Mathematics - I
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Lecture - 17
Riemann Integral

Today we are going to start with Riemann Integration. You perhaps, already know this, as the definite integral. Today, we are going to start with the theoretical construction of this integration. It was first done by the German mathematician, Bernhard Riemann. That is why, this integration called Riemann integration. Now, historically speaking, the theory of integration precedes the theory of differentiation. It was actually known to the Ancient Greek and it is essentially their method, which Riemann analytically established.

So, first, we will start with the usual technique of integrating as conceived by the Greeks. It is essentially to find out areas of certain regions, in terms of the areas of certain regions, which we already know. For example, we know, what you mean by area of triangle? We know, what is area of rectangle? But, using this, can we find area, let us of a disk. That is what the Ancient Greek has done. So, let me explain for that to you in the modern language.

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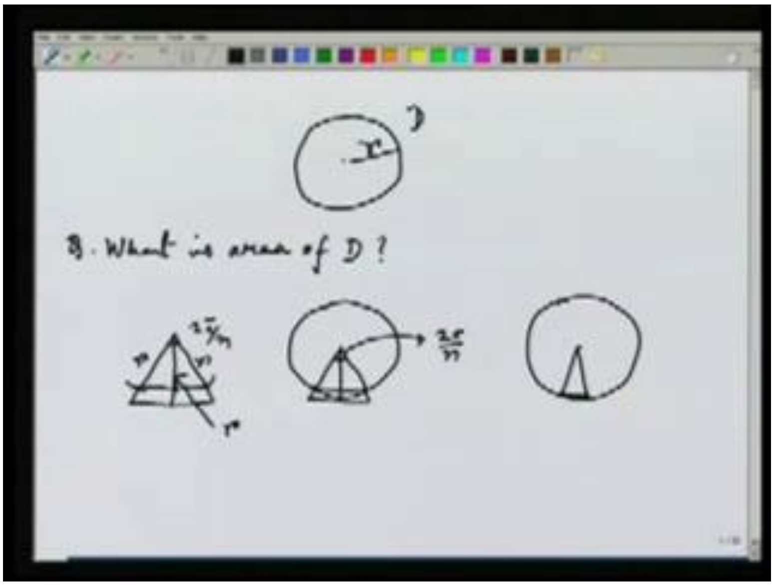

So, the problem is this. I look at a disk, whose radius is r, so this distance is r, call this disk D. Question is what is area of D? When you ask this question, surprisingly it makes sense to everybody, that we can try to find the area of the disk D. But, the question is, it has not been defined a theory, that what exactly is area of a disk. What is the definition of the area? We will see we will proceed intuitively and see that the Riemann integration, actually defines the notion of area for us also.

It is, not only calculating areas, it defines the area also. So, we follow the Greek method, which goes as follows. I draw a bigger picture of the disk and then, this is the center. And what I do is, I look at this triangle, where this angle, let us say is 2 pi by n. That is, I am actually trying to write down this disk as the union of triangle. Where, the top angle is 2 pi by n. So, I am going to use n many triangles. And obviously, I can see from the picture, it is not going to fill up whole disk, some portion will be left.

But, notice just one thing, if I take l bigger N, that means, l lesser 2 pi by N. Then, the picture would look something like this. And then, I can go on increasing the N's and perhaps, these triangles, which I am drawing, it is getting almost is equal to the disk. But, we are not satisfied by just this. What I will do is, I will draw another triangle, which is slightly bigger. It is this.

So, separately the picture looks like, first triangle is this, where the top angle is 2 pi by n. This is 2 pi by n and then, there is slightly bigger triangle, with the same angle. I know this height is r, so is this side. And if I draw this line in the picture, here it is the this line, I know this is also r up to this is r up to this is r, and this whole thing is also r. Now, I want to find the areas of the smaller triangle and the bigger triangle.

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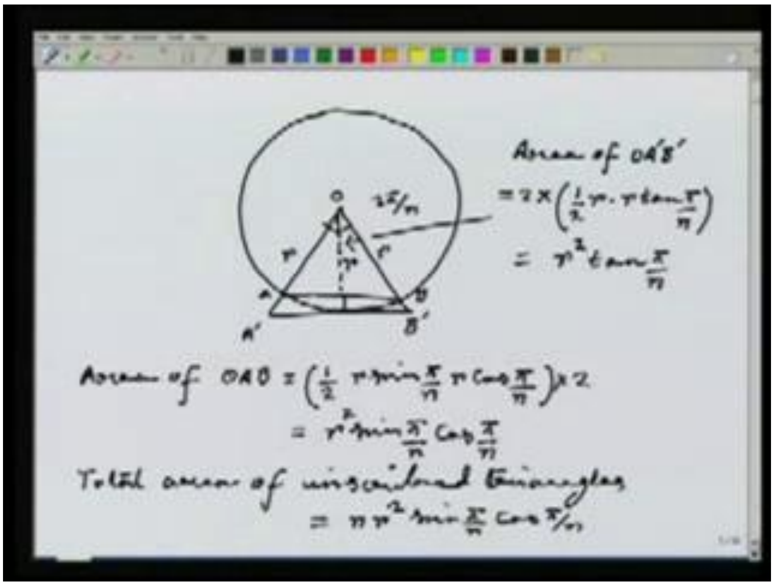

So, I will draw another picture of this. Now, I write as a bigger disk, this is the center; I have this smaller triangle, which is the inscribed 1. And then, I have a bigger triangle, which is super scribed 1. This line I know as length r, this line has the length r, this line has the length r, this side and whole angle is 2 pi by n. Now, if I try to find out, first the area of the smaller triangle. So, let me given a name O, A, B and the bigger triangle is O, A prime, B prime.

