is 1 by n plus 1 into n plus 1, which is less equal to 1 by n n plus 1, but I have already noticed that summation n from 1 to infinity 1 by n times n plus 1 that converges. So, this implies by comparison test summation n from 1 to infinity 1 by n plus 1 whole square converges, and this is same as saying certainly that summation n from 1 to infinity 1 by n square converges.

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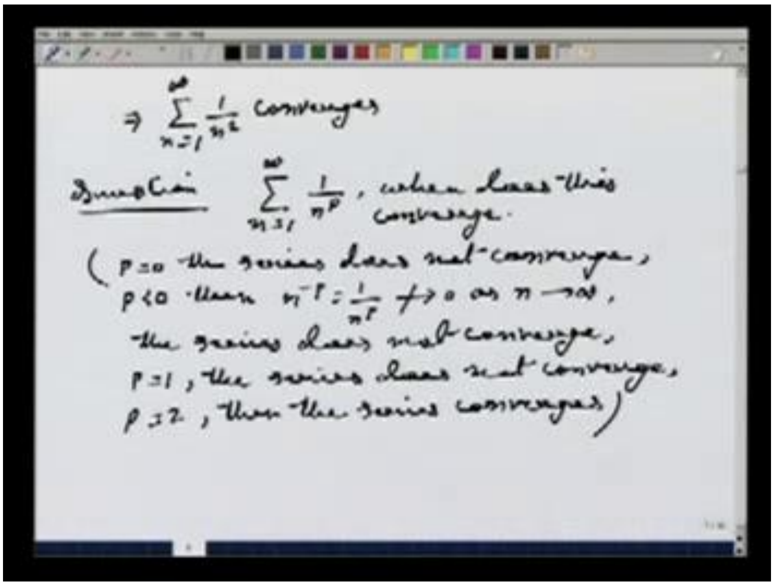

The difference is in the previous case, if I look at the first term here is, 1 by 2 square and the first term here is 1, but adding only the term in a convergent infinite series does not change the behavior of the convergence of the infinite series at all that we have observed. So, since the previous series converges that is summation n from 1 to infinity 1 by n plus 1 whole square, so does the series.

Notice that summation n from 1 to infinity 1 by n that does not converge that we have seen in the previous lecture, but instead of 1 by n if I have 1 by n square, then it is converges. So, then the question arises if I look at this infinite series summation n from 1 to infinity 1 by n to the power p when does this converge, certain cases are easy to guess suppose p is equal to 0, then you get the constant terms that is always 1 and then terms go to 0.

So, for p is equal to 0 the series does not converge and the same observation will be true if p is less than 0 if p is a negative number, then 1 by n to the power p actually mean n to the power minus p where p is positive, but then n to the power minus p that is1 by n to the power p it does not go to 0 as n goes to infinity. So, the series does not converge and we have already observed that if p is equal to 1 the series does not converge and if p is equal to 2, then the series converges.

But what about other values of p that is the question we are interested in studying that is, what are the precise values of p for which summation n from 1 to infinity 1 by n to the power p converges. Now, for this we need another test which is very fundamental in theory of infinite series most of the time it will help you it is called Cauchy condensation test.

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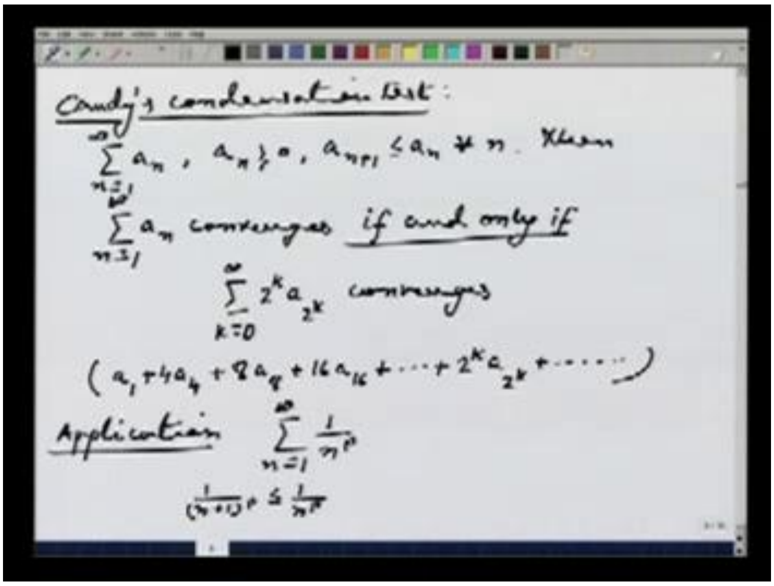

