**Mathematics-I**
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**Lecture – 15**
**Test of Convergence**

In today's lecture, we are going to learn some more tests, about convergence of an infinite series, one is called the ratio test, and other is called is root test; both are wide application in testing whether an infinite series converges or not. And then we are going to deal with series, which are not necessarily of non negative terms. We are going to deal with a particular kind of series called alternating series, and try to understand the notion of convergence for that kind of a series.

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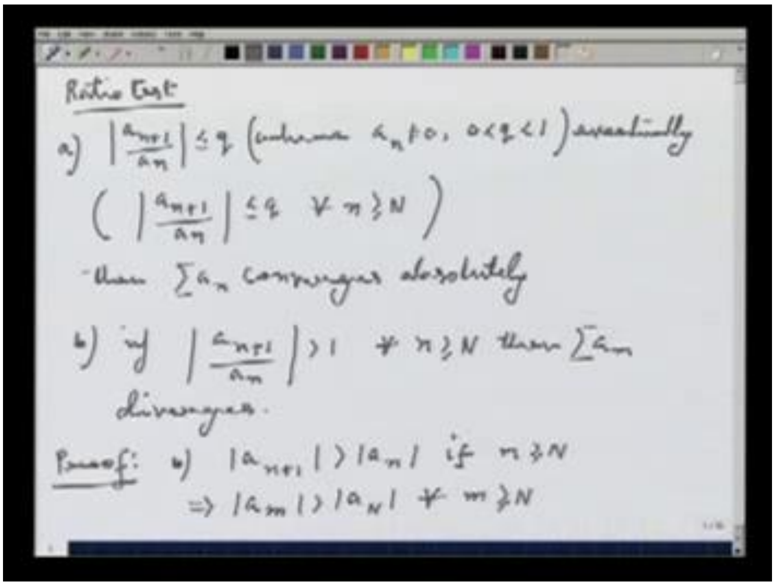


So, we first start with ratio test. The ratio test is actually outcome of the comparison test, we will see how it related to the comparison test as follows. Let suppose I have a n, which as the following property, that modulus of a n plus 1 by a n, is lesser equal to q. Where first of all, these quotient should make sense, so I will say a n is not equal to 0. And I put a condition 0 strictly less q strictly less 1. And I assume, that this inequality is true, eventually, that is after one stage, this a n plus 1 by a n is lesser equal to q. So, in particular, what I mean is that mod a n plus 1, divide by a n is lesser equal to q.

Let us say for all n, bigger than or equal to some capital N, if this happen then, summation a n, converges absolutely, this is part a. Then the part b is, if modulus a n plus 1 by a n, is strictly bigger than 1, for all n bigger than or equal to capital N, then summation a n diverges. So, let us see, once again, what I say that suppose I have an infinite series summation a n. I want to test, whether it converges or not, suppose all the a n is non zero, that means all the terms are non zero.

Then, I look at modulus of a n plus 1 by a n, if it happens than this quantities are bounded by some positive quantity, which is strictly less than 1 eventually than means after some stage, then the series converges. And, if it happens, that modulus of a n plus 1 by a n it is strictly bigger than 1 eventually after some stage, then the series diverges. Now, coming to the proof of it, it is very easy to prove it you will see, it is just simple application of the comparison test. The easiest part to prove b first, the given condition says, that mod a n plus 1, is strictly bigger than mod a n, if n is bigger than or equal to capital N.

In particular I can say, that mod a m, is strictly bigger than mod a N for all m bigger than or equal to capital N, this simply means, that the coefficient a n of the infinite series they do not converge to zero. Because, if a n converges to zero, then after some stage, they have to be less than epsilon, for arbitrary choice of epsilon. In other words, they can be made arbitrarily small after some stage. But here we see, it neither be, made smaller than, modulus of a n, because it always bigger than modulus of a n.

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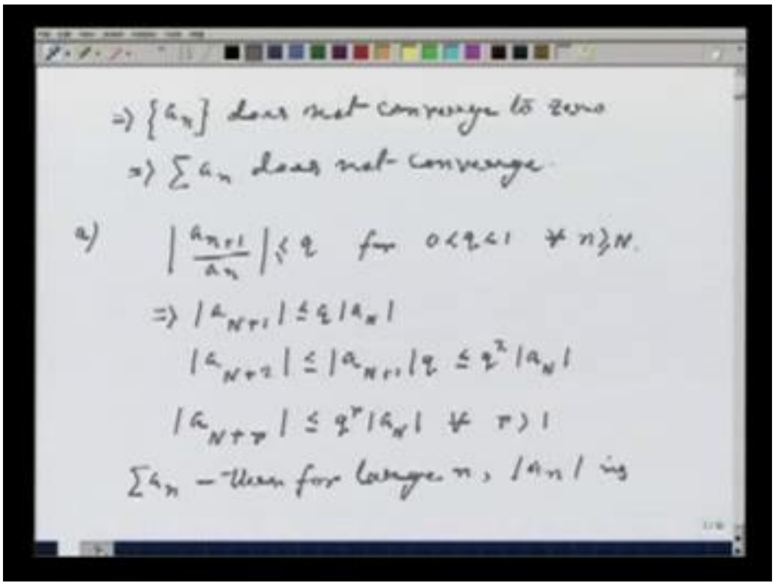


That implies the sequence a n does not converge to zero, this implies, summation a n does not converge. So, b is simple, now coming to a, what is given to me, it is given that modulus of a n plus 1 divided by a n, this is lesser than or equal to q, for some q, which lies between 0 and 1. This q is a constant, this does not depend on n, this is true, for all n bigger than or equal to capital N. Now, this then implies, that mod a n capital N plus 1 is lesser equal to q times, modulus a capital N.

And also, modulus a N plus 2, I can look at that, that is lesser equal to, modulus of a N plus 1 into q, which is further lesser than or equal to, q square into a mod, mod a N. This way I will get finally, that mod a N plus r is lesser equal to q to the power r, times mod a N, this is true, for all r strictly bigger than 1.

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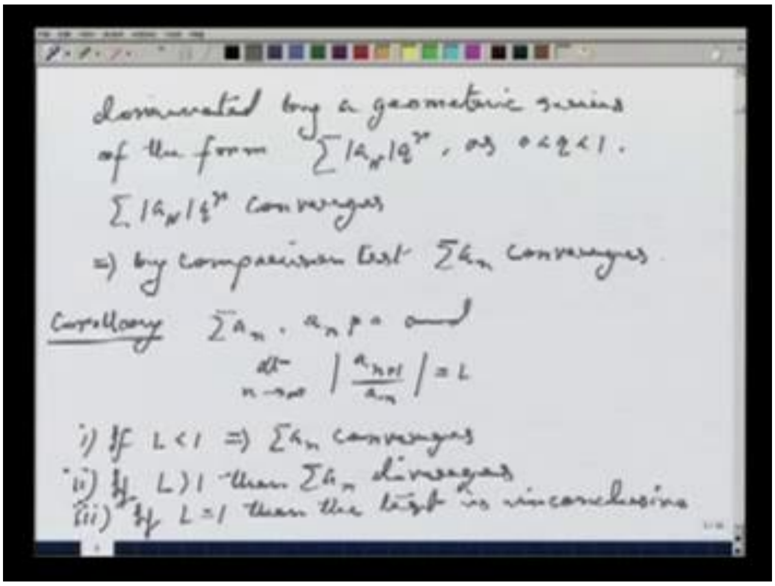


What does this mean, that means the infinite series, if I look at summation a n, then for large n, mod a n is dominated by, a geometric series of the form, summation mod a N times, q to the power r. Because of my assumption, that q lies between 0 1, 0 and 1, this geometric series converges. Then by comparison test, summation a n converges, it is just a simple application of the comparison test, but what fundamental here is, that q lies between 0 and 1.

