Mathematics - I
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Lecture - 18
Riemann Integrable Function

In the previous lecture, we have defined, what is meant by Riemann Integrability of a Function. Now, comes the next question as we said in the last lecture, that are, there enough functions, which are Riemann integrable. That is, does there exists, examples of a large class of functions, which are Riemann integrable.

Now, so called nice functions of calculus? That is continuous functions or differentiable functions. We want to know, what they have to do with Riemann integrability. That is, our continuous functions Riemann integrable and so on. Now, for that, we need criteria, by which, we can check whether a function is Riemann integrable. As we said in the last lecture, something likes Cauchy criteria of convergence of sequences. We cannot always check convergence of a sequence, by finding the limit. What we do is, we show that the sequence is Cauchy sequence. We are searching similar kind of criteria for Riemann integrability.

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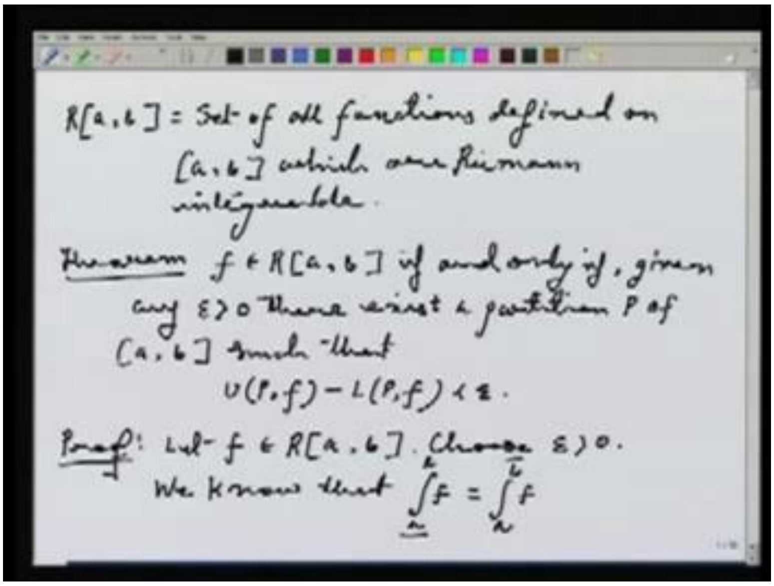

Now, first let us define R a, b. What you mean by that, this is the set of all functions, defined on the closed interval a, b, which are Riemann integrable. So, in mathematical symbols, we want to know, how large R a, b is. Now, for that, the first theorem is a necessary sufficient criteria for Riemann integrability. So, the theorem is this. That is a function f, belongs to R a, b. If and only, if the following is given any Epsilon bigger than 0. There exists a partition P of a, b, such that, U p f minus L p f is less than Epsilon.

So, the criteria is given any Epsilon bigger than 0. One can find the partition P, such that, partition P, if I look at U p f and L p f. Then, the difference between those two numbers is less than Epsilon. We will see that, it satisfies the criteria have necessary and sufficient for Riemann integrability of a function. So, for the proof of this, let us first assume f is Riemann integrable.

And then, given any Epsilon bigger than 0. I will find a partition P, such that, U p f minus L p f is less than Epsilon, so let f belongs to R a b and choose Epsilon bigger than 0. Now, for the first question is, how do I get hold of the partition P, which is going to satisfy the criteria? Well, it is very simple what I do is, since f is Riemann integrable I know the lower Riemann sum and upper Riemann sum, both are same. Because f is Riemann integrable, so we know that integral a bar to b f is same as integral a to b bar f, that is known to us…