So, suppose I have a n infinite series a n, where a n is non-negative and they are decreasing that is a n plus 1 is lesser equal to a n for all n, notice that this conditions are modeled after summation 1 by n to the power p 1 by n to the power p is always non-negative, if I am taking natural numbers as n increases for a fix p 1 by n plus 1 to the power p is always less equal to 1 by n to the power p.

So, we have a sequence of numbers 1 by n to the power p, which is decreasing, so we are just putting this condition here, then the test states the following summation n from 1 to infinity a n converges if and only if, so this is important summation k from 1 to infinity 2 to the power k a 2 to the power k converges well in this I would start from k is equal to 0 not that it matters really; that means, the series need to look at is first I put k is equal to 0, then I get a 1, then I put k is equal to 2; that means, plus 4 a 4, then I have to put k is equal to 3; that means, 8 a 8 plus 16 a 16 and so on, this is the series I need to look at to understand the convergence of summation a n.

Now, this new series look much more complicated than the original series given to us. So, you might feel looking at the statement this perhaps looking at the statement more complicated, but in many cases you will see that because of the 2 to the power of k factor throw in with the term a 2 to the power k actually makes matter easy for us well.

Let us first try to use this result in certain cases and then come to the proof of it, I say that I can use this result to prove the deal with the question I was posing what happens to summation 1 by n to the power p I am going to use this test for that series. So, first I am going to look at, so this is application first I am going look at summation n from 1 to infinity 1 by n to the power p, notice that the term here are non-negative that is one thing, second this is certainly true that 1 by n plus 1 whole to the power p is lesser equal to 1 by n to the power p; that means, the terms are decreasing.

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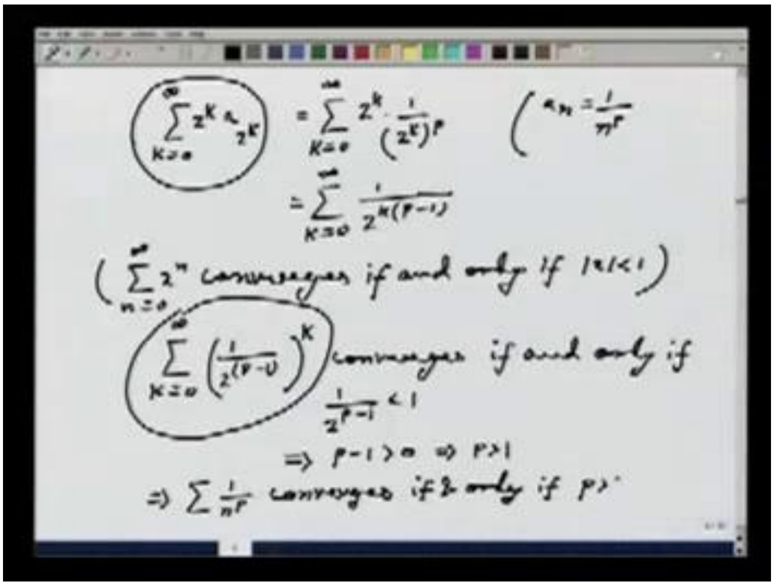

Now, I am going to use Cauchy condensation test; that means, what I have to look at I need to look at this series, now 2 to the power k a 2 to the power k, where k varies from 0 to infinity. But, I know what is a n a n is just 1 by n, so this series is same as summation k from 0 to infinity 2 to the power k times 1 by 2 to the power k whole to the power p, because a n is equal to 1 by n to the power p this implies a 2 to the power k is 1 by 2 to the power k whole to the power p; that means, this is same as summation k from

0 to infinity 1 by 2 to the power k p minus 1, do you see the advantage of using Cauchy condensation.

