Mathematics-I
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Lecture - 16
Power Series

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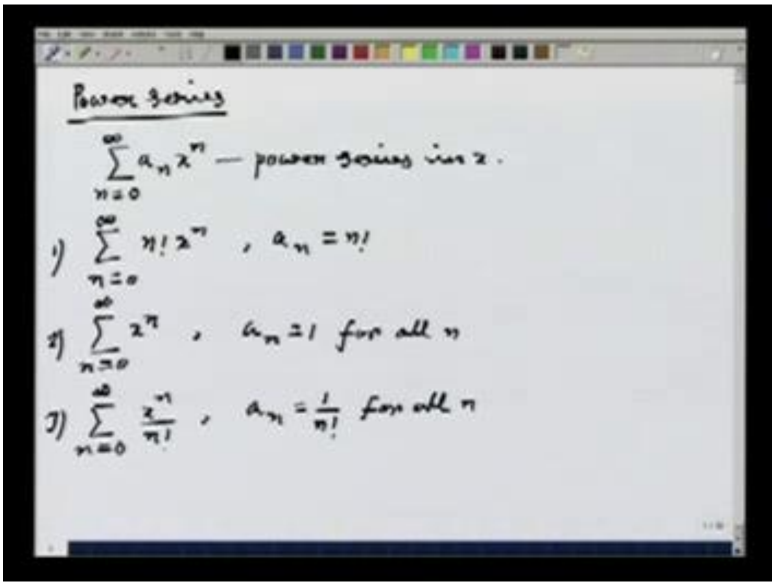

In today's lecture, we are going to deal with Power Series. So, what is a power series? A power series is an expression of the following form. Summation n from 0 to infinity, a n times x to the power n. Here, an's are some given numbers, some real numbers, so it is a sequence of real number and x is the variable. So, if the sum makes sense, I can view this as a function of x. And what do we like to understand is, that for each x, this function makes sense. So, to start with, this formal expression, because I do not know the meaning of this, is a power series in x.

Notice this, that if I gave a fixed value to x, then what it gives me is an infinite series. And now, we very well understand the meaning of convergence of an infinite series. So, we can check x by x if we can, that is put the value of x in the series. You get any infinite series, check that, whether it makes sense. So, to start with, let me look at some examples. So, first let us look at this series, summation n from 0 to infinity, factorial n times x to the power n.

So, here a n is equal to factorial n. So, this is a power series in x, another example would be, summation n, from 0 to infinity, just x to the power n. That is, here, a n is equal to 1, for all n. Another example would be, summation n from 1 to infinity. Well I can still take n from 0 to infinity, divided by factorial n this times. So, here a n is 1 by factorial n, for all n. That I will like to test now, 1 by 1 is for which x, this three infinite series, which I have written they makes sense.

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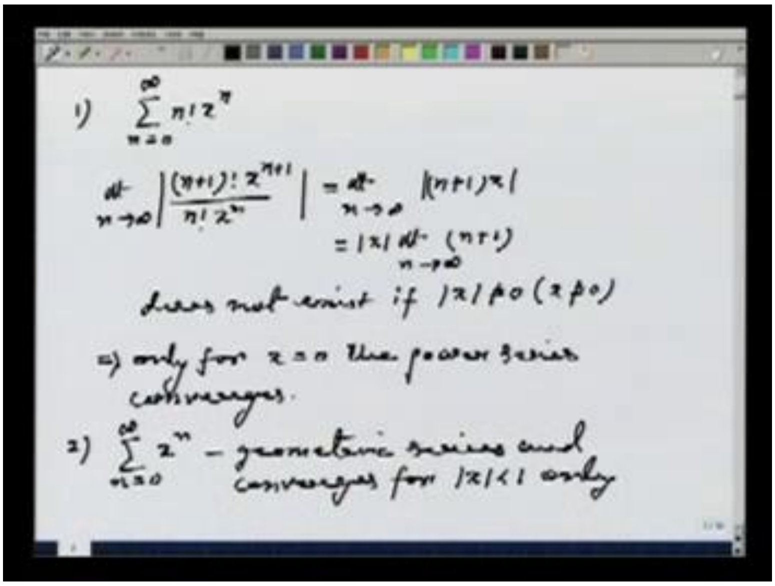

So, we start with example 1 first. So, here the infinite series is, summation n from 0 to infinity, factorial n times x to the power n. So, I fix x and I want check for which values of x this would makes sense. So, I try to apply my ratio test. So, that means, I have to look at modulus of the n plus 1th term. That is n plus 1 factorial times x to the power n, divided by the nth term. That is, factorial n times x to the power n. And then I have to look at the limit, as n goes to infinity. Then, what I get is, limit n going to infinity, modulus of n plus 1 times x.

Notice my x is fixed. So, it can be taken out of this limit. So, it is mod x times limit n going to infinity. Now, this limit will certainly be infinite, it does not exist finitely. So, that way, I do not get any x it look likes, for which the series makes sense. Notice that, this limit does not exist, if mod x is not equal to 0; that is x naught equal to 0, on the other hand what happens; if I choose x to be equal to 0. Then look at the terms of the power series, it is factorial n times x to the power n.

But, if I choose x to be equal to n, then all the terms of the power series is 0. So, that certainly converges. So, the conclusion then is, that only for x equal to 0, the power series converges. Now, let us look at example 2, that is summation n from 0 to infinity x to the power n. Now, if I apply ratio test or by knowledge, whatever I know, this is the geometric series and converges, for mod x less than 1 only, because if I take, mod x to be bigger than or equal to 1, the series does not converge.

So, this power series, it makes sense as long as modulus x is strictly less than 1, so at least, if I choose x in the open interval minus 1, 1.Then, the series certainly represents a function, I can write f x equals to the sum, as long as x is minus 1, 1, otherwise not. If you look at the previous example, if I want to write f x, I cannot unless I choose x only to be equal to 0. Let us look at the third example now.