So, first try to calculate the area of O, A, B. What is that area? Well, the half angle in the top is certainly pi by n. And then, I know this length is r. That is the hypotenuse. So, the area is, half times the base. The base is r sin pi by n times the altitude, what is the altitude here, well it is just r cos pi by n. So, half times sin pi by n times r cos pi by n. Now, how many such triangles are there? But, I have two triangles. So, the area of the O, A, B is, actually what I have written as half, is actually half of O, A, B.

So, area of O, A, B is this times 2. So, it is r sin pi by n times cosine pi by n, well r square. Now, how many such triangles are there, n many? Because, I am looking at the angle 2 pi by n the whole angle is 2 pi. So, the total area of inscribed triangle is equal to n times r square sin pi by n into cosine pi by n. Now, I will concentrate on the area of the bigger triangle, that is O, A prime, B prime. Let us see what is the area there, I know length of this line r, this is r, that I know. And then, I know angles, those are pi by n's.

So, again it is half base into altitude. So, let see area of O, A prime B prime. That is equal to 2 into half. In any case, I know the altitude here. So, it is r into, now the base, well the base is clearly r tan pi by n. So, the total area trans out to be, in this case r square tan pi by n. Then what is the total area of the superscribed triangle.

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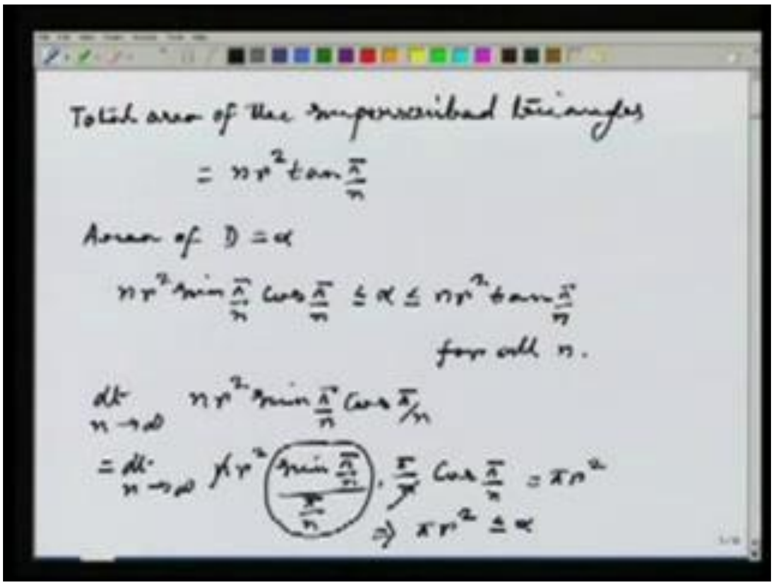

So, total area of the superscribed triangle is equal to n times r square n pi by n. Now, let us assume that area of D is equal to alpha. Our intuition suggests that if such some alpha exists, then this should be true. That n r square sin pi by n into cosine pi by n is lesser equal to alpha. Because, I am looking at the left hand side represents the total area of the inscribed triangle. So, the total area is certainly smaller than the total area of the disk. So, it is lesser equal to…

So, on the other hand alpha is lesser equal to n r square tan pi by n. Because, now the right hand side represents the superscribed triangles. So, the total area must be bigger than the area of D. Now, this is true for all n. As I said that if I go on taking larger n, larger and larger n, it looks like that triangles are almost filling up the whole disk. So, I want to take the limit as n goes to infinity and see what happens?

But, whatever the limit is, inside both the limits, the alpha remains, alpha still stands between both the limits. So, once I take the limit, what is this quantity, limit n going to infinity n r square sin pi by n times cosine pi by n? I just write it in the form, limit n going to infinity n r square sin pi by n divided by pi by n into pi by n times cosine pi by n. Now, these two n's cancels each other.

Notice that this portion as n goes to infinity has the limit 1. Because, pi by n goes to 0 and cos pi by n, then goes to 1. So, the next result is pi r square. So, one thing that certainly clear, that this implies, that pi r square is lesser equal to alpha. Now, let us look at the right hand side.

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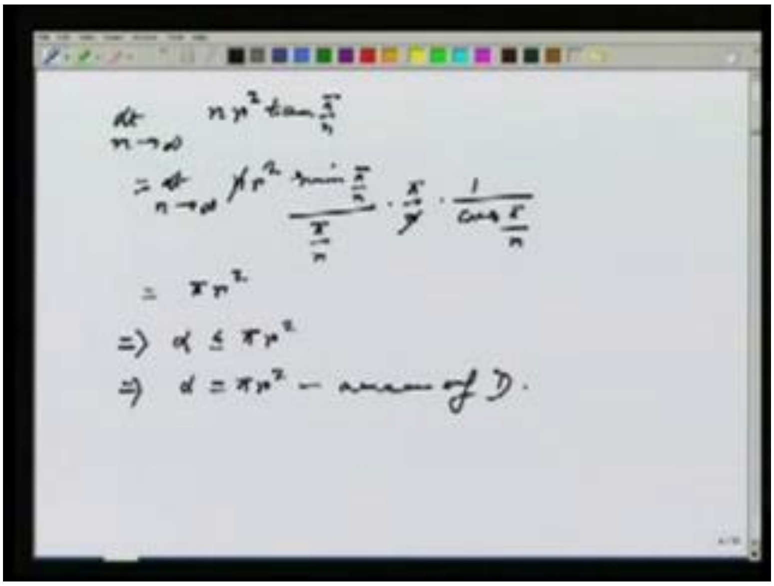

That what is this limit? Limit n going to infinity, n r square tan pi by n. Well, I write in the following form, limit n going to infinity n r square sin pi by n divided by pi by n into pi by n into 1 by cosine pi by n. As in the previous case, these two n's cancels each other. And what I get is, pi r square. So, this implies now, that alpha is lesser equal to pi r square. These two together imply then, that alpha is equal to pi r square, which we know this is the radius of the disk of radius r. So, this we call the area of D.