Because, I started with the series summation 1 by n to the power p, because of the substitution 2 to the power a 2 to the power k the problem has reduced to geometric series; that means, summation x to the power n, you know and that kind of series we know very easily, how to calculate we have already got our result that summation n from 0 to infinity x to the power n converges, if and only if mod x is less than 1 we know that that what I am going to use.

Now, because the above series actually be written as in the form summation k from 0 to infinity 1 by 2 to the power p minus 1 this whole to the power k and then by the previous observation this converges if and only if 1 by 2 to the power p minus 1 is strictly less than 1. Now, what are those p's for which this is true that this quantity is strictly less than 1 the answer is very simple that this implies p minus 1 is strictly less than 0 that is that p is strictly bigger than 1 notice that if p is equal to 1, then I get 1 I demand strictly less than 1, so p is strictly bigger than 1, so if p is strictly bigger than 1. So, if p is strictly bigger than 1, then this series this converges; that means, this converges if p is strictly bigger than 1 then Cauchy condensation test tells me summation 1 by n to the power p converges, if and only if p is bigger than 1 for all other p's the series does not converge.

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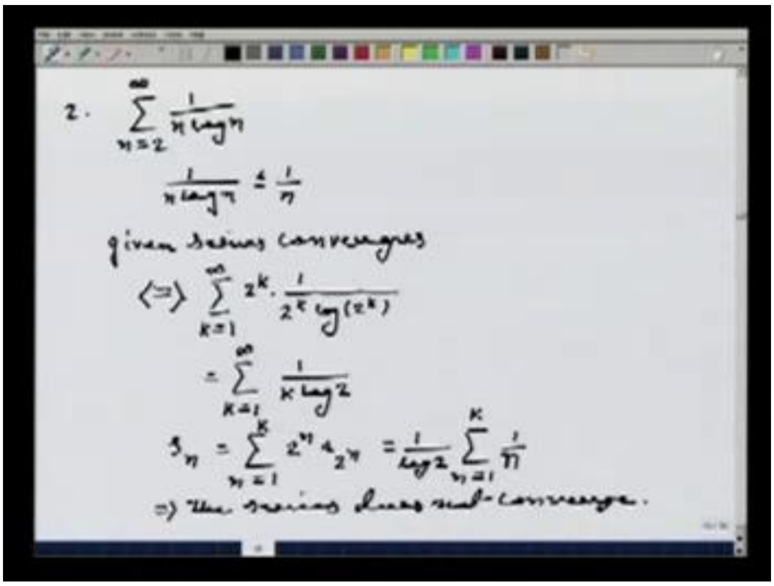

Let us look at one more example, what happens to this infinite series, summation n from 2 to infinity 1 by n log n notice that n log n grows faster than n. So, 1 by n log n certainly goes to 0, but we know that going to 0 is not enough to make the infinite series converge, but is going to 0 faster than n does that make it convergent, that is the question to check whether convergent or not if I apply the comparison test all I get is that 1 by n log n is lesser equal to 1 by n.

But, summation 1 by n diverges from that neither I can that summation 1 by n log n converges or not, because less than something convergent gives me convergent, but if it is less than some infinite series does not converge, that I cannot say whether my series converges or not. So, to tackle the problem, what we do is we use Cauchy condensation test, because the terms here are non-negative and again at the same time it is decreasing. So, Cauchy condensation test can be applied.

So, the given series converges if and only if by Cauchy condensation test summation 2 to the power k into a 2 to the power k, what is a n here, here a n is 1 by n log n, so a 2 to the power k is 1 by 2 to the power k log 2 to the power k. So, k from we will from 2 to infinity for may k from 1 to infinity we cannot take k from 0 here, now this then same as 1 by k log 2 log 2 is a constant, so any partial sum of this if I call it s n summation n from 1 to k.

Then, from our calculation we can see this is same as 1 by log 2 summation n from 1 to k 1 by k in this case 1 by n. Now, we know that this diverges, because this just partial sums of the infinite series of 1 by n and hence s n does not converge that implies the given series does not converge. So, it turns out if I put instead of n summation 1 by n log n if I just use the Cauchy condensation test using 2 to the power k a 2 to the power k it becomes the partial sums exactly like partial sums of summation 1 by n, which we know diverges, hence this also diverges.