Now, once we have this, as a corollary, we can have the standard statement of the ratio test, which you can have many books on that, it is enough form it is applicable. So, that forms says, that summation a n I look at a n is not equal to 0. And limit n going to infinity, modulus of a n plus 1 divided by a n, is equal to L. Suppose I have this, I have an infinite series a n, all the an's are non zero, and limit n going to infinity, modulus of a n plus 1 by a n exists, and it is equal to L. Then if L is less than 1, this implies summation a n converges, number 2 is, if L is bigger than 1, then summation a n diverges. But what happens if L is equal to 1, then the test is inconclusive, that is the series might converge, it may diverge also.

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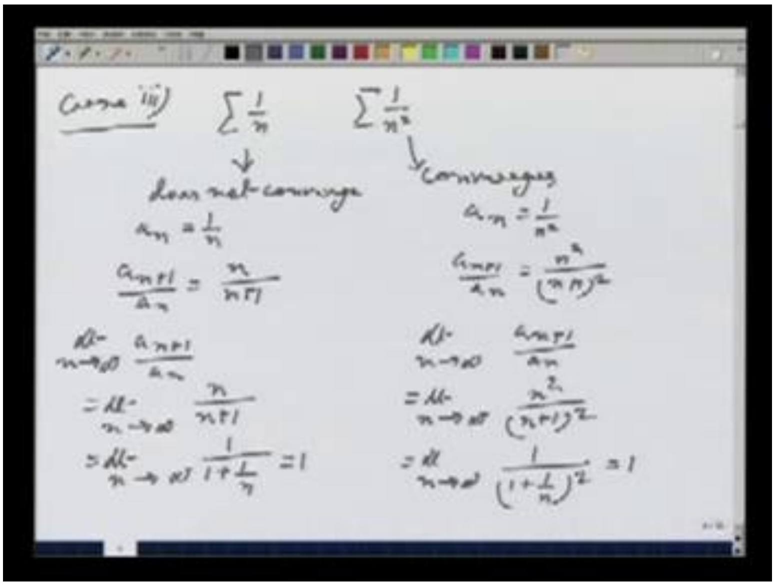

That is easy to see in the following example. So, first we illustrate 3, so case 3 look at the series, summation 1 by n, also look at the series summation 1 by n square, I know, that this series does not converge and I know that this is converges. That is a n here, a n is equal to 1 by n. And here, a n is equal to 1 by n square, then what is a n plus 1 by a n, that is, n by n plus 1, here a n plus 1, by a n that is equal to n square. So, what is then limit, n going to infinity a n plus 1 divided by a n. This is same as, limit n going to infinity n by n plus 1, that is limit n going to infinity 1 by n plus 1 by n, which is equal to 1.

And here, so you see, in both the cases, the required limit of a n plus 1 by a n, turns out to be 1. But in the case, the series converges, and in the 1 case the series does not converge. That means, the case L is equal to 1 ((Refer Time: 12:31)) does not reveal any about the infinite series. So this test, will only in the cases, when L is less than 1, or L is bigger than 1. L is less than 1 implies, the series converges that is, what is given in 1, and L is bigger than 1, the series a n diverges.

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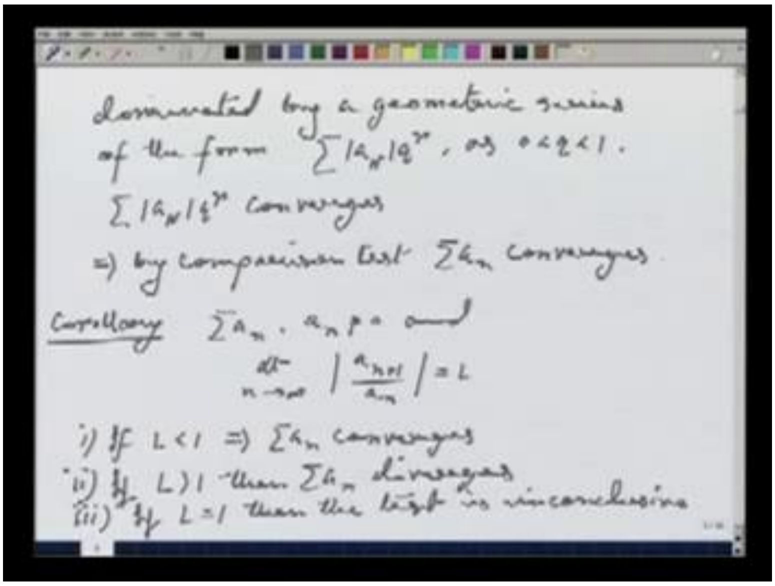


Now let us see a quick proof of this, so we start with 1, given that limit n going to infinity, mod a n plus 1 by a n is equal to L, which is less than 1, I know that. Now choose, epsilon bigger than 0, such that L plus epsilon is less than 1 and L minus epsilon bigger than 0. Then by the definition of convergence of sequence, there exist capital N. Such that for all n bigger than or equal to capital N this quantity is less than L plus epsilon and bigger than L minus epsilon. Now, I just concentrate on this part, this is true for all n bigger than or equal to capital N.

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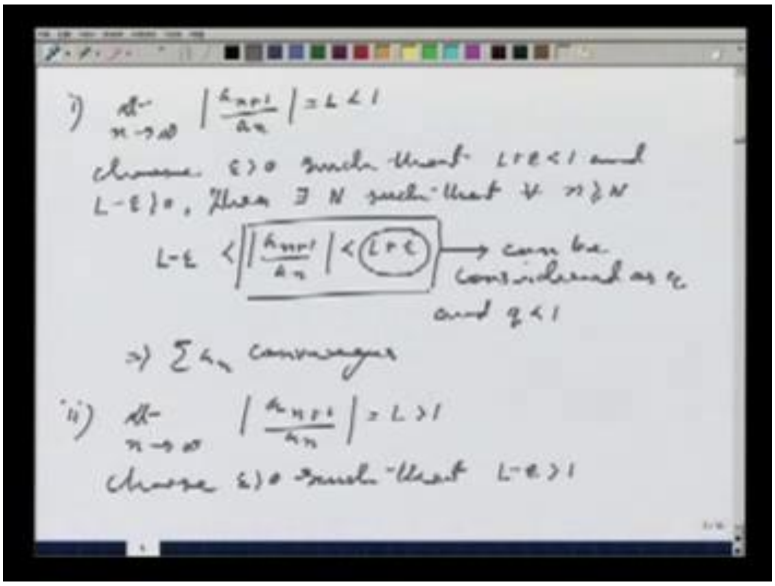

So, according to our previous result, which I have proved, this quantity l plus epsilon, can be considered as q. And, since my choice L plus epsilon is less than 1, this q is less than 1 this implies summation a n converges by the previous result. The previous result was, if after some stage, modulus of a n plus 1 by a n is less than q, where q is less than 1 then the series converges. Here, I got the exactly the same thing, instead of q i got epsilon, but which is less than 1, this can be considered as q, and hence the previous result applies.