But, there are certain hypothesis, we are using classically here. Number 1; that the disk D has something called area, which has been guaranteed. Second is this area, satisfies some kind of inequalities, that the inscribed triangle had less area, than disk, which looks bigger. And disk has lesser than the superscribed triangle, which looks bigger, this we are assuming. Under this assumption, which is natural assumption, we will like our area to possess, these properties. It comes out that the area of a disk of a radius r is pi r square.

So, whatever we have done is variation mathematics, done before cries by the Greek. Now, let us see what Riemann did to analytically establish, this technique in the present mathematical frame work. So, we start with the following. Again, I will first try to informally tell you, what exactly Riemann's ideas doing integration.

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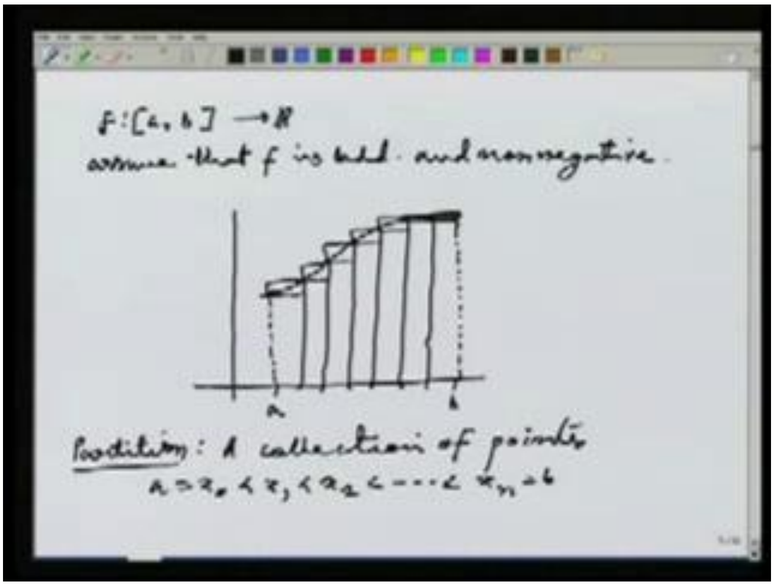

For that let us say, F is a function on the closed interval a, b to R. Assume that f is bounded and non negative. Let us assume, I can draw the graph of the function f. So, these are the axes, this is my interval a, b. Let us say, the function look like, very simple kind of function I have taken. What I want is, let us draw the vertical lines first. What I want is, I want to find the area between the enclosed regions, I want to find this area.

Now, notice, that has not been predefined, but as in the disk case. If we follow our intuition, some time area comes out. So, it try to do exactly same thing here. So, for that I need certain things to define first. First, I define something called partition. Partition, just means, a collection of points x 0, x 1, x 2 and so on, up to x n equal to b. That is, it is the collection of points between a and b.

So, what I do is, I mark them. Let us say, these are the partition points. At this partition points, what I do is, I first draw vertical lines and till intersect the graph of the function. I go on doing this at each of the partition points. They look like rectangle, but you can see one side of the rectangle is not it really straight line, it is curved line, because it is on the function side.

Now, I start from this line and draw this, then I draw this, then I draw this and I go on doing this. You see that I get certain rectangles, but these rectangles are below the functions. If I found out the area of these rectangles and add them together, what does our intuition say? This added sum of areas of the rectangles is less than the area of the region below the graph of the function. That is intuitively very clear, because rectangles are inscribed inside.

Similarly, I can draw some superscribed rectangles also; they are drawn in this fashion. So, took the higher point on the curve of the graph of the function and draw the bigger rectangles. And then if I find out the area of the bigger rectangles and sum them up. It is certainly going to be bigger than the area of the region, under the graph of the function. Now, how to I draw this rectangles. That is the next questions.

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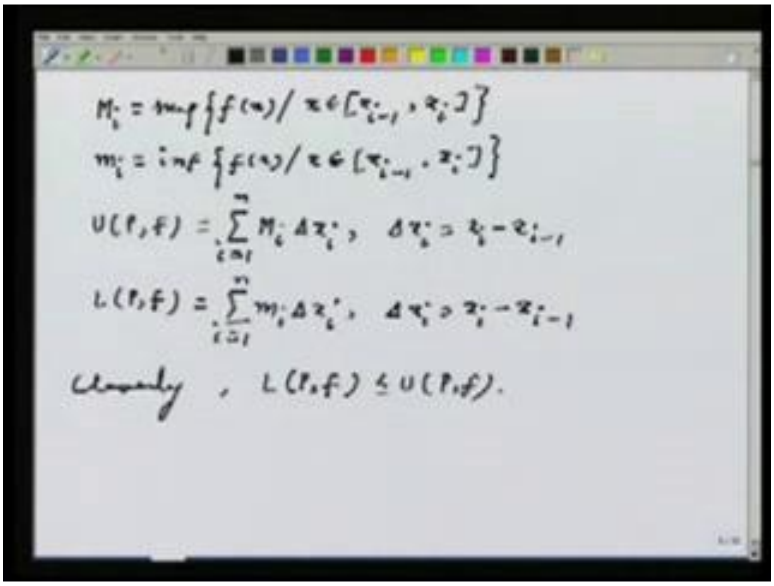