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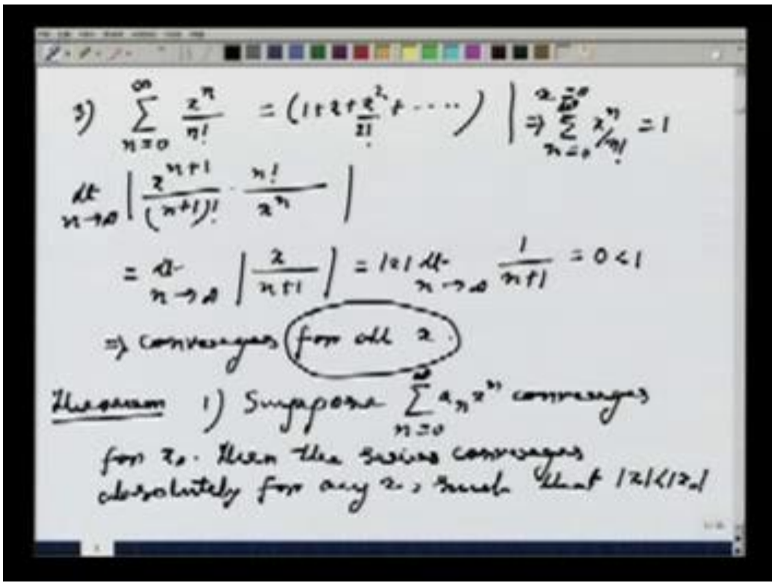

Here the power series was, summation n from 0 to infinity, x to the power n by factorial n. So, again to understand the behavior of the series, for different possible x is, all I do is, I look at the ratio test, so that means x to the power n plus 1, divided by factorial n plus 1 into factorial n divided by x to the power n modulus, I have to take the limit as n goes to infinity, well, I choose x to be not equal to 0 also, otherwise this, x to the power n in the denominator will be a problem.

But, if I choose x to be equal to 0 in the series, in any way you get 0, well not quite, if I put x to be equal to 0, then all the terms bigger than the first term, that is the n equal to

0 case, they are all 0. It is the only first term which survives, that means, I can write this series, if I look at it clearly, it is 1 plus x plus x square by factorial 2 and so on. Notice that, if I put x to be equal to 0, then I get x equal to 0 implies, summation n from 0 to infinity x to the power n, by factorial n is equal to 1.

So, the problem of convergence is not there if x is 0, it is only there, when x is nonzero, so in that case, I can certainly divide and go to the ratio test. What I get in this case is, limit n going to infinity, x divided by n plus 1, then the modulus, again x is independent of n, so it is mod x times limit n going to infinity, 1 by n plus 1 which is 0. Whatever, my x is, notice for convergence in the ratio test, the limit is strictly less than 1, in this case I get 0, which is strictly less than 1.

That is, this series converges for all x, whatever x you choose, so that means, given a power series, summation an x to the power n, for each x, it will converge that certainly depend on the coefficients an, I am choosing. It may happen, that it does not converge for any x nonzero, it might happen, that it converges for all x, it might happen converges for some x. Now, let us go to more deep into the power series, it can usually never happen, that you have discretely any exist, for this power series converges.

The convergence if it happens, it is always be an some interval, if you look at the example, which we were doing in 2. In 2 ((Refer Time: 10:13)) the x is for which, it converges is modulus x less than 1, other case is x equal to 0, it can happen, that converges only on a point and nowhere else. In the third case, I got, that it converges for all x, so except a point it is always an interval, so let us go to the next theorem, which will explain that, why such a thing is happening.

That means, if convergence happens at nonzero points, it is always happening in some interval. So, the result is this, suppose summation n from 0 to infinity, an x to the power n converges, for x naught, the particular point, then the series converges absolutely, for any x, such that mod x is less than, so what does this mean.

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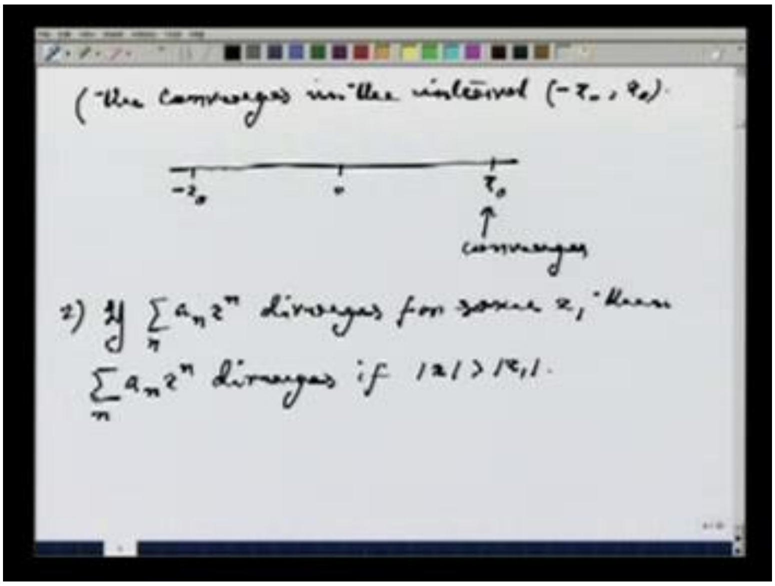

It means that the series converges in the interval, minus x naught, x naught, I am assuming my x naught is positive, so minus x naught is negative or it is the other way around. If your x naught is negative, that can be considered as minus x naught, so x naught is positive in that, minus not, x naught is positive in that case, but what is happening is, that whenever you have some point. So in picture, what it means is this, let us say 0 here and x naught here and suppose the power series converges at this point. Then immediately, what you know is, look at minus x naught which is here, then every point between minus x naught and x naught.

Whatever point you choose between minus x naught and x naught, the power series converges not only that, it converges absolutely. That is the reason, why I am always getting intervals as the domains of convergence that means, I am always getting intervals of x s from which, the power series converges. Now, there is a second part, it says, that if summation over n, an x to the power n diverges for some x 1.