Now, let us come to the second part, this says, the limit n going to infinity, modulus of a n plus 1 by a n is equal to L, which is strictly bigger than 1. What I do is, choose epsilon, bigger than 0, such that capital L minus epsilon, strictly bigger than 1. Since L is strictly bigger than 1, I can always choose an epsilon, such that L minus epsilon strictly bigger than 1, and epsilon is positive.

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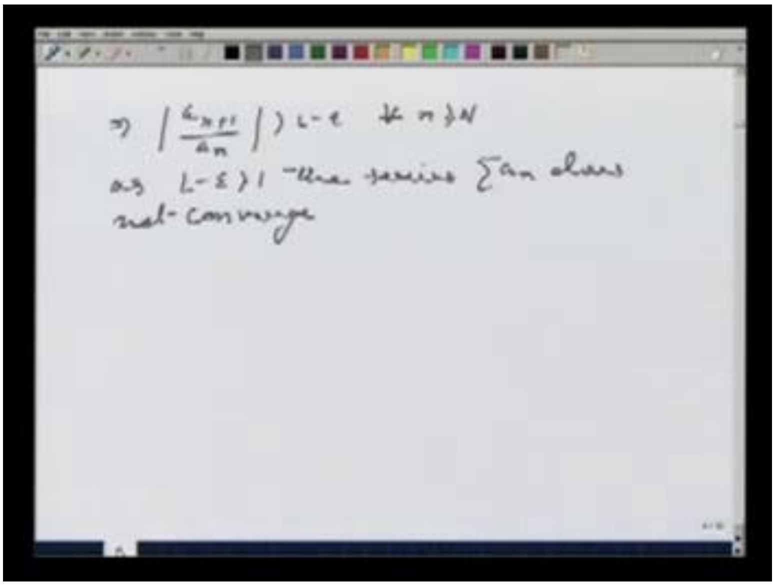

Then by the definition of limit, this implies modulus a n plus 1 divided by a n is bigger than L minus epsilon, for all n bigger than or equal to capital N. But, then I can compare this, again with the previous result, that as l minus epsilon is strictly bigger than 1, the series diverges, summation a n does not converge. Look at the previous result, where I have proved ((Refer Time: 16:33)), that this is the statement, I am looking for…

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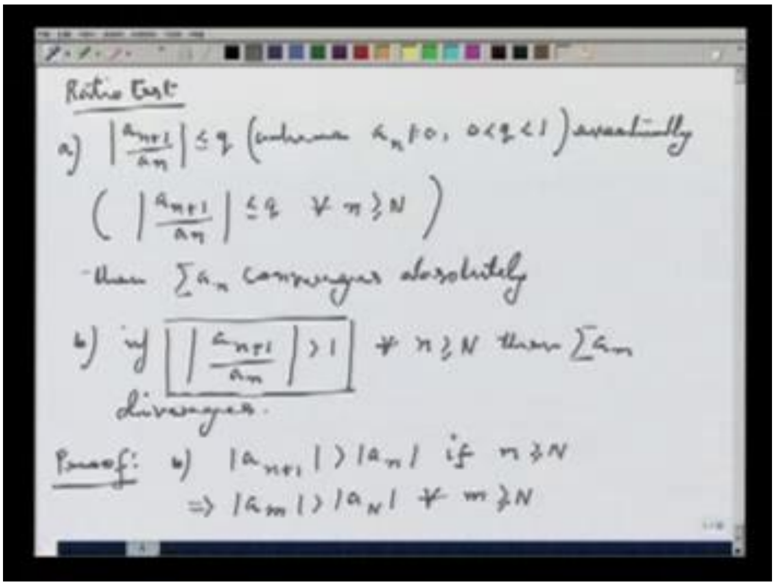


See if n is bigger than or equal to N and modulus of a n plus 1 a n is bigger than 1 for all n bigger than or equal to N, then summation an diverges. It does not converge, because the term do not go to zero, that is what is happening.

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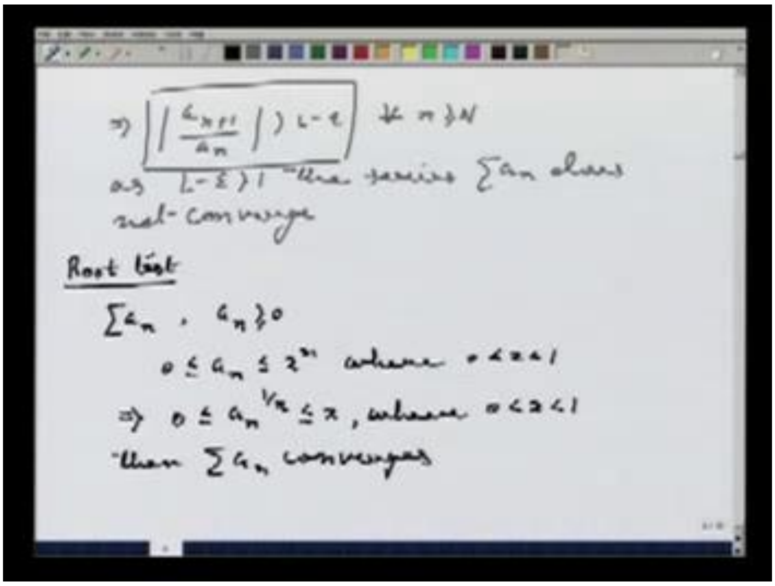


Now, the situation I have at hand is precisely this, that modulus of a n plus 1 a n, this is bigger than L minus epsilon. But, L minus epsilon by my choice, strictly bigger than 1, then the series, summation a n does not converge, because it is terms do not go to zero, that gives you the ratio test.

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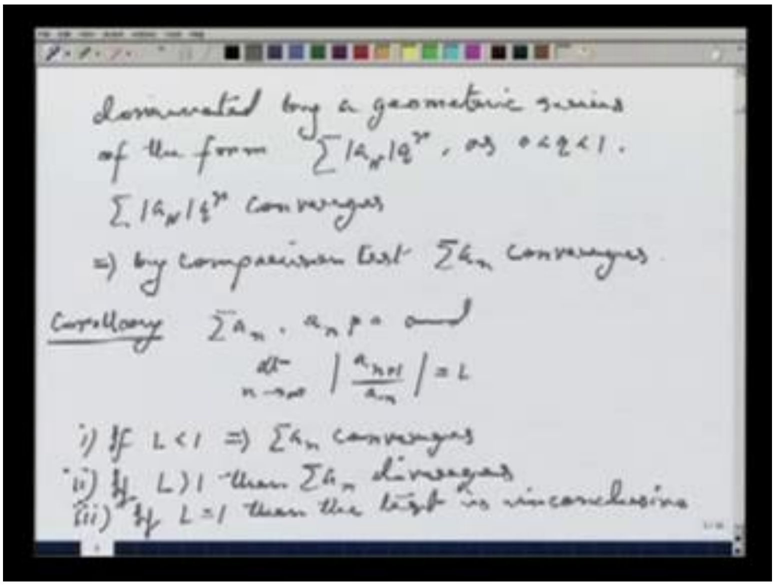

Let us look back again at the statement of the ratio test, it says what that first, have six series summation a n, the k has to be taken, all the terms are non zero. Otherwise, it is difficult to apply the ratio test, because we are dividing by something. So, if all the terms are non zero, I look at modulus of a n plus 1 by a n, and then I look at the limit. Suppose the limit exists, then I have to check, whether limit is less than 1, if it is then the series converges. If the limit is bigger than 1, then the series diverges, if L is equal to 1, then we cannot say anything, and we have to try for something else.