So, I define something like this. I define capital M i is equal to supremum of f x, when x lies between x i minus 1 and x i. Since, the supremum depends on i, I called it M i. Similarly, I can define little m i. That is infimum of f x, when lies between x i minus 1 and x i. Notice that if I draw a rectangle, whose base is the interval x i minus 1 x i and whose height is little m i. Then, I get a rectangle, which lies the below the graph of the function. It just touches the function at one point, that is at little m i

If I draw a rectangle with base x i minus 1 and x i interval and whose height capital is M I, then rectangle goes above the graph of the function. Now, I define U, p, f equals to summation i from 1 to n, capital M i times delta x i, where delta x i is the length of the base with interval x i minus 1 and x i. That is, it is x i and x i minus 1. And I define L p, f is equal to summation i from 1 to n, little m i times delta x i. Where, delta x i is same, it is x i minus x i 1. From the construction, one thing is very clear, L p f is less or equal to U p f. This is simply because, L p f uses the infimum and L p f uses the supremum.

It just means, if you notice that what is U p f. It is just some of areas of the rectangles, which super scribe the region under the curve. Because, each terms capital M i times delta x i is height time's base. It is area of the rectangle and then, I summing them up. So, the partition is actually gives me, the base of the rectangle, which I am choosing and then the M i gives me the height. Then M i times delta x i gives me the area of the single rectangle. And then, I sum them up, over all, the base which I have chosen, similarly L p f.

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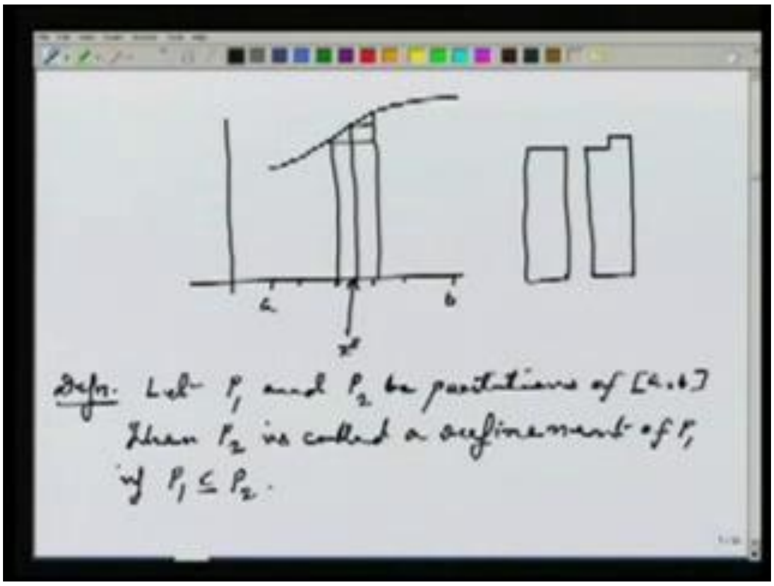

Now, comes the next observation, which is again clear from the picture. Let us again draw picture of the function, which I had taken earlier. This is a, this is b, my function look something like this. Suppose, I mark the partition points as earlier, now I add some extra points. Let us say here, I choose another point x star. So, original L p f rectangle was this, this was my original rectangle.

Now, if I assume that the x star is also the point of the same partition, let us assume that. Then, I should have done this; I should have drawn this line and then, this line. So, the earlier rectangle, which I was working with it, is this. And now, the new rectangle which I am getting it looks exactly like this. You can very easily see now, that by putting an extra points, what I have achieved is that the new rectangle, which I am getting. They are giving a bigger area.

So, mathematically, what I drawn so far, can be explained in the following. I first, define some called refinement of a partition. Let P 1 and P 2 be partitions of a, b. Then, P 2 is called a refinement of P 1. If P 1 is a subset of P 2, it means, what it just means the given the partition P 1, I have inserted some more points in the partition to create P 2. So, P 2 is actually breaking the intervals in a more refined fashion than P 1. That is why; it is called the refinement of P 1.

So, one thing is very clear from the definition, that P 2, after all any partition is a collection of points. So, P 1 is also a collection of points, so is p 2. But, P 2 has more points in it than P 1 as a set, not just number of points. All points, which appearing P 1, they also appearing P 2, but P 2 has more points perhaps. That is why; P 2 is called refinement.

Now, the basic question, which we are going to ask ourselves, that given any partition P, I can talk about L p f. So, I can talk about L p 1 f and I can also talk about L P 2 f, these are two numbers. What is the connection among these two numbers, which one is bigger? If you look at the picture, then perhaps you can guess, that L P 2 f is bigger than or equal to L p 1 f. Because, you are putting more points there, so the rectangles are gaining in height. So, the total area will be bigger. So, let us try to prove it analytically.