Then summation over n an x to the power n, diverges if mod x is bigger than mod x 1, so once you get one point where it diverges, then quickly what could you do is, look at the interval with the end points of that point and minus of that point. That is, if the point is x 1, where the power series does not converge, look at the interval minus x 1 and x 1, look at this closed interval, outside that closed interval, certainly the power series going to be diverged, that is what it says. So, let us come to the proof of this.

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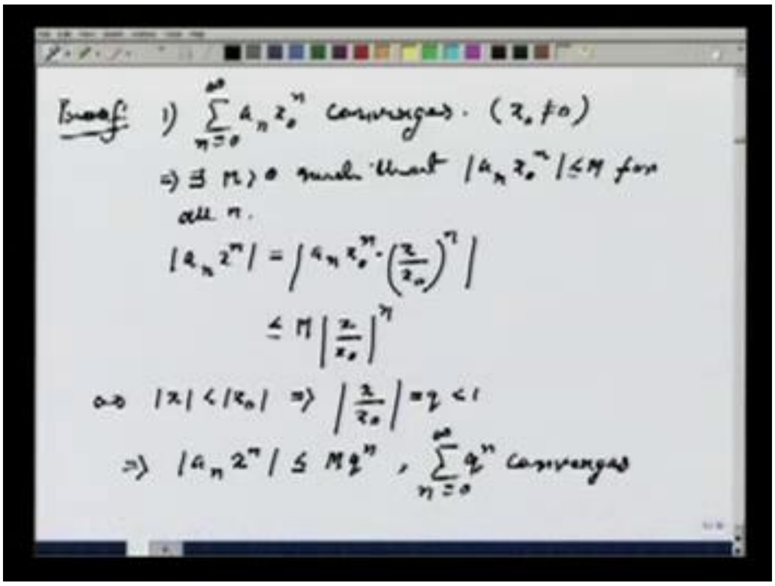

So, first I try to prove 1, so what I know, I know that summation n from 0 to infinity, an x naught to the power n converges, that is my hypothesis and if, an infinite series converges then, it is term has to be bounded, because I know that after some stage all the terms go to 0 and before that stage, there are any way finitely many terms. So, the sequence of terms, of the convergent infinite series is always a bounded set, so this implies, there exist some M bigger than 0, such that mod an, x naught to the power n is less than or equal to M, for all n.

Now, I am going to use the simple trick, I look at modulus of an, x to the power n, let me assume, that x naught is nonzero, because if I assume x naught is equal to 0, then the theorem is trivial. So, I assume, x naught is not equal to 0, because if x naught is equal to 0, then mod x is less than modulus of x naught means, x is also 0, then the power series obviously, converges. So, the point is, if x naught nonzero, then what happens, so I look at an x to the power n, I write this as modulus of an, x naught to the power n into x by x naught, whole to the power n.

If I separate the modulus, it would give me using the previous inequality, that modulus of an x naught to the power n is always bounded, it is lesser equal to m times, mod x by x naught whole to the power n. Now notice, as modulus x is less than modulus of x naught, this implies, modulus of x by x naught, if you allow me, I will call it r, let me call it q, instead of r. This is strictly less than 1 and that means, what I got is, that mod an times x to the power n is less than or equal to M into q to the power n.

Now, as q is less than 1, I know, that summation n from 0 to infinity, q to the power n converges, being the geometric series. If I have a geometric series, where the common ratio is less than 1, then the geometric series converges, that we have already seen, using that, let them follows by comparison test.

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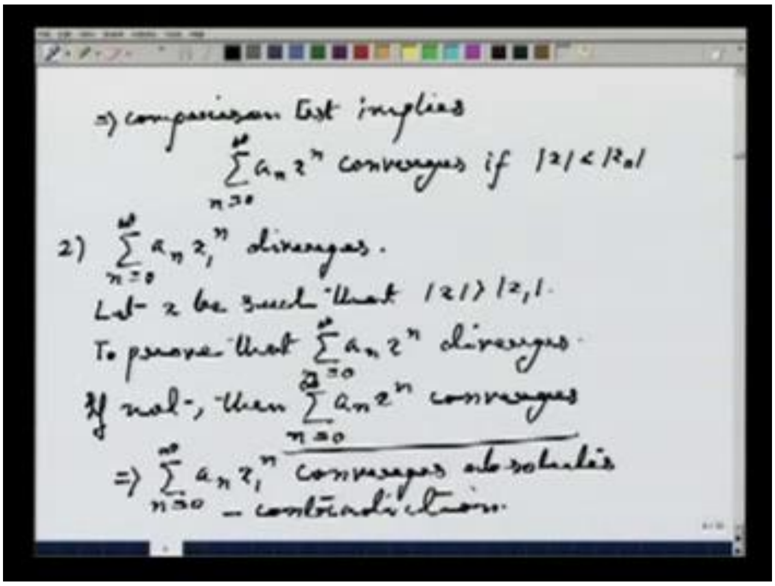

This implies by comparison test, that summation n from 0 to infinity, an x to the power converges, if modulus of x is strictly less than modulus of x naught. Now, let us go to the part 2, what it says, that if at some point x 1 it diverges, then mod x is bigger than modulus of x 1, the power series diverges. So, let me write down the hypothesis first, the given hypothesis is summation n, from 0 to infinity, an x 1 to the power n diverges, let me choose an x, let x be such that, that modulus of x is bigger than modulus of x 1.

I want to prove, that summation an x to the power n diverges, to prove that summation n from 0 to infinity, an x to the power n diverges. This, I need to prove, if not, when summation n from 0 to infinity, an x to the power n converges, but notice that modulus of x 1 is less than modulus of x and I know, that summation n from 0 to infinity, an x to the power n converges, then I apply the first part of the result. Since, modulus of x 1 is less than modulus of x and summation an x to the power n converges.