And hence, the given series does not converge, because summation 2 to the power k a 2 to the power k does not converge well here time. And again I am using the terms diverges, it means it is a n infinite series it does not converge, let just say that a n infinite series diverges means just means that it does not converge.

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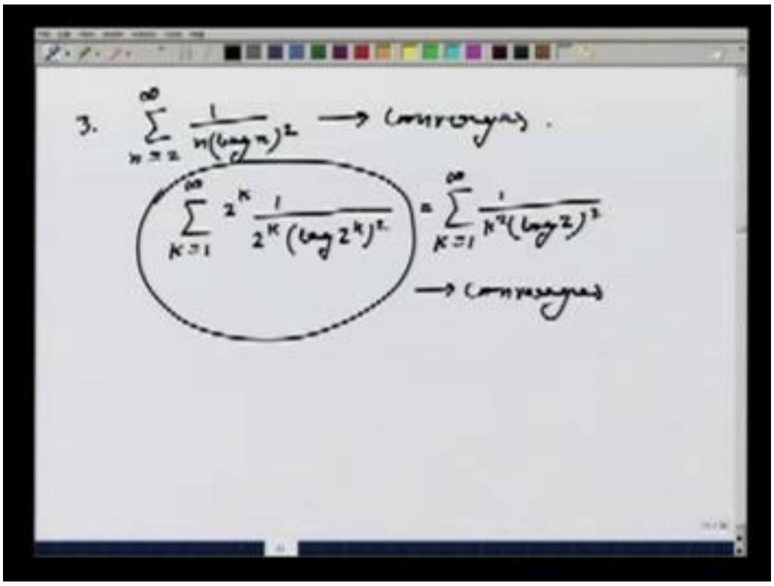

Now, let us look at the another example summation n from 2 to infinity 1 by n log n whole square. Here also I try to use Cauchy condensation test, and then let us see what happens I will get summation k from 1 to infinity 2 to the power k, then a 2 to the power k means 1 by 2 to the power k log 2 to the power k whole square all we do is instead of a n write 2 to the power k times a of 2 to the power k instead of a n just put 2 to the power k and multiply it by 2 to the power k that is the series, which appears in the Cauchy condensation test.

So, here this then same as summation k from 1 to infinity 1 by k square times log 2 whole square, but log 2 whole square is a constant. So, it is just summation of 1 by k square times 1 by log 2 square, but summation 1 by 2 square converges that we have seen because summation n to the power p converges if p is bigger than 1, here the case is p is equal to 2, so this converges; that means, this infinite series converges by Cauchy condensation test that would mean this converges. So, using Cauchy condensation test or comparison test there are many series, which we can prove they converge.

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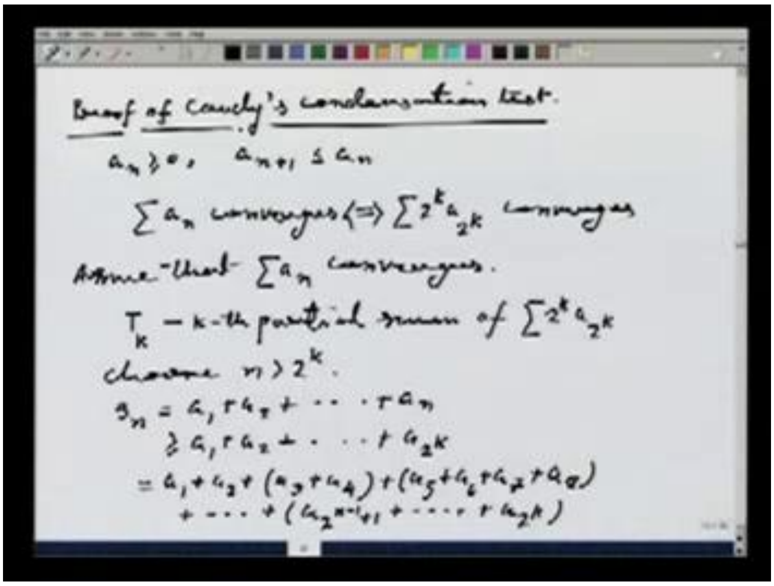

Now, let us go to the proof of Cauchy condensation test, so let us recall the statement again that a n is non-negative decreasing, then I have to prove that summation a n converges if and only if first, let us assume that summation a n converges I have to show that summation 2 to the power k a 2 to the power k converges; that means, I have to look at the partial sums and try to show that it converges.