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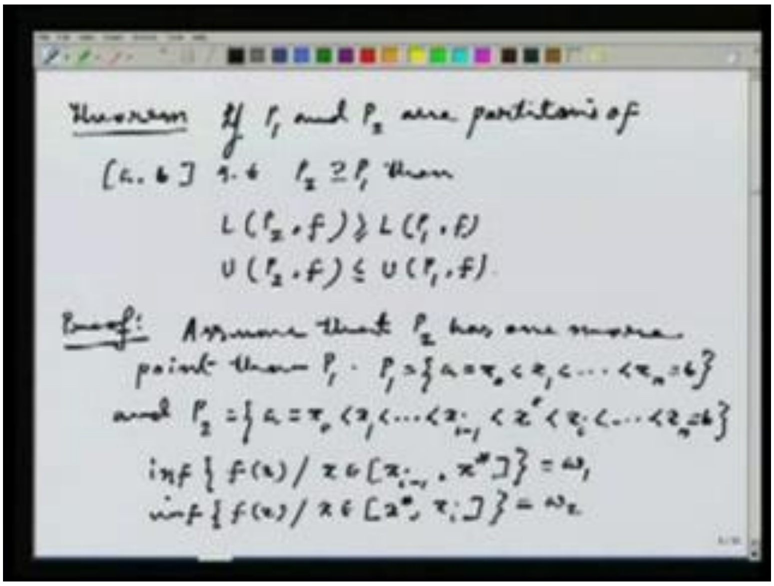

So, this is our first theorem. If P 1 and P 2 are partitions of a, b such that P 2 is a refinement of P 1. Then, for L P 1 f and U p f's, the total relations actually changes. That U p 2 f is lesser or equal to U p 1 f. So, now I want to prove this. In the proof, I will just assume that P 2 has only one more point than P 1. Because, then one can inductively proceed with the same argument. With one point more, if you can prove the result, certainly I can prove the result with two points more and so on.

So, assume that P 2 has one more point than P 1. Let us say the point is x star and it appears. Well, let us assume that P 1 equals to x 0, x 1 up to x n and P 2 is x i minus 1, then the additional point x star, then x i, then up to x n. So, what P 2 actually look like, that if you look at the points x i minus 1 and x i, which appears in P 1. Then, there is additional extra point x star between those two, which is there in P 2, but not in P 1. So, P has exactly one more point, then P 1 and that point is x star.

Now, we are trying to calculate the L p x first. The proof for U p f is analogues. Now, what I am interested in first is this set infimum f x, where x lies between x i minus 1 and x star. I call this w 1. Similarly, I can look at infimum of f x, where x belongs to x star x i plus 1, this interval, x star x i. I call this w 2.

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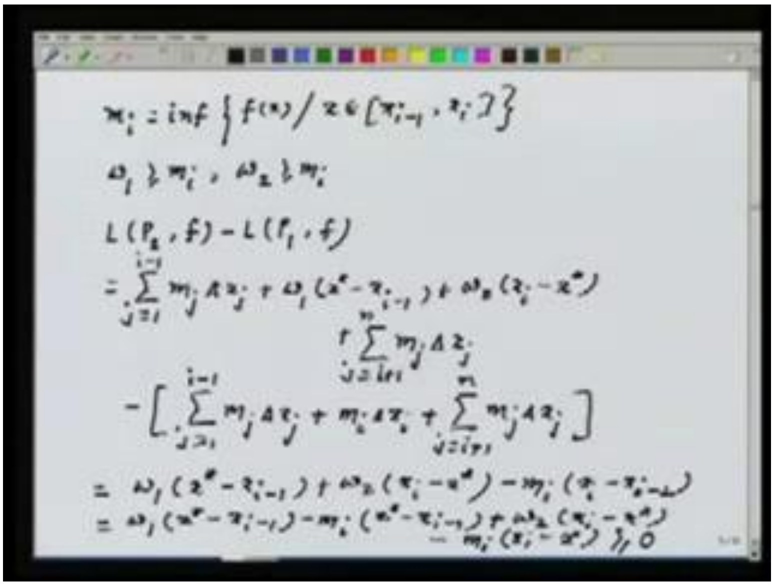

Now, observe one thing, if m i is equal to infimum of f x, when x belongs to x i minus 1 x i. Then w 1 is bigger than or equal to m i, w 2 is also bigger than or equal to m i, why is so... Because, infimum of a smaller set is always bigger than the infimum of the bigger set the set f x x in x i minus 1 x i is certainly bigger than the set of all x. Such that x is in x i minus 1 x star. Because, x star is less than x i.

So, that is the smallest set. So, in the smaller set, if I look at the infimum, that is certainly bigger than the infimum of the much bigger set. That is why, I we get the inequalities w 1 and w 2 both are bigger than or equal to m i. Now, I look at this quantity L P 2 f minus L P 1 f. So, I just write down them down. Remember, there is difference of only one point in P 2 than P 1.

So, I can write summation j from 1 to i minus 1. I will write this is as m j delta x j plus w 1 times x star minus x i minus 1 plus w 2 times x i minus x star plus summation j from i to n m j delta x j this is L P 2 f. Now, comes minus L P 1 f that is summation j from 1 to i minus 1 m j delta x j plus m i delta x i plus summation j from i plus 1 to j m j delta x j. I have just written in the definition of L P 1 f and L P 2 f.