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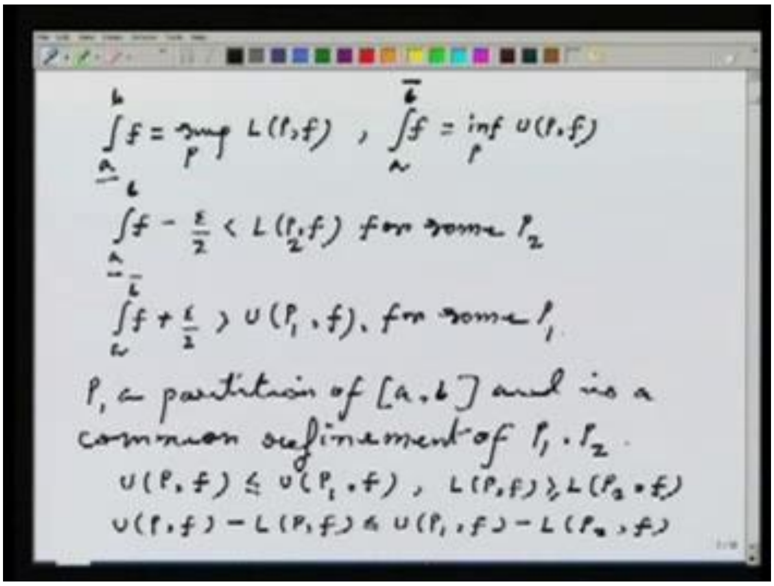

Now, what is the definition of this quantity a bar to b f. If you recall, this was defined as supremum, over all partitions P of the numbers L p f. And a to b bar f was defined as infremum over P of all the U p f. Since, Epsilon was chosen; I will start with Epsilon by

2. And I can say that integral a bar to b f minus Epsilon by 2. That is, if I decrease the supremum a little bit. Then, there is some member of the set, which is bigger than this decrease supremum.

So, this is then less than L p f, I call it L p 2. For some P 2, which is a partition of a, b. Similarly, if I increase the infremum of the U p f by Epsilon by 2, then I know it is bigger than some U p 2 f for some partition P 1, where P 1 and P 2 are partitions. Well, this is just from the definition of supremum and infimum. So, what we have done, I know that the lower Riemann sum is the supremum of the L p f. So, I decrease the lower Riemann sum by an Epsilon by 2.

But, since it is a supremum, then this supremum minus Epsilon by 2, will be super ceded some L P 1 f. That is, what I have written. Then, I look at the infremum of the U p f and increase that by Epsilon by 2. Then, there is one U P 1 f, which is less than increased infremum. That is, nothing I have done. Now, I look at P, which is the partition, a partition of a, b and is a common refinement of P 1 and P 2. What does this mean, P 1 is a sub set of a, b, so is P 2.

I take another sub-set of a, b which is bigger than P 1 and P 2, so that is the refinement. And then, I know the following thing, U p f is less than or equal to U P 1 f. And I also know that, L p f is bigger than or equal to L P 2 f. This is the result, which I have proved in the last lecture. Now, what does this mean, I look at the quantity U p f minus L p f. This then is certainly less than or equal to U P 1 f minus L P 2 f.

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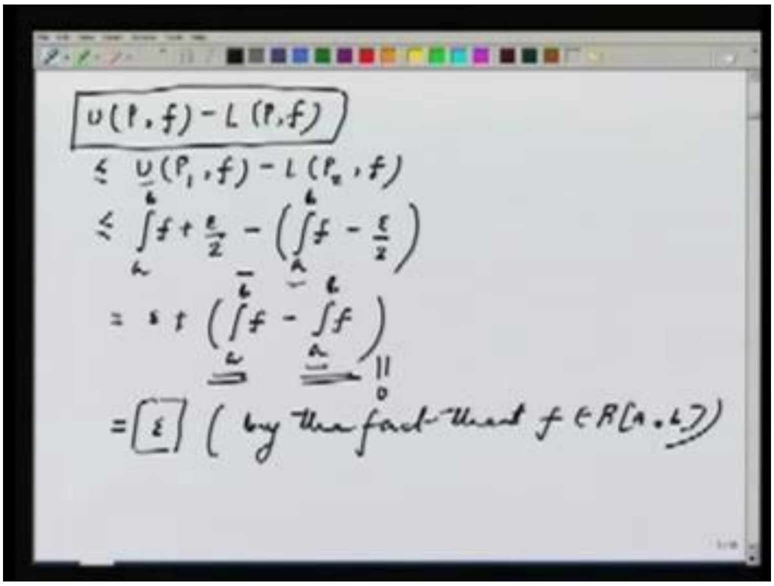

Now, from the previous relation, I know that U P 1 f. So, I will write it again U p f minus L p f is less than or equal to U P 1 f minus L P 2 f, which is further less than or equals to. Let us look at the previous page, I know this relation, that U p 2 f is less than. So, will use that, this is less than or equals to a to b bar f plus Epsilon by 2, then minus of a bar to b f minus Epsilon by 2. From the previous relation, which then is equal to Epsilon plus integral a to b bar f minus integral a bar to b f.