Now, the next test, what we are going to deal, with is called the root test ((Refer Time: 18:05)), this is also consequence of the comparison test, but here the comparison being made with geometric series. For example, suppose the situation is this, that I have summation a n I know that all the an’s are bigger than or equal to 0. And let us say, 0 lesser equal to a n is less than or equal to x to the power n, where 0 less x less 1. Then, it is very clear from the comparison test, that the series summation a n converges, because the geometric series summation x to the power n converges.

This implies, if my condition was, that 0 lesser equal to a n to the power 1, by n lesser equal to x, where 0 less x less 1, then summation a n converges. Because, a n to the power 1 by n lesser equal to x, it would imply an is lesser equal to x to the power n, then I can compare the root test is essentially the same thing, written in different language in terms of limit.

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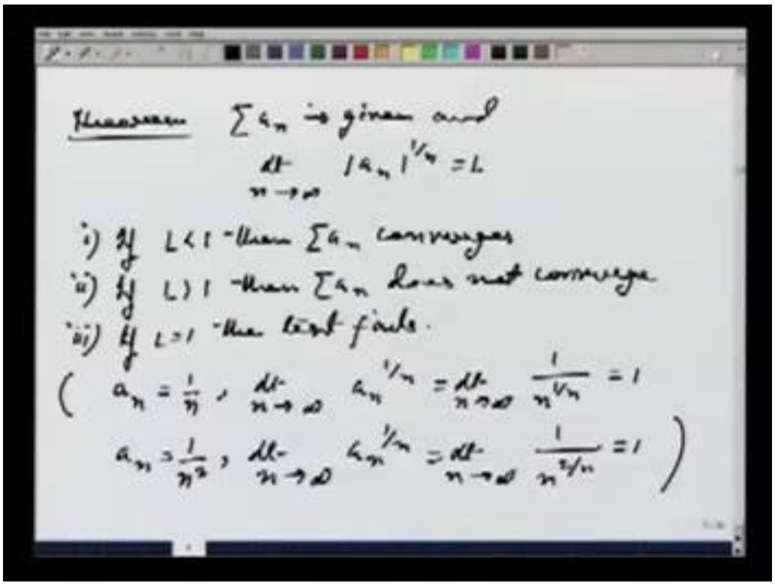


So, let me first, give you the statement. So, I write it as a theorem, summation a n is given, and limit n going to infinity, modulus of a n whole to the power 1 by n is equal to L. Then, if L is less than 1, then the series converges, if L is bigger than 1, then series does not converge. And third is, like the ratio test, if L is equal to 1, the test fails, that is no conclusion, may be drawn. Again as an illustration of 3, what we have to do is take an to be equal to 1 by n. Then limit n going to infinity, which is certainly, is equal to 1.

But I start with a n is equal to 1 by n square, then also, this limit is also is equal to 1, but summation 1 by n diverges and summation 1 by n square converges. So, in this case, L is equal to 1, and hence the test is not conclusive. So again, we just need to prove the case 1 and case 2.

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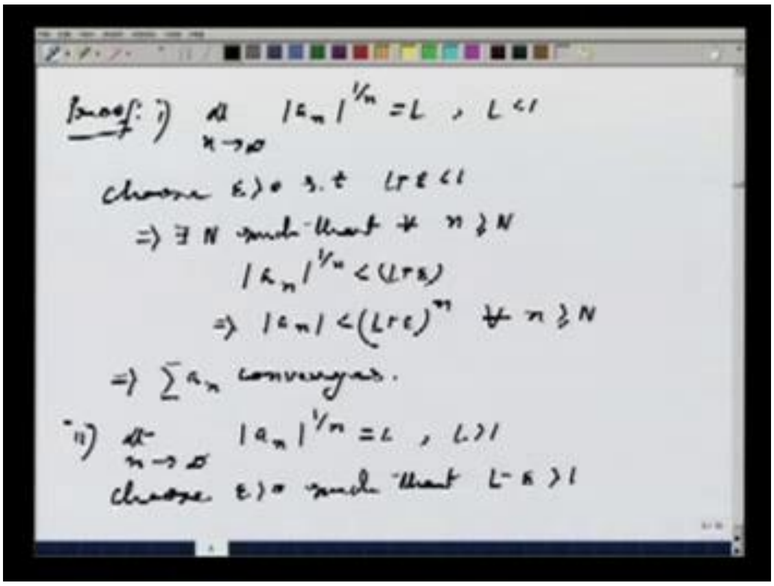

So, let us come to the proof, what is given to me it is given, I limit n going to infinity, modulus of a n, whole to the power 1 by n is equal to L and L is less than 1. Choose epsilon again, such that L minus epsilon L plus epsilon is strictly less than 1, this then would imply, that there exist, some capital N. Such that, for all n bigger than or equal to N, modulus of a n to the power 1 by n, is less than L plus epsilon ((Refer Time: 23:15)). It imply, that modulus of a n is less than L plus epsilon whole to the power n, for all n bigger than or equal to capital N.

Now, notice that L plus epsilon is strictly less than 1, so L plus epsilon to the power n, if I look at the sum, that gives me a geometric series, which converges, and modulus of a n is less than that, by comparison test. Then, summation a n converges. In fact, a n converges absolutely, so in this case, we again comparing, and again with geometric series.

Now, the second case, the limit n going to infinity, modulus of a n to the power 1 by n, that is L and L is bigger than 1. Now choose, epsilon bigger than 0, such that L minus epsilon is still bigger than 1, since L is bigger than 1, I choose some such epsilon.