Now, if I compare, I see the term, there cancels each other. And what remains is w 1 times x star minus x i minus 1 plus w 2 times x i minus x star minus m i times x i minus x i minus 1. So, what I have done is, it look so complicated, I have just look concentrate on the i th term in the sum of L P 2 f and have broken it written it in terms of x star. And while looking at L P 1 f, I have just separately written down the i th term.

All the terms, I have kept in other term. I have kept in the summation. And then I can see that because of that the cancellation. Only, those i th interval is going matter, other are canceling themselves. Well, the final step, then is I can write this as w 1 times x star minus x i minus 1 minus m i times x star minus x i minus 1 plus w 2 times x i minus x star minus m i times x i minus x star.

So, what I have done is, when I look at the quantity m i times x i minus x i minus 1. This is x i minus x i minus 1, I am just writing as x star minus x i minus 1. And then, x i minus x star. And then, since I know the w 1 is bigger than m i and w 2 is also bigger than m i. I get that this quantity is bigger than or equal to 0. That settles the case. Because, that means, then L P 2 f is bigger than or equal to L P 1 f.

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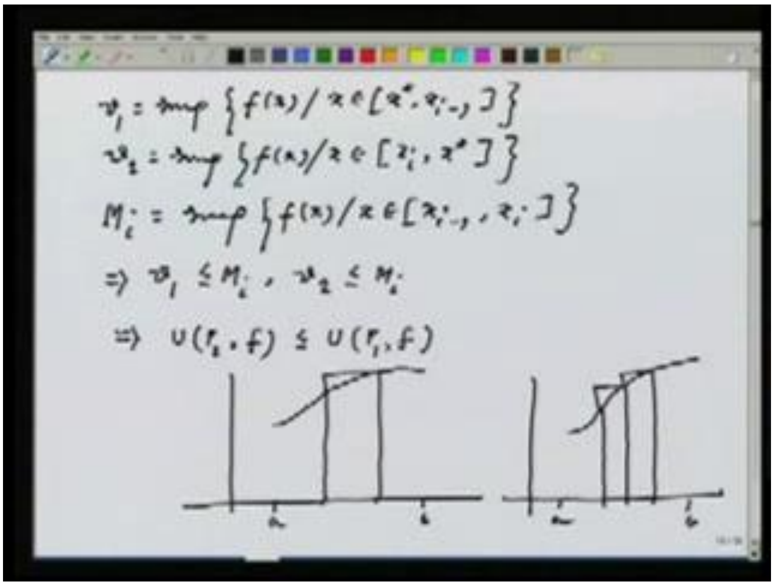

Now, if you look at the U p f, what you think will happen. The only difference is observed in the following relation. That if I define v 1 to be equal to supremum of f x, when x belongs to x star x i minus 1. And v 2 supremum f x, when x belongs to x i x star and I define capital M i is equal to supremum of f x, when x belongs to x i minus 1 x i. Then, what is the connection between v 1 and v 2 and M I, because while dealing with L p f. I dealt with the connection between w 1 and w 2, little m i. So, analogously, I have to deal with v 1, v 2 and capital M i here.

Well, here the obvious think is the supremum of a smallest set is smaller than supremum of the bigger set. This implies that v 1 is lesser equals to M i and v 2 is also lesser equal to M i this. Then, would imply just exactly as in the previous case, that U P 2 f is lesser equal to U P 1 f. Again I can explain it by picture the case of U p f I have a here, I have b here, I draw the graph of the function as this, I look at two partitions points. Then, the corresponding bigger triangle I have to draw, because I dealing with U p f.

Now, the bigger rectangle is this. I draw the same picture again here, just show you the difference, these are the two points. Suppose, there is a partition point inside, draw the graph of the same function, then I have to draw three vertical lines, 1, 2. This is the 3rd one. And I am drawing the bigger rectangle, this is the first one, this is the second one.

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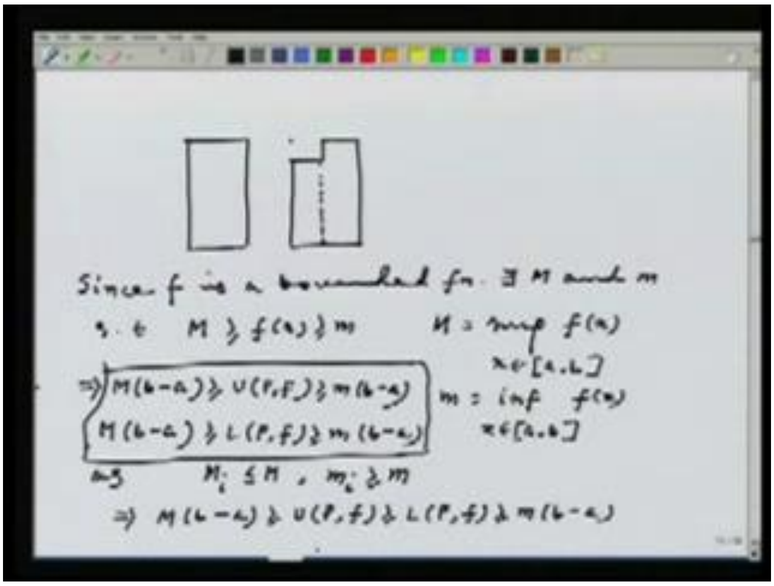

Now, you can compare in one case, I had this rectangle in the next case. What I am getting is, go back to the picture, you see, what I am getting. Compare it, with the previous page. This was my first rectangle without the partition points. Then, what I get is this one; this fellow certainly has a smaller height. See, the next page the first one is of the smaller height. That means, this area is certainly less, this is what is actually explain by saying that if I look at the refinement of partition. Then U p f are actually getting smaller.