Now, notice that, f is Riemann integrable. It means, the lower Riemann sum is same as the upper Riemann sum. That means, this first quantity is same as the second quantity. That means, the portion in the bracket is actually is equal to 0. That means, this is equals to Epsilon by the fact f belongs to R a, b. But, this is what we wanted to prove, because I found a partition P, which is actually the refinement of P 1 and P 2.

And that P satisfies that U p f minus L p f. This quantity is less than or equal to Epsilon. This is precisely, what we wanted to prove. But, now we have to look at the converse. So far what we have proved is that, if f is Riemann integrable, then given Epsilon bigger than 0, I can find a partition p such that, U p f minus L p f is less than Epsilon. Now, what is the converse of this, I will assume that given Epsilon bigger than 0. There exists a partition P, which satisfies the given condition. And from that, I need to show that f is Riemann integrable.

Now, if I want to show that, f is Riemann integrable. The only way, we can show it is by showing that the lower Riemann sum is same as the upper Riemann sum. So, let us try that.

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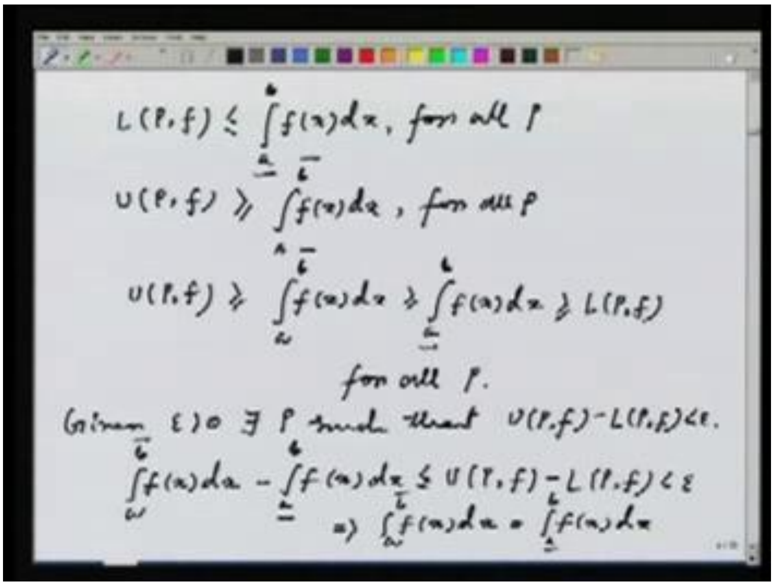

Now, what is the relation we have? Anyway, we have that L p f is always less than or equals to integral a bar to b f x d x. This is the lower Riemann sum, because we have defined the lower Riemann sum as supremum of all L p f's. So, this is true for all P. Similarly, I have U p f is bigger than or equal to integral a to b bar f x d x for all P. Because, the upper Riemann sum, defined as infimum of all the U p f. Well, this means what, it means, that U p f is bigger than or equal to a to b bar f x d x.

But, the upper Riemann sum is always bigger than or equal to the lower Riemann sum. So, this is bigger than or equals to integral a bar to b f x d x, which by definition bigger than or equal to L p f. This is true for all partitions P. Now, the given criteria tells me, that given Epsilon bigger than 0 there exist P such that, U p f minus L p f is less than Epsilon. Now, from the previous inequality, what we get is that a to b bar f x d x minus a bar to b f x d x is less or equals to…

So, from the previous inequality, now we get that the left hand side is less than or equals to U p f minus L p f. But, I have chosen the partition P in such a way that U p f minus L p f is less than Epsilon. That means that, this difference between the upper Riemann sum and the lower Riemann is less than Epsilon. But, this upper Riemann sum and the lower

Riemann sum are two absolute constant. They do not depend on anything given the function f; that is the constant.