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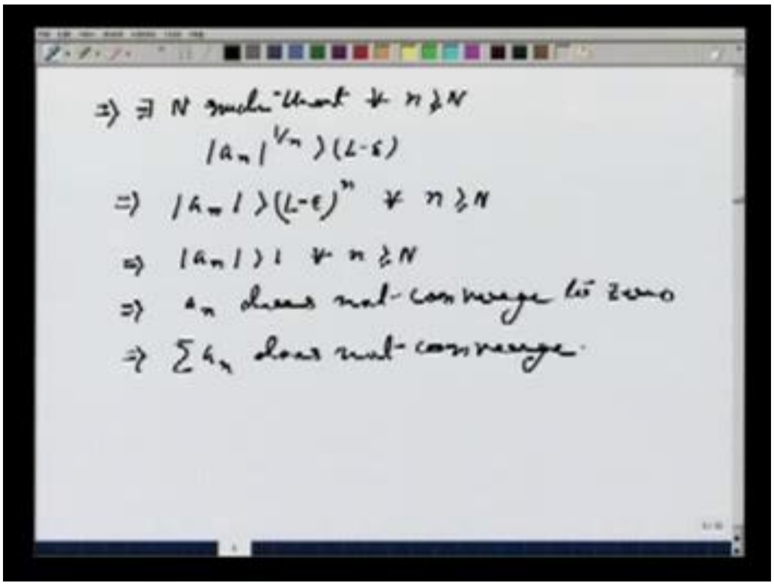


This implies then, by the definition of limit, that there exist capital N such that, for all n bigger than or equal to N. Modulus of a n to the power I by n is bigger than L minus epsilon, this implies modulus of a n is bigger than L minus epsilon, whole to the power n, for all n bigger than or equal to capital N. Now, notice L minus epsilon being bigger than 1, L minus epsilon to the power is always bigger than 1, this would then imply, that modulus of a n is bigger than 1, for all n bigger than or equal to capital N. This certainly implies, that a n does not converge to zero, this implies, that the summation an does not converge.

So again the statement is very simple, what I do is, a n is given, I look at modulus of a n to the power 1 by n, and I just calculate that limit, If that limit, is less than 1, then the infinite series converges, if that limit is bigger than 1, then the infinite series does not converge. But, if the limit is equal to 1, then I have to careful, because in that case, infinite series may converge, and it may not converge.

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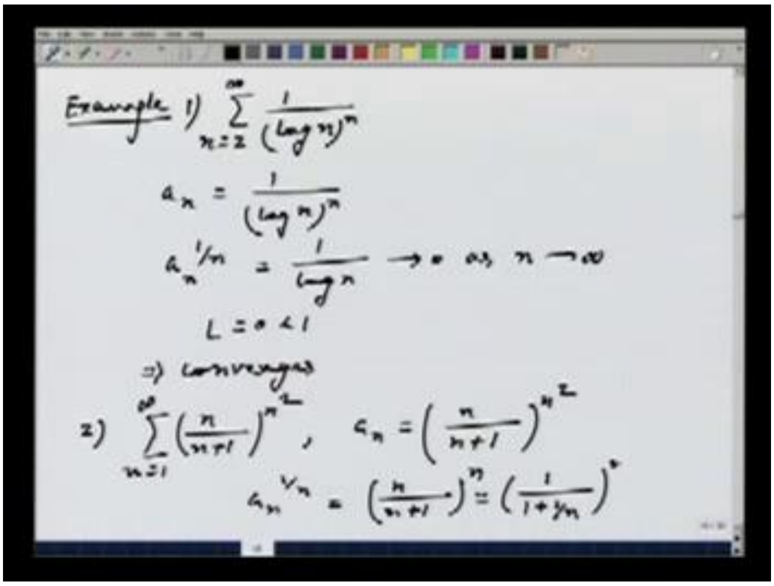


Now, let us see, some examples here, let us look at summation, so this is the first example, 1 by log n whole to the power n, n from 2 to infinity. So, here a n is equal to 1 by log n whole to the power n, that means, a n to the power 1 by n, that is 1 by log n, this goes to 0, as n goes to infinity. So, in this case, L is equal to 0, which is less than 1, that implies the series converges, similarly, I can look at summation n by n plus 1 whole to the power n square. So, in this case, a n is equal to that means, a n to the power 1 by n trans out to be, which is same as…

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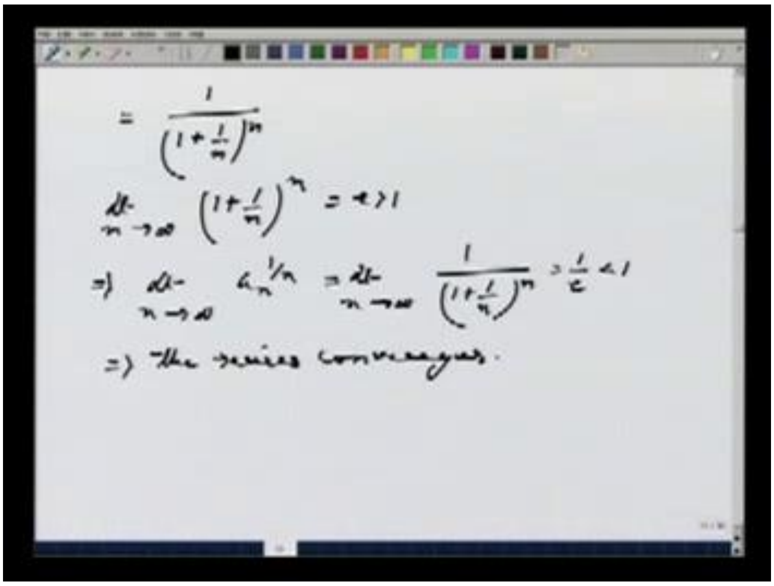


So I can write this as, and since we known that limit n going to infinity, 1 plus 1 by n whole to the power n, is e which is strictly bigger than 1, this then implies, since I have 1 by it implies, that limit n going to infinity, a n to the power 1 by n, this is 1 by e, which is strictly less than 1, this implies then the series converges. So, given an infinite series, now we have certain techniques, which we can apply to test whether the series converges or not.

The first one, the most effective 1, is the comparison test, you try to compare it with geometric series, or summation 1 by n to the power p. Then we have the limit comparison test, then we have the ratio test, now we have seen root test, after this, I am going to look at series, which are not always series of non-negative terms, that means, here, negative terms can also come. And, we want to look at certain test, which will tell me the convergent of this kind of series.

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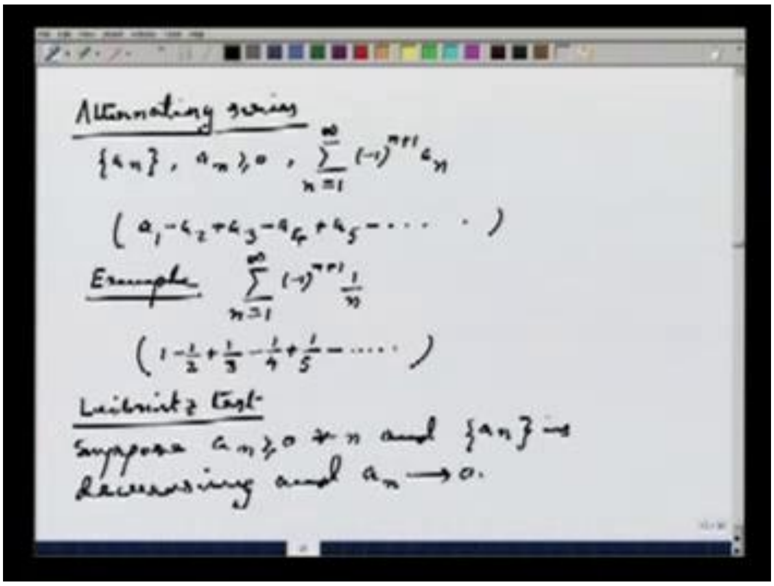


Now, we come to something called alternating series, so what is an alternating series, suppose I have a sequence a n, and a n is bigger than or equal to 0. Then, I look at the infinite series of this form, summation n from 1 to infinity, minus 1 to the power n plus 1, times a n. Notice then, what is the first term of the infinite series that means, when n is equal to 1, this is a 1. So, the series looks like, the first term is a 1, then I take n to equal to 2, that means, minus 1 to the power 3 times a 2 minus 1 to the power 3 is minus 1, so it is minus a 2.