Now, let us notice the following thing, that since, I have assumed that since f to be bounded. Since, f is a bounded function there exists capital M and little m such that M is bigger than or equal to f x is lesser equal to little m. This capital M is actually the supremum of the f x, this is supremum of f x, x belongs to a, b. And little m equals to infimum of f x, x belongs to a, b.

Notice, when that this implies that capital M into b minus a is always bigger than or equals to U p f and it is bigger than or equal to m into b minus a. Similarly, capital M into b minus a is bigger than or equal to L p f this bigger than or equal to little m b minus a. This follows from the fact, that capital M i is always lesser equal to m and little m i is always bigger than or equal to little m.

Now, this would then imply that capital M into b minus a is bigger than or equal to U p f, which I anyway know bigger than or equal to L p f. But, L p f which is constructed out of m i is certainly bigger than or equal to m into b minus a. This is how get this, which is going to be a very fundamental for us…

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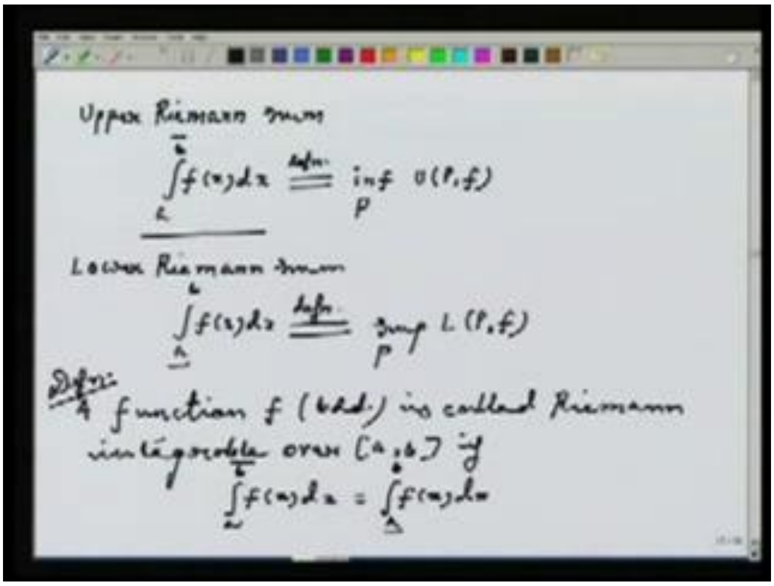

Because, now I am going to define something called upper Riemann sum, it is denoted as integral a to b bar f x d x, it is just involves, the meaning of this is. So, this is definition this is infimum of all the U p f's. Because, if I go on varying the partitions P, I get different U p f and I have already noticed that this set of U p f's, it is bounded by little m into b minus a and capital M into b minus a. So, it has a infimum, that infimum I define as the upper Riemann sum, which is the symbol is a to b bar f x d x.

Similarly, I define lower Riemann sum, which I denoted by integral a bar to b f x d x. Notice that right now the d x has absolutely no meaning, the meaning of this symbol is explained by this is supremum over all P, L P f. That is I construct U p f and then I look at their infimum. I construct L p f and I look at their supremum, because again if you think back at the picture as I go on taking finer and finer partitions U p f are coming down. So, finally, they suppose to colloid with something and that thing is the infimum.

Similarly, for L p f as I go on taking the refinement of the partitions L p f is increasing. So, they will colloid with something that thing is the supremum, which is the lower Riemann sum. Now, the function f, a function f this is the definition, a function f, which is certainly bounded. Notice that boundedness is very important, otherwise capital M i and little m i will not exists. For the existence of the supremum and infimum, I need boundedness of the function is called Riemann integrable over a, b. If the lower Riemann sum is same as the upper Riemann sum, that is if a to b bar f x d x is same as integral a bar to b f x d x.

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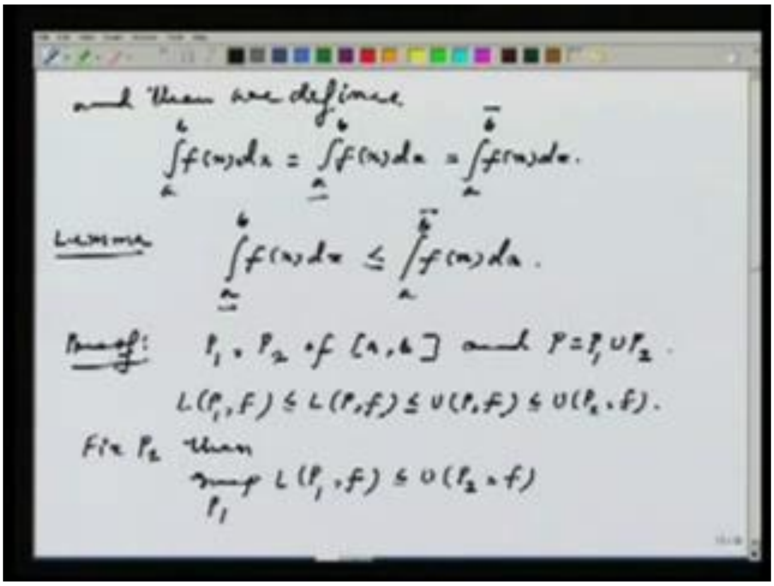

And in this case, we define the Riemann integration of f as integral a to b f x d x, which is the common value of the upper Riemann sum and the lower Riemann sum that is. So, the whole possesses again this that I look at all the partitions of the closed interval a, b. Corresponding to each partition, I look at U p f's, I collect those U p f's look at infimum of that set.