By the first part, this implies that summation n from 0 to infinity, an x 1 to the power n converges absolutely, in particular that means, it converges, but my hypothesis says that summation an x to the power n diverges. So, that is a contradiction and this contradiction happens, because I have assumed, that n from 0 to infinity an x 1 to the power n converges, that cannot be true. That is why the contradiction happens and that suddenly proves our result.

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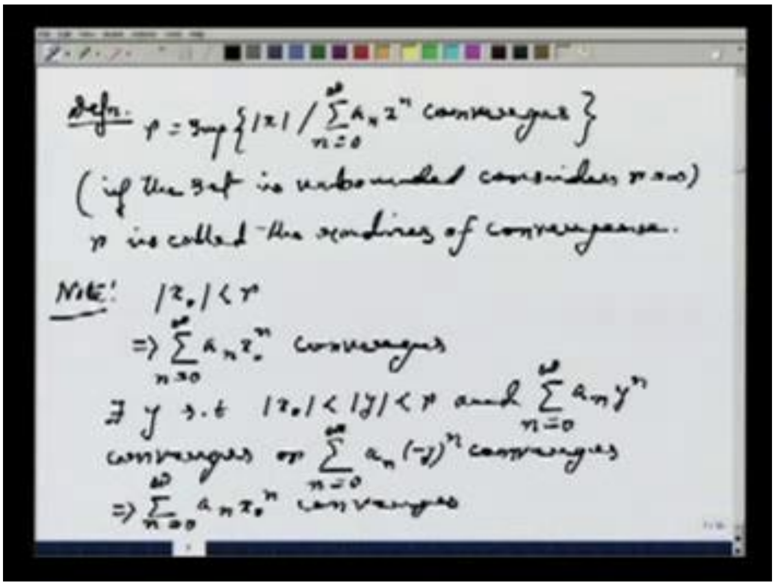

Now, I am going to define a concept, which is the most fundamental one, in power series that the radius of convergence. So, we say, let us look at all the x, for which the power series converges, so I am going to look at mod x, such that n from 0 to infinity, an x 1 to the power n converges. Notice that, it can happen that a power series, an x to the power n, it converges for x, but it does not converge for minus x, that can happen, we have see some such an example.

I look at all the xs for which x to the power n converges and then I look at the modulus and then I look at the supreme over this, I call it r. Now, notice here, this set is an unbounded set, so if the set is unbounded, then I take my supreme of the equal to, infinity. So, let me note it here, that if the set is unbounded, that can also happen, consider r equal to infinity, this r is called the radius of convergence, now the very definition implies certain things.

Assume, suppose, I will call it a note, suppose mod of x naught is less than r, then I want to show, that this implies, summation n from 0 to infinity, an x to the power, x naught to the power n converges. How does this follow, since modulus of x naught is less than r, I have, there exist y in the set, such that modulus of x naught less than modulus y, less than r by the definition of supremum and summation n from 0 to infinity, an y to the power n converges or an minus y to the power n converges, one of this has to be true.

But then, apply the condition 1 of the previous theorem, that modulus of x naught is less than modulus of y and summation an y to the power n converges or summation an minus y to the power n converges, wherever converges, I will consider it as y and modulus of x is less than modulus of y. So, summation an x naught to the power n also converges, by one of the previous theorem, so this implies, summation n from 0 to infinity, an x naught to the power n converges.

In fact, I am writing it, this minus y to power n for extra clarity, if I just write, there exist y, such that modulus of x naught is strictly less than. Notice here, for extra clarity I am writing minus y to the power n, but minus y actually can be considered also as y, what is the statement of there, it says that, there exist in y, such that modulus of x naught is less than modulus of y, which is strictly less than r and summation an y to the power n converges.

That can as well apply to the minus 1 also, if it is y or minus y, I can just choose that one of them. So, any way it follows that if I take a point x naught, whose modulus is less than the radius of convergence, then the power series suddenly converges there.

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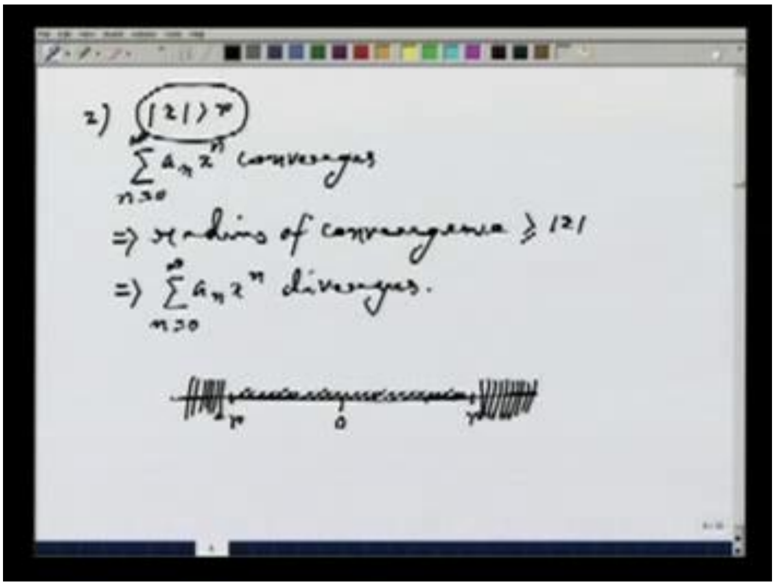

The next observation is, what happens, if modulus of x is bigger than r, that is, it is bigger than the supremum, suppose summation n from 0 to infinity, an x to the power n converges, if this happens, then by the definition of radius of convergence. I will get that, this implies that the radius of convergence must be bigger than or equal to mod x, because radius of convergence is supremum of all the mod xs for which summation an x to the power n converges.

So, that it would mean, since mod x is 1 such, I already have an x, for which an x to the power n converges, the radius of convergence must be bigger than or equal to mod x, but I already said, that the radius of convergence is less than mod x. This implies, an x to the power n, then diverges, so fairly clear picture is coming out, so I draw the real line, this is 0. Since, r is the supreme of modulus of x s, it is a positive number, well it is nonnegative, it cannot always strictly positive, I look at r then somewhere, here is minus r.