Then comes, plus a 3 minus a 4, plus a 5 and so on, So, you see, the signs actually alternate 1 to another, first term comes with the positive sign, second term comes with the negative sign, third comes with the positive sign, fourth term comes with the negative sign and so on. That is, either terms alternating series, all we are interested, in that under which criteria, this kind of a series converges, the particular example we have in mind is, this series, summation n from 1 to infinity, minus 1 to the power n plus 1 times 1 by n.

So, the series look likes, if I look at first n is equal to 1, the first term is 1, then minus half, plus 1 third, minus 1 fourth, plus 1 fifth and so on. Notice that, this series is not absolutely convergent, because if I take modulus of the terms, all I get is summation 1 by n, which I know does not converge. But, the question still remains “does the series converge”, because absolute convergent implies convergent, but not the other way, so we are interested to know, that this particular series converges or not, that we will get, by something called Leibnitz test.

The statement is like this, suppose a n is non-negative for all n, and the sequence a n is decreasing, and a n converges to 0. So, what are the properties, I have a sequence a n, such that elements are non-negative, it is decreasing, and it converges to 0.

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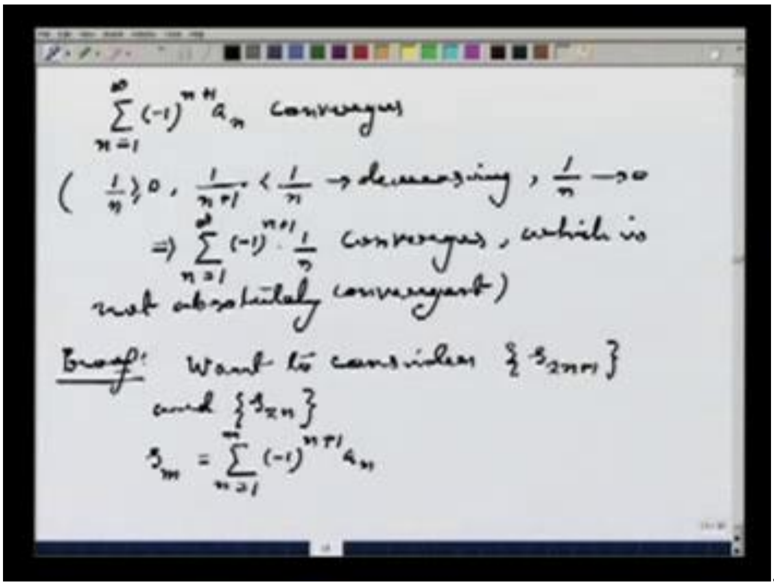


Then, the alternating series summation n from 1 to infinity, minus 1 to the power n plus 1 a n converges, before we go to the proof of this, come back to the example, 1 by n is bigger than or equal to 0, 1 by n plus 1 is less than 1 by n, this is the condition decreasing, and also 1 by n goes to 0. Hence, by Leibnitz test, it would imply, that summation n from 1 to infinity, minus 1 to the power n plus 1, into 1 by n, converges, which is not absolutely convergent.

So, in particular it is also gives me an example of a series, which converges, but not absolutely convergent. So, the properties are very simple to remember, for this Leibnitz test as a model, you should always remember the series, minus 1 to the power n plus 1 times 1 by n. The conditions are exactly analogues to this series, that means, a n are non-negative, decreasing and goes to 0, as n goes to infinity, then the alternating series converges.

Now let us come to the proof of this, so first we will look at, partial sums of the series, but the even partial sums, and the odd partial sums. So, we want to consider, the odd partial sums s 2 n plus 1, and s 2 n, so what are the definitions, very simple, usually s m means, summation n from 1 to m minus 1 to the power n plus 1 into a n.

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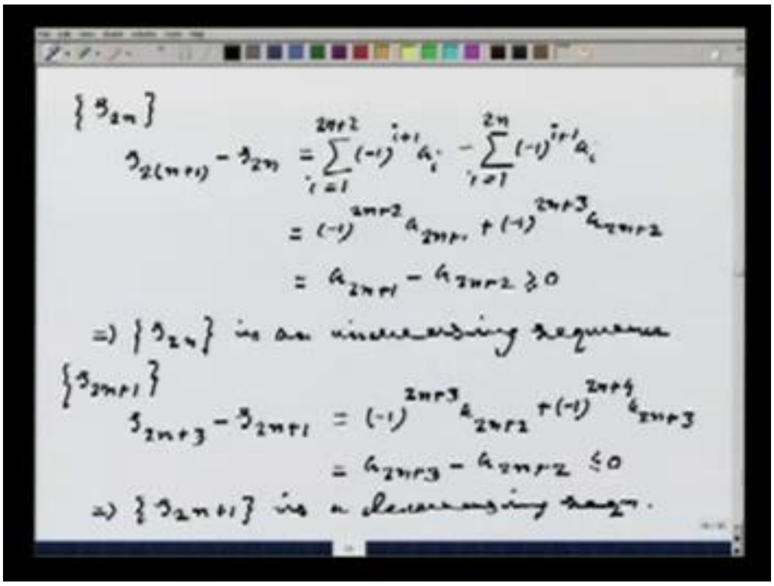

Now, let us see, what kind of sequences this, this partial sums are, let us first start looking at, s 2 n, whether it is increasing or decreasing or what. So, first let us check, s 2 n plus 1, minus s 2 n, what is this, this is summation i, from 1 to 2 n plus 2 minus 1 to the power i plus 1 a i, minus summation i from 1 to 2 n, minus 1 to the power i plus 1 a i. Once, I write this, many terms cancel except, the last 2 terms that means, what remains is minus 1 to the power 2 n plus, that means, the terms i is equal to 2 n plus 1.

So, it is 2 n plus 1 plus 1, that is 2 n plus 2 a, 2 n plus 1, then the next terms plus minus 1 to the power 2 n plus 3 a, 2 n plus 2, that means, what I get is, a 2 n plus 1, minus a 2 n plus 2. Now, I know, that the sequence a n decreasing, that property given to me, that means, a 2 n plus 2 is less than a 2 n plus 1, that means, this is bigger than or equal to 0. What does this prove, this implies, the sequence is 2 n is and increasing sequence, because I have seen, then the next term is bigger than the previous term, and that is happening for each n, so the sequence is increasing.