Notice here, 1 thing since I am dealing sequence of series of non-negative terms if I look at partial sums as the n increases the terms of the sequences are going bigger and bigger. So, if I want that the sequence to converge it is enough it is bounded, so if I have a n infinite series consisting of non-negative terms to show that the infinite series converges it is enough to prove that the sequence of partial sums is a bounded sequence we will use it you see.

So, first we will look at the partial sums, let us say t k of the series this is the k th partial sums of and choose some n is bigger than 2 to the power k that I can always do given any k I look at 2 to the power k and I choose a natural number bigger than that now, what is s n? s n is a 1 plus a 2 plus, so on up to a n.

Now, since the terms are non-negative I can certainly say this is bigger than or equal to a 1 plus a 2 plus up to a 2 to the power k, because the terms here are non-negative. Now, this I write in the following form a 1 plus a 2 plus I club a 3 and a 4, then I club a 5up to a 8 I go on like this, then the last clubbing is a 2 to the power k minus 1 plus 1 up to a 2 to the power k. I can certainly count that how many terms are there in each club for example, in the first club there are 2 in the second club there are 4; that means, 2 square elements and in the last there are 2 to the power k many elements.

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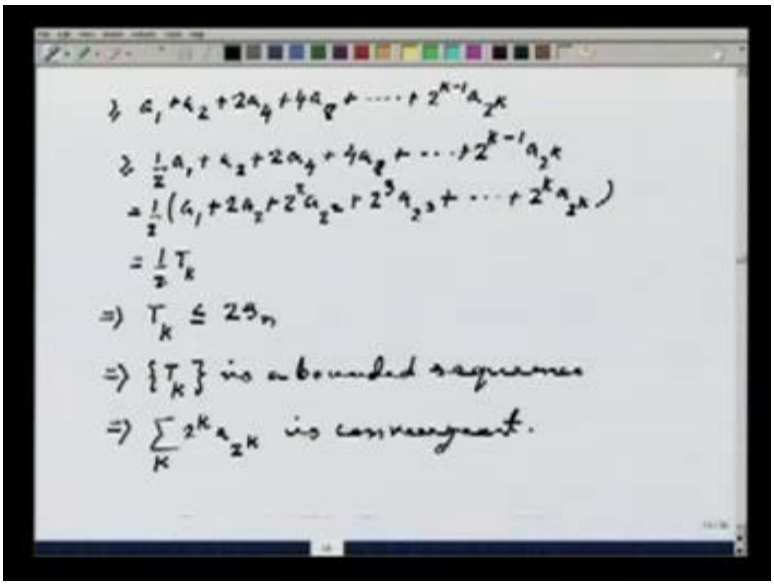

Now, I say that this is bigger than or equal to let us write a1 plus a 2 then I add a 3 plus a 4, but a 3 is bigger than or equal to 4, because it is decreasing. So, this is I can write twice a 4, then I can write 4 a 8 it goes on like that I finally, get this is bigger than or equal to half a 1 plus a 2 plus twice a 4 plus 4 a 8 plus 2 to the power k minus 1 a 2 to the power k, which is precisely half common half of a 1 plus 2 a 2 plus 2 square a 2 square plus 2 cube a 2 cube and so on, up to a 2 to the power k, which is nothing but half t k this, then implies that t k is lesser equal to 2 times s n.

Now, since a n s are non negative and I have assume that summation a n converges that implies that the sequence of partial sums s n that is a bounded sequence; that means, the sequence 2 s n also a bounded sequence this implies that t k is a bounded sequence. And, since summation 2 to the power k a 2 to the power k is also a series of non-negative terms, once I show that its partial sums forms a bounded sequence the series converges, but I have shown that the sequence t k is a bounded sequence this implies. So, we got the result in 1 direction that if summation a n converges, then summation 2 to the power k a 2 to the power k also converges.