Then, at all the points inside, the power series converges and all the points after r, here the power series diverges, here also the power series diverges, only thing we did not talk about is, is what happens at r or minus r. I draw the real line, this is 0. Since, r is the supreme of modulus of x s, it is a positive number, well it is nonnegative, it cannot always strictly positive, I look at r then somewhere, here is minus r.

Then, at all the points inside, the power series converges and all the points after r, here the power series diverges, here also the power series diverges, only thing we did not talk about is, is what happens at r or minus r. For that I will look at the examples, so look at this example.

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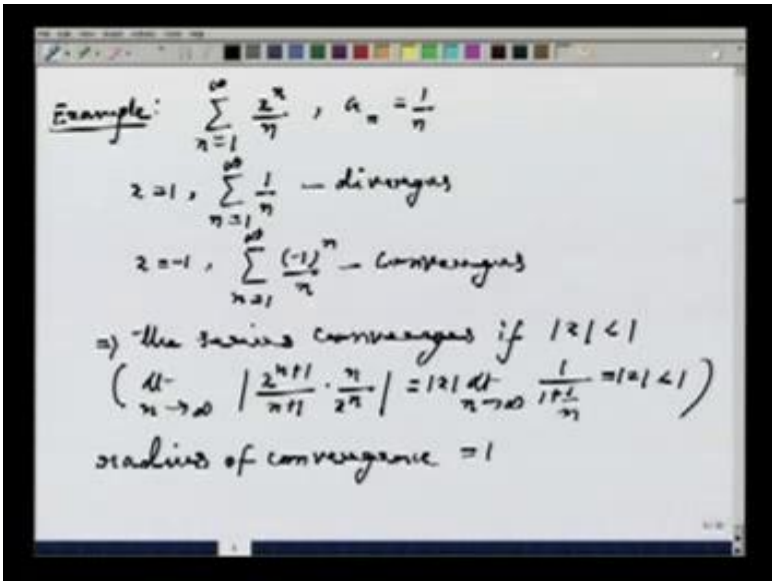

Now, let us consider this power series, summation n from 1 to infinity, x to the power n divided by n. That is, here an is just 1 by n, we like to find out the radius of convergence or we like to investigate the behavior of the power series at r and of course, minus r. So, first notice, that if x is equal to 1, then the series is summation n from 1 to infinity, 1 by n, which diverges and if I take x to be equal to minus 1, then summation n from 1 to infinity, I get minus 1 to the power n by n, which by Leibnitz test convergence, because this is an alternating series.

Then, if I apply my previous theorem, what should it tell me, what should be the radius of convergence. You see, what is the supremum of the x s for which it converges, now if I take any number x, whose modulus is bigger than 1, there, the series cannot converge, because at x equal to 1 it diverges. So, all the x s follows the series, can converge by the previous theorem must satisfy, modulus of x is less than 1, so this implies the series converges, if modulus of x is less than 1, that is by the previous theorem.

That means theorem, the first theorem 1 and 2, which I am applying, now if you do not believe this, what we do is, you just go for ratio test. You just look at limit n going to infinity, modulus of x to the power n plus 1, divided by n plus 1 into n divided by x to the power n, that would give me, modulus of x times limit n going to infinity, 1 by 1 plus, 1 by n which is certainly mod x. So, this less than 1, implies the series converges by ratio test, which I already have.

So that means, radius of convergence is equal to 1, but notice at the radius of convergence, that is when r is equal to 1, the series diverges, but r is equal to minus 1. That means, minus r, there the series converges. So, the situation, what is coming out is precisely like this.

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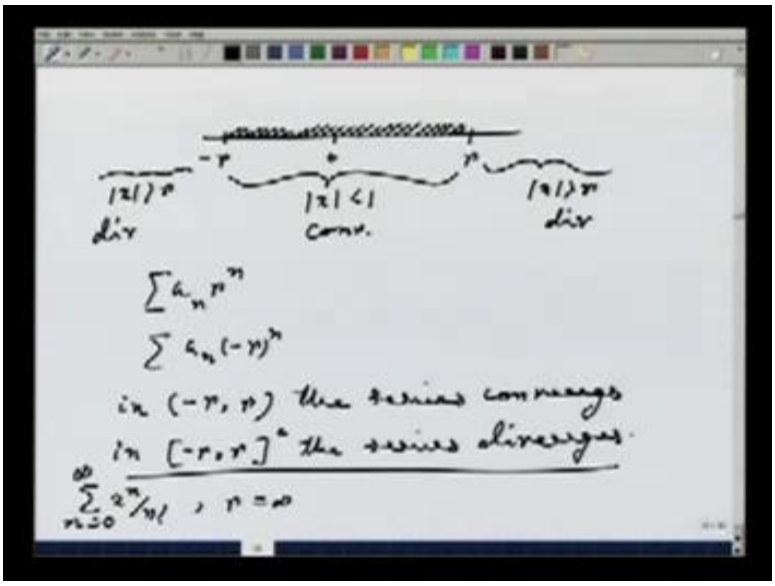

This is 0, this is r, this is minus r, so the possibilities are this, certainly inside the series converges, no problem about that, this is the region mod x less than r, outside I have mod x bigger than r. So, this is where divergence happens, here convergence, now it can happen, now anything can happen for this quantities, an r to the power n and an minus r to the power n, it can converge, it can diverge, that means, both of them can diverge, both of them can converge, one of them can converge, others can diverge.

So, I gave an example of x to the power n by n, there r was equal to 1, for that I have seen, that for r equals to 1 the series diverges, but for minus r it converges. So, at the radius of convergence, anything can happen, you have no hold on that situation, series can converge, it can diverge and there are two ends, there is an interval after all, but what you know is, that inside that open interval, the series always converges and outside that open interval, outside that closed interval, I should be careful here, outside the closed interval minus r, r, outside that the series is always diverges. So, I will note it here, that in minus r, r, the series converges and in the complement of the set minus r, r, if that makes sense, the series diverges.