Now, if I start with, the odd sub-sequence of partial sums, the odd sequence of partial sums is 2 n plus 1. Then I would certainly look at s 2 n plus 3 minus s 2 n plus 1, then the calculation exactly like the previous one tell me, that what remain is, minus 1 to the power 2 n plus 3 a, 2 n plus 2 plus, minus 1 to the power 2 n plus 4, a 2 n plus 3, that means, a 2 n plus 3, because minus 1 to the power 2 n plus 4 is 1, minus a 2 n plus 2, but notice, that the sequence is decreasing, that means, a 2 n plus 3, is less than a 2 n plus 2, which is less than or equal to 0.

So, the behavior changes, this implies then, the sequence s, 2 n plus 1, is a decreasing sequence. So, the even sub-sequence is increasing, that is what i got, and the odd sub-sequence is decreasing, fine. Now, I want to compare between s 2 n and s 2 n plus 1, what is the relation among them.

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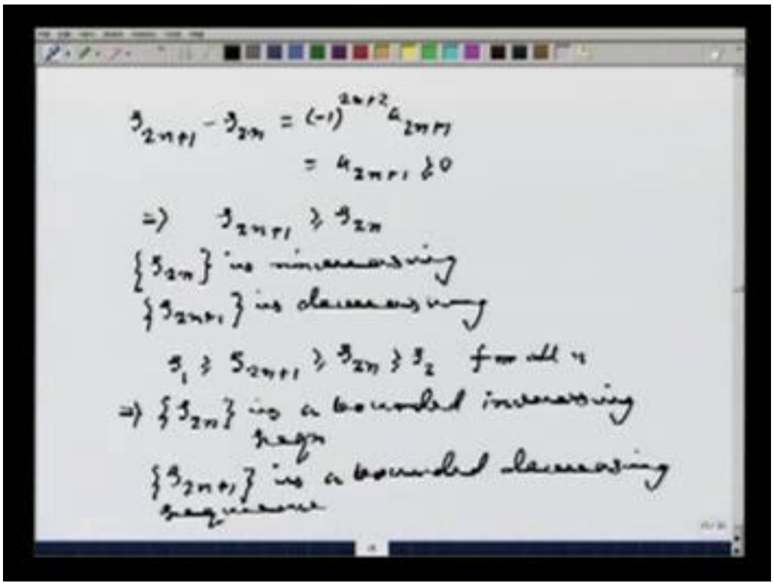

So, let us look at, s 2 n plus 1 minus s 2 n, if you write down this quantities, you will see, what will get is, minus 1 to the power 2 n plus 2, a 2 n plus 1, that is a 2 n plus 1, because 2 n plus 2 is an even number. So, the minus 1 to the power 2 n plus 2 is any way 1, this is bigger than 0, because all the terms are non-negative, so this implies, s 2 n plus 1, is bigger than or equal to s 2 n. Now, I have two more information, about this s 2 n and s 2 n plus 1, I have already seen.

So, let me write here, s 2 n increasing, and s 2 n plus 1 is decreasing, now notice one thing, i can write it in this form, s 2 n plus 1, that is bigger than or equal to s 2 n, I got that. Since, it is increasing, it is certainly bigger than or equal to s 2, because s 2 n is increasing, so all the higher terms are bigger than the first term, the first term is s 2. So, s

2 n is bigger than s 2, on the other hand, if I look at s 2 n plus 1, that is decreasing, that is all the terms less than or equal to the first term. So, that means, which is less than or equal to first term, which is s 1, and this is true, for all n, this implies, then that the s 2 n, is bounded increasing sequence, similarly s 2 n plus 1, is a bounded decreasing sequence.

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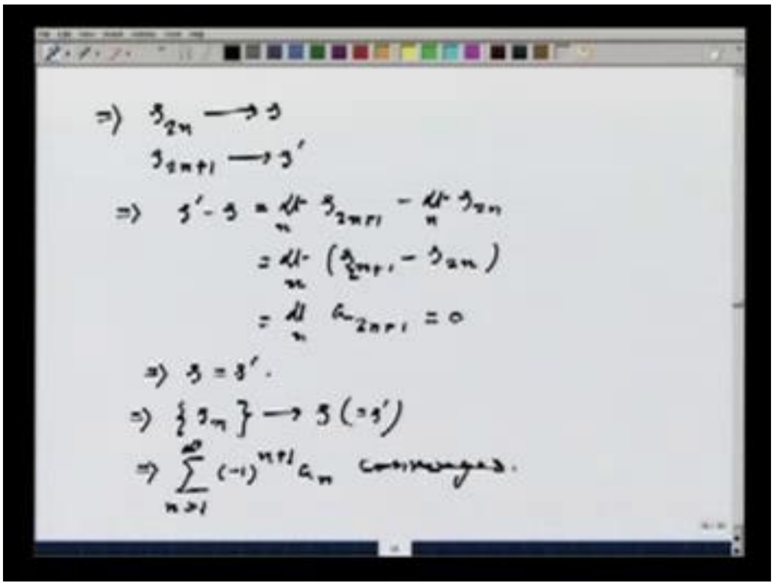


But, we know, that every bounded increasing sequence, and bounded decreasing sequences, they converge to some numbers, using that this implies, that s 2 n, converging to some number s, and s 2 n plus 1, that converge to some number s prime let us say. Because, these are bounded increasing and decreasing sequences, now this implies, now that if I look at s prime minus s, that is limit n s 2 n plus 1, minus limit n s 2 n, which is same as, limit n s 2 n plus 1, minus s 2 n, that is limit n, a 2 n plus 1.

Now, there is another condition on that an's, that as n goes to infinity, a n goes to 0, as n goes to infinity a 2 n plus 1 is also goes to 0. Because it is a sub-sequence of the, sequence of an's, so this is 0, this would then imply, that s is equal to s prime, but since, s is equal to s prime, this would imply, that the sequence of partial sums s n if I look at, converges to s, which is same as equal to s prime, this simple fact about the sequences, which we have used earlier also that if we have the sequence s n, look at it is even sub sequence, and look at its odd sub sequence.

If, the even sub sequence, and odd sub sequence has the same limit, then the whole sequence, also has the same limit, in particular it converges, by that observation. Now, sequence of partial sums of the alternating sequence s n converges to a number s, this implies the whole series converges.

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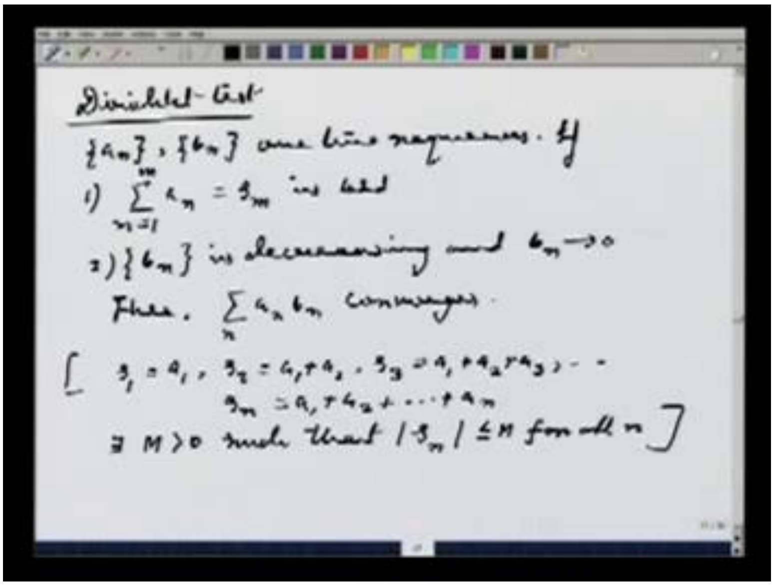


Now, i am going to tell you about, another test, this time without proof, but it comes very handling and many practical situations, you have to use that test, it is called the Dirichlet test. Suppose, a n and b n are 2 sequences, if number 1, that summation n from 1 to m a n, which i call s m is bounded, that means, the partial sums of a n is a bounded set. Number 2 is b n is decreasing and b n converges to 0, then summation over n, a n b n converges, this is called Dirichlet test.