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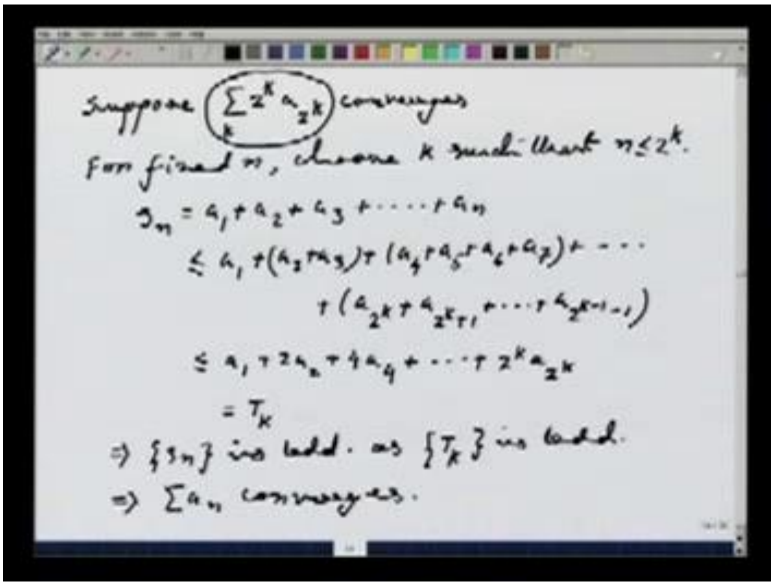

Now, it is the other way that suppose summation over k a 2 to the power k a 2 to the power k converges assume that I have to show that summation a n converges; that means, just I have to show that the sequence of partial sums s n of summation a n that converges that is a bounded sequence that is good enough.

So, what we do is for fixed n choose k such that n is lesser equal to 2 the power k and then start with s n that is a 1 plus a 2 plus a 3 plus up to a n, then I write this as this is lesser equal to a 1 plus a 2 plus a 3 plus a 4 and so on. Since, 2 to the power k bigger than n I am taking many more terms and since every term is non-negative here certainly right hand side much bigger than.

Now, this I further write as this is less than or equal to a 1 plus a 3 is lesser equal to a 2, so this lesser equal to twice a 2 plus, then I get it lesser equal to 4 a 4 and so on. For example, the last term which, I have written here it turns out to be 2 to the power k a 2 to the power k, which is precisely t k. Now, since I have assumed that 2 to the power k a 2 to the power k converges and it is series of convergence non-negative terms that implies s n is bounded as the sequence t k is bounded as I said if I have a n infinite series consisting of non-negative terms, then for it to converge it is necessary as well as sufficient that it sequence of partial sums is a bounded sequence.

Now, since summation 2 to the power k a 2 to the power k converges by my assumption; that means, t k is a bounded sequence. Now, since every terms of s n is lesser equal to some t k and t k is a bounded sequence that implies that the sequence s n is bonded this implies since this is increasing sequence of nonnegative terms; that means, the sequence s n converges this implies summation a n converges.

So, this completes the proof of Cauchy condensation test now in the next lecture we will continue with infinite series we will talk about some more test which will be needed in many practical situation, where you are dealing with some infinite series those are called limit comparison test or ratio test things like that that we will continue in the next lecture and then see some more examples of convergent infinite series.

So, in this lecture we have essentially concentrate only on two tests, which will give examples of convergent infinite series one is the comparison test very fundamental one, we will see when we go to ratio test again using the some form of the comparison test then we have understood the nature of very fundamental infinite series given by summation 1 by n to the power p. We will see this is one of the fundamental infinite series, because most of the times, when you deal with some infinite series to test its convergence you use what you actually do is you compare those infinite series with the series summation 1 to the power p whose behavior, now we know very well. And the third one is Cauchy condensation test that is very another fundamental test for convergence of a infinite series these tests actually have produced convergence summation 1 to the power p. So, this for today in the next lecture we will continue infinite series again.