Notice that, the second situation it does not occur, if you know your r to be equal infinity. So for example, if I look at the series, x to the power n by factorial n, as I have observed already n from 0 to infinity, in this case, r is infinity, that means it converges for all x. So, if r is equal to infinity, then the second situation does not make sense, so with this, we finish our discussion, on the elementary properties of power series. Now, you are going to tell you certain methods of how to find the radius of convergence of the power series. So, now let us try to see how to find the r.

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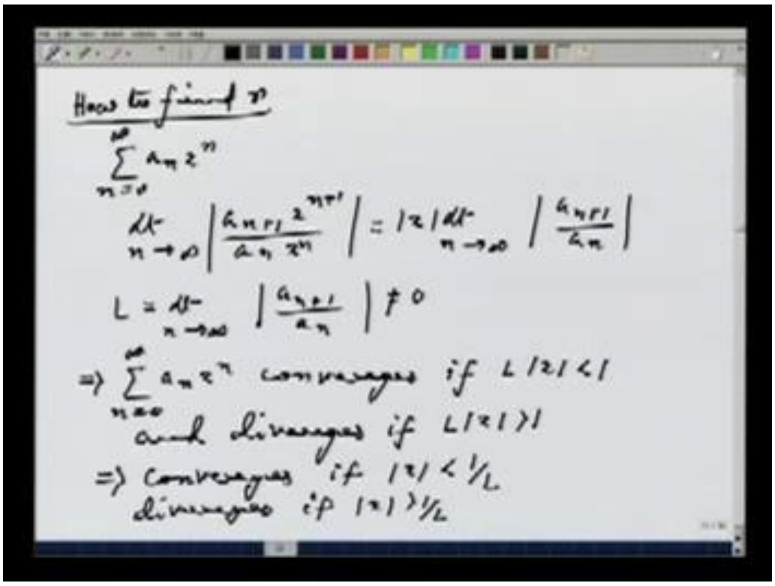

I will tell you one method of finding r, it will work in most of the cases, but suddenly there are other cases, where it will not work. So, I look at the series, summation n from 0 to infinity, an x to the power n, the idea is just use somehow the ratio test, so I look at the quantity, limit n going to infinity, an plus 1 x to the power n divided by an x to the power sorry, plus 1 here, x to the power n modulus of that. So, this is modulus of x times, limit n going to infinity, modulus of an plus 1 divided by an. I said in most of the cases it will work, what I meant is, that you know all these ns are nonzero, because if there are infinitely many ns, for which a n s are 0, then it would be difficult to apply this test. For example, you cannot directly apply this test, if you know, that a of 2 n is 0 for all n; that means, all the even position we have 0, so you cannot divide, you can apply this test, if you know that after some stage all the terms are nonzero, then it will go throw, but otherwise, not.

In that case, we have to use certain tricks, any way let us say, suppose all the ns are non zero, then I look at this limit and let us say L is this limit, L is limit n going to infinity, modulus of an plus 1, divided by an, assume that this is non 0, suppose so. Then, this implies, that summation n from 0 to infinity, an x to the power n converges, if L times mod x is strictly less than 1, this is by ratio test and diverges, if L times mod x is strictly bigger than 1. That implies that the series converges, if mod x is less than 1 by L and diverges, if mod x is bigger than 1 by L.

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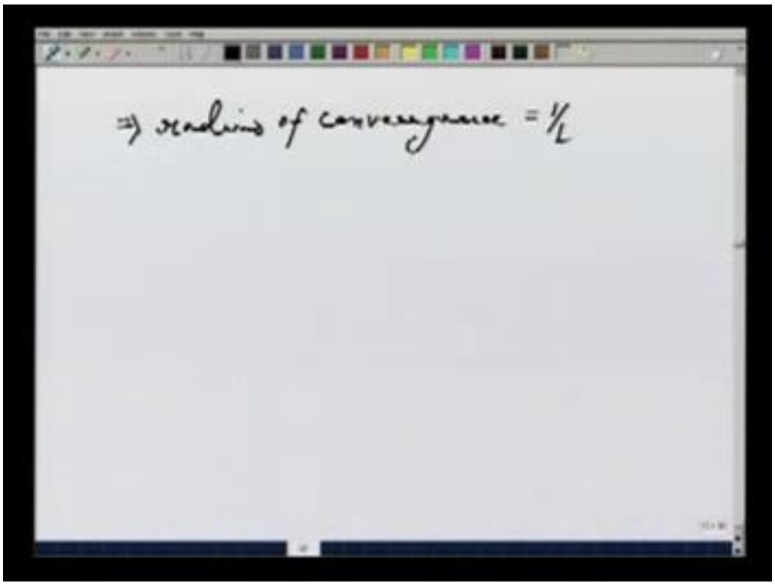

That certainly implies, by the definition, coupled with the theorem 1, that radius of convergence is equals to 1 by L. So, let us again look back, ((Refer Time: 38:02)) if you have a power series, for all the a ns are nonzero, look at limit n going to infinity, modulus of an plus 1 by 1 an, if that limit is nonzero, then 1 by that limit is the radius of convergence. The problem happens is, some of this a n s are 0 and this limit does not exist, those are complicated power series for which this test will fail.

But there is an analogue of this result for, which we have done here, using the root test, in most of the cases that will work, but in most of the practical situations, you will find out, that it is the ratio test analogue of finding the radius of convergence, which works, which is precisely what I am described here. That is you look at the coefficients form the terms, an plus 1 by an, look at the limit, whatever you get, if that is nonzero, look at the reciprocal of that, that gives you the radius of convergence.

Now, equate to the knowledge of power series, I am going to talk about Taylor's series, which is a power series, which we already have encountered with. Let us talk about the convergence of Taylor's series, so let us recall first about Taylor polynomial of a function.