Let us, elaborated on condition one, what we mean is, we look at s 1, which is just a 1, then we look at s 2, which is a 1 plus a 2, then you look at s 3, that is a 1 plus a 2 plus a 3 and so on, so that s n is equal to a 1, plus a 2, plus a 3 up to a n look at this numbers. Then, partial sums are bounded, which means there exist, some m bigger than 0, such that, modulus of s n is less than or equal to m, for all n. This is what we mean, by saying that s m is bounded, that means there exist a number capital M, such that, in the modulus the partial sums of a n are less than or equal to, that number M. If that happens, bn’s are decreasing, and b n is going to 0, then the series summation over n, a n, b n converges this is called Dirichlet test.

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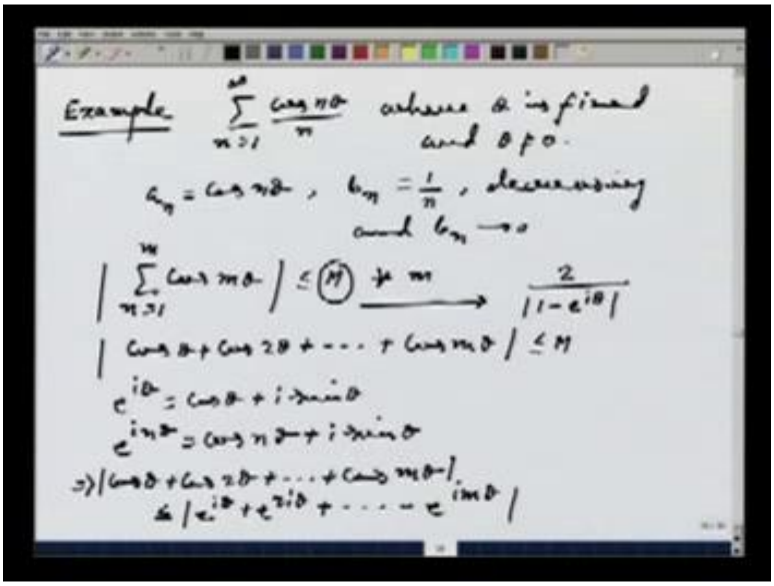


So, let us just, illustrate it by the following example, look at the following series, summation n from 1 to infinity, cosine of n theta divided by n, where theta is fixed number, and theta is not equal to 0. Notice that, if theta is equal to 0, then this series does not converge, because cosine of n theta is 1, and summation 1 by n does not converge, but if theta is non-zero, then we are going to prove, this series converges so what we do is, we want to apply Dirichlet test here.

So, I put a n, is equal to cosine n theta, and b n is equal to 1 by n, now bn's are decreasing and decreasing to 0, that is clear. So, this is decreasing, and b n goes to 0, now all I need to prove is that summation n from 1 to m, cosine n theta, if I look at the mod, it is less than or equal to some number capital M for all m. So, essentially now, i am bother about this kind of a finite sum, that is cosine theta, plus cosine 2 theta, plus up to cosine n theta. I want to show this is lesser equal to capital M.

What I do is I view cosine theta as, the real part of the complex number, and that is very well complex number all of us know it, if I look at e to the power i theta. This is cosine theta plus I sin theta, and we also know by de moivre formula, that e to the power i n theta, is cosine n theta, plus i sin n theta. Now, it is always it is true, that real part of the complex number in the modulus is always less than or equal to modulus of the complex number.

So, this, then implies, the modulus of cos theta, plus cos 2 theta, plus up to cosine n theta, is certainly less than or equal to, modulus of e to the power i theta, plus e to the power twice i theta, plus up to e to the power i m theta. Now, notice the right hand side sum, I can actually calculate, because it is a geometric series, with common ratio e to the power i theta.

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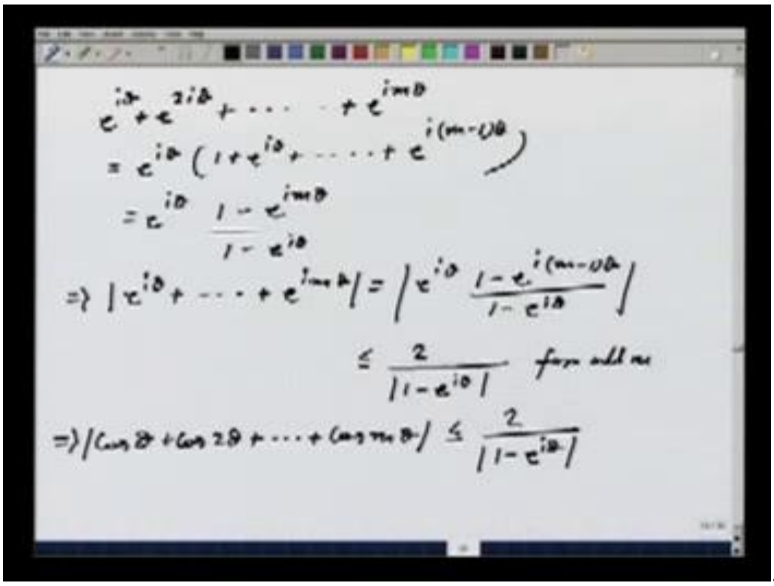

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So, I am going to use that, e to the power i theta, plus e to the power twice i theta, plus up to e to the power i m theta, that I can write as, e to the power i theta, into 1, plus e to the power i theta, plus e to the power i m minus 1 theta, which then by a well known formula, g p series is 1 minus. Now, this implies then, that modulus of e to the power i theta, plus up to e to the power i m theta, is modulus of e to the power i theta, into 1 minus, by the previous formula.

Now, since modulus of e to the power i theta, or e to the power i n theta is always 1, what I get is, this is lesser equal to using the triangle inequality 2 divided by modulus 1 minus e to the power i theta. The point is, now think are independent of m, this is true, for all n, this then imply modulus of cosine theta plus cosine 2 theta plus cosine m theta is lesser equal to this well defined as theta is not equal to 0.

And hence the Dirichlet test applies, ((Refer Time: 54:40)) I go back to the previous step, this m I have actually found out, this m Trans out to be 2 divided by modulus of 1 minus e to the power of i theta. It depends on theta, it seems, that is it should not depend on little m's, which it does not, so the partial sums of an's are bounded, bn's decreasing and going to 0 this then implies by Dirichlet test, that the series converges. This is all we had to cover in the infinite series. In the next lecture we are going to talk about something called power series.