Then corresponding to each partition, I look at the L p f I look at that set. I look at the supremum if the supremum and the infimum are same. Then the function is called the Riemann integrable. And the Riemann integrable of the function is defined to be that supremum it is same as that infimum. Because the definition of Riemann integrability demands that the supremum and infimum are same.

Now, in general what is the connection between the upper Riemann sum and the lower Riemann sum? That will be needed in the sequel. So, I just include it as a Lemma in general, what we know is that integral a bar to b f x d x, this is the lower Riemann sum. It is always less than or equal to integral a to b bar f x d x. In general, this is the relation between the upper Riemann sum and lower Riemann sum. When these two quantities are same, then the function is Riemann integrable.

Now, how to prove this, well I take any partition P 1 and P 2 of a, b and P is the common refinement of P 1 and P 2 means, P is just P 1 union P 2. That is, then another partition. Then, anyway know that L P 1 f is lesser equal to L p f as I have observed, if you put more points in a partition L p f increase. But, L p f is anyway is lesser equal to U p f, but has more points than P 2. So, U p f is lesser equal to U p 2 f. Now, what I do is fix P 2, then supremum over P 1, L P 1 f. That is, certainly lesser equal to U P 2 f, because the above inequality is true for all P 1 and P 2.

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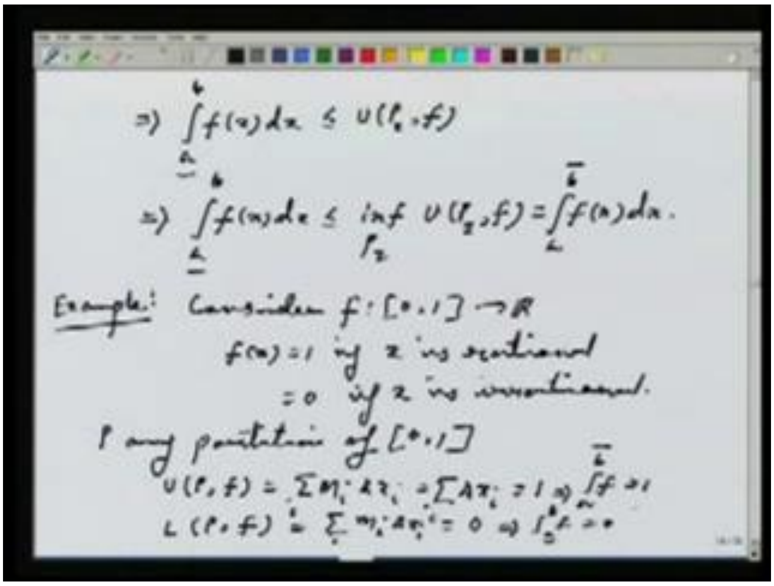

But this then implies that integral a bar to b f x d x, which is by definition the supremum of all L p f s is lesser equal to U P 2 f, this is true for all P 2. This then implies that integral a bar to b f x d x is lesser equal to infimum over P 2, U P 2 f. But, this is by definition a to b bar f x d x, so the relation is proved. Now, I will just give you an example to show not every bounded function is Riemann integrable, consider the function f from 0 1 to r given by f x equals to 1. If x is rational is equal to 0, if x is irrational.

It is very easy to see, that this function is not Riemann integrable. Simply, because of the following, take any partition P, P any partition of 0, 1. Then my question is what is U p f well this is summation i capital M i delta x I, but if you take any sub interval of x i minus 1 x i. There will be some rational point there and at other irrational point f is 0, but at the rational point f is 1. So, the supremum is 1; that means, capital M i is 1 for all i.

That means, it is just summation over i delta x i, that is the length of the interval, which is 1. Now, if I look at L p f, that is summation over i little m i delta x i, what is little m i is the infimum of the function over a i th sub interval. Well, fix any i, look at the x i minus 1 x i, it will certainly contained the irrational number, but then f is 0 there. That means, the all sub interval the infimum of the function is 0. That means, little m i is 0 for all i, that means, the final sum is 0.

Now, if you take the supremum of all partitions P. That means the upper Riemann sum that is 1. This implies the upper Riemann sum is 1, but for all partitions P, L p f is 0. That means, the in supremum of the L p f that is also 0, that means, the lower Riemann sum is 0, which is not equal to the upper Riemann sum. And hence, the function is not Riemann integrable.

So, there exist functions, which are not integrable. So, the next question, which we like to ask, is well, what are good examples of integrable functions. That is, what are Riemann integrable functions? Do we have examples of large class of functions, which are Riemann integrable? For example continuous functions, differentiable functions are they Riemann integrable the so called good functions should be Riemann integrable.

So, that is the next thing, we are going to examine. So, in the next lecture, we will continue with our investigation of findings examples of large class of Riemann integrable functions. But, the problem, there is we need a criteria to determine, when a function is a Riemann integrable. What we have, so far is, just the abstract definition of Riemann integration. It turns out that with this definition, it is difficult to show that certain functions are integrable.

Just like, if you look at sequences, just the definition of convergence of sequence is not good enough to check. There is sequence actually converges. Similarly, here we need a necessary and sufficient condition which one can handle to use. To check whether functions are Riemann integrable that is the next thing; we are going to do in the next lecture.