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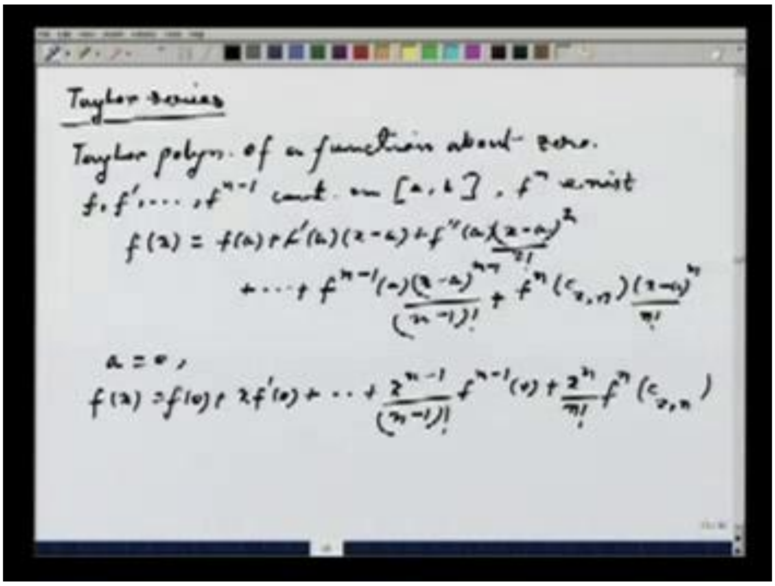

Around 0, let us say, it is usually called the Maclaurin polynomial, about zero, what is that, that is we have proved that if, f, f prime, up to f n minus 1, these are all continuous on a, b and f n exists. Then I can write, f x equal to f a, plus f prime a into x minus a , plus f double prime a x minus a whole square by factorial 2 plus f n minus 1 a x minus a power n minus 1 by n minus 1 factorial plus the last term, that is f n, instead of a, here we have c, which certainly depends on x.

It will also depend on n times, x minus a power n by factorial n, this is the Taylor's polynomial about a. So, put a equal to 0, this is called the Maclaurin polynomial or the Taylor polynomial for simplicity. So, in that case what happens is either that f x equal to f 0, plus f prime a, times x plus f double prime, sorry f prime 0 times x plus f double prime 0 times x square by factorial 2 and all that.

So, equal to 0 means, f x is equal to f 0, plus x f prime 0, plus x to the power n minus 1 by n minus 1 factorial into f n minus 1 0, plus x to the power n by factorial n into f n, at c x n., so the point c which we are getting here, it depends on x as well as n. Now, suppose f is a function, for which all possible derivative exist.

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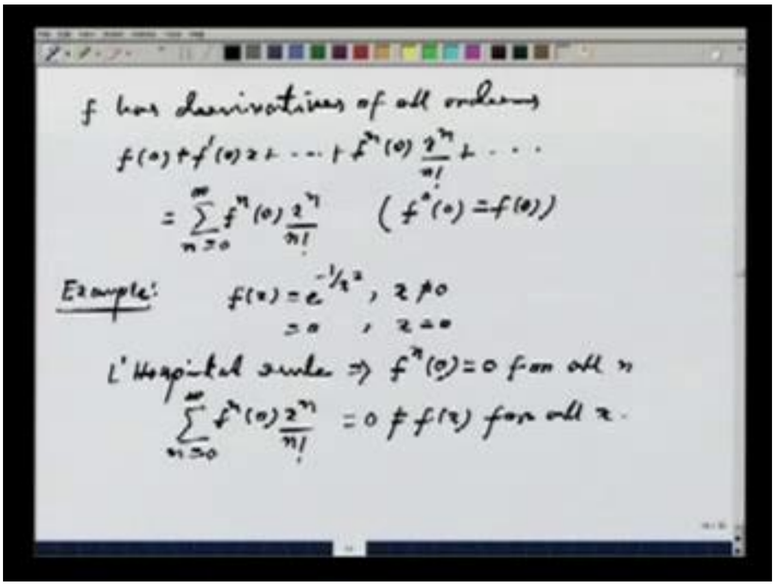

Suppose, the situation is f has derivatives of all orders, then if you just go on writing the Taylor polynomial, which you are getting for all n, you immediately get a power series, because what you are going to get is, f 0 plus f prime 0 at x, I can look at all derivatives, so it will go on, plus f n 0 times x to the power n, by factorial n and so on. So, in short I get summation n from 0 to infinity, f n 0 into x to the power n by factorial n, with the understanding, that if 0, 0 is just exide.

So, this is the power series, now you can suddenly ask, for which x, this power series converges and if it does, does it converge to the function. Well, the bad news is, that there are functions, for which this series makes sense, that for all x it converges, but it does not converge to the function that can happen, but for most of the function, which are good function, for that we will see, that many of those functions, this Taylor series exist and it is equal to the function.

So, first the bad thing, so we can look at an example, it is kind of a difficult example to conceive, but it is true, that I look at this function, fx equals to e to the power minus, 1 by x square, when x is nonzero and it is 0 and x is equal to 0. What happens is, if we use L’Hospital, this implies, that f prime that derivative of f at the point 0 exist, so whatever derivative of f you look at, it 0, at 0 it exist. That is, f n 0 exist, not only that it is always equal to 0.

This is true for all n, this is not very difficult to show that happens, so this function has the property, it has derivatives of all order existing at the point 0 and at the point 0, all those derivatives are 0. So, if you look at then, the Taylor’s series, that is summation n from 0 to infinity, f n 0 into x to the power n by factorial n, this is always equals to 0, because all the f n 0s are 0.

So, it cannot be equal to f x for all x, because the function is nonzero, so if you have a function, so this is a particular example of a kind of a function, which has the property, that it has the derivatives of all order, but at the point 0, all it is derivatives are 0. Then, for those kind of functions, you can always write down the Taylor’s series around 0, but it does not make much sense, you will always get 0, that is not the function f x, but for certain other functions, the Taylor’s series exists. I will show some such example, look at the function.

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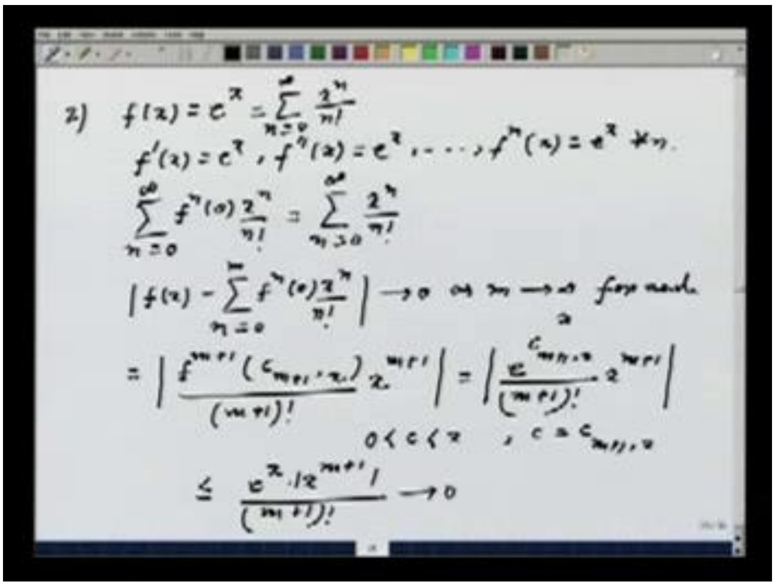

The most famous one, f x equals to e to the power x, it has derivatives of all order and f prime, at x is e to the power x, f double prime at x is e to the power x, it is always e to the power x, for all n. Then, what does the Taylor's series look like, summation n from 0 to infinity, f n 0 into x to the power n, by factorial n, you can see from the previous calculation, that f n 0 is e to the power 0, which is 1. I get, summation n from 0 to infinity, x to the power n, by factorial n.

This power series we have encountered before, the radius of convergence of this power series you know, is actually equal to infinity, so for every x it converges. Now I want to show, that this converges to e to the power x. So, what is the idea, you look at modulus of f x, minus summation n from 0 to m, f n 0 x to the power n by factorial n modulus, what I want to show is, that I want to show, that this goes to 0 as m goes to infinity. For each x, that will prove, that the function f x is equal to the power series.

Now, what to do with this inside thing, f x minus this summation, n from 0 to m mean, I know from Taylor's theorem, this is nothing but, modulus f of m plus 1, at a point c m plus 1, x. The point depends on x, as well as m plus 1 divided by, into x to the power m plus 1, where x is fixed. Now, if I write down all the quantities here, what I get is, e to the power c, m plus 1 x, divided by m plus 1 factorial, because all the derivatives are e to the power x.

So, at the point c m plus 1 comma x, it is e to the power c m plus 1 comma x, times x to the power m plus 1, but on with, on c we have the property, that 0 less c, less x, well this c is c m plus 1 comma x. So, that means this whole quantity is lesser equal to, e to the power x, into x to the power m plus 1, divided by m plus 1 factorial. Now this is, something which we have seen before that as m goes to infinity, these goes to 0, that means, the function f x equal to e to the power x is represented by it is power series. So, now I can finally write, that this is summation n from 0 to infinity x to the power n by factorial n. So, in general if you want to show, the Taylor's series of a function at a point x, converges to the function, all you have to do is to prove.

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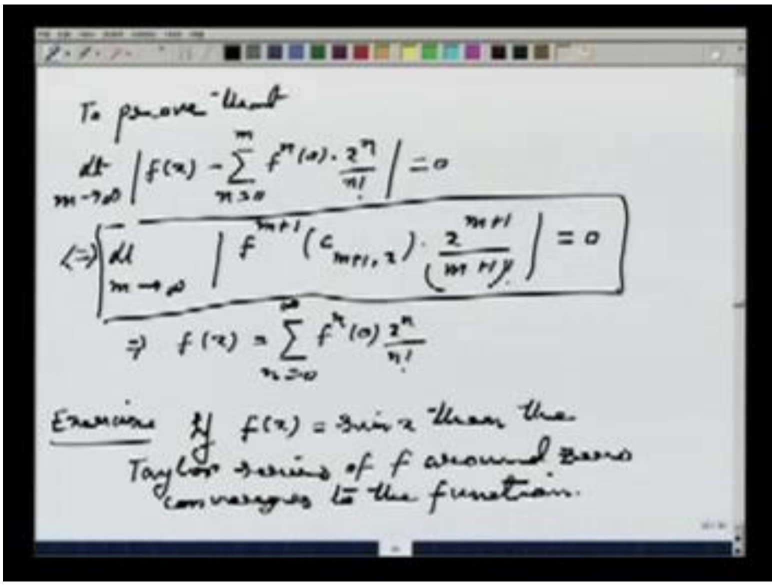

That, modulus of f x minus summation n from 0 to m, f n 0 into x to the power n by factorial n, limit n going to infinity is equal to 0. That is, in other words we have to prove, that limit m going to infinity, modulus of f m plus 1, at c m plus 1 comma x times, x to the power m plus 1 divided by m plus 1 factorial, this is by using Taylor's theorem, at this quantity is equal to 0. If this happens, then at the point x, the function is given by Taylor series.

That is you can write then, that this implies f x equals to, summation n from 0 to infinity, f n 0. If, f has differentiate derivatives of all orders, of course x to the power n by factorial n, so this is the method, we have done it for x equals to e to the power x. Now, I will leave it as an exercise to check, that if f x equals to sin x, then the Taylor's series of f around 0, converges to the function, so to prove this, all we have to prove is, whether this is true, should not be very difficult, because f m plus 1 you can find out, given the function, sin and then try to estimate and see look at the limit goes to 0. So, with this we finish our discussion on power series and Taylor's series.