The difference between those two numbers is less than Epsilon. For all Epsilon means, what it means both the numbers are same. So, this implies integral a to b bar f is same as integral a bar to b f. But, this is precisely what is meant by f is Riemann integrable. So, the result follows. So, here what we have done, we have just use the fact, that given any arbitrary Epsilon bigger than 0. I can find a partition P, such that U p f minus L p f is less than Epsilon.

And using that, I have shown that the difference between the upper Riemann sum and the lower Riemann sum is less than Epsilon. That means, upper Riemann sum is same as the lower Riemann sum. Because, Epsilon I can make as small as I like that means, f is Riemann integrable. Now, we are going to use these criteria to prove integrability of certain functions.

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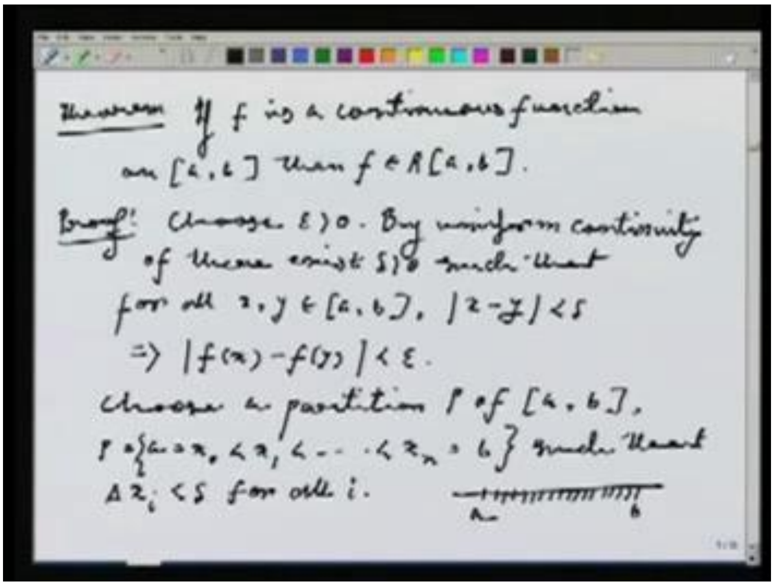

So, the first one is the most important one. This incidentally will prove that the classes of Riemann integrable functions are quite large. The only property, we are using about Riemann integrability of functions are bounded functions remember that. Time and again, I am using the fact, that f is bounded. It does not make sense to talk about Riemann integrability of a function, which is not bounded.

We are talking about integrability of bounded function only. Otherwise, in the very definition the supremum and infimum, which we are using may not exists. Now, the theorem says that, if f is a continuous function on the closed interval a, b. Then, f is Riemann integrable. So, that I will explain by saying f belongs to R a, b the set of all Riemann integrable functions.

So, it proves that every continuous function is Riemann integrable. So, how to prove this we have to go by the necessary sufficient condition of Riemann integrability of a function, that is what we are going to use. That is the previous result. Remember, the previous result says, that if I can somehow show that given any Epsilon. There is a partition P, such that, the difference between U p f and L p f is less than Epsilon. Then, f is Riemann integrable. That is, what I need to show here.

So, I start with Epsilon bigger than 0. Now, my job is to find the correct partition for which the relation would be true. Here, at this point I use uniform continuity of the function. This is a result, which we proved in some previous lecture. That if I have a continuous function defined on some closed interval a, b. Then, the function is actually uniformly continuous. The difference between the uniform continuity and continuity is as follows.

Continuity says, continuity at a point x naught says that given Epsilon. There is a delta bigger than 0, such that, certain things happen. Now, the delta, which exists because of the Epsilon, depends on the point x naught. Uniform continuity means this choice of delta works for all x, it is independent of the x naught that is what I am going to use now. So, by uniform continuity of f, there exists delta bigger than 0. Such that, for all x y in the closed interval a, b modulus of x minus y less than delta implies mod of f x minus f y is less than Epsilon.

This is the uniform continuity; notice that, this delta does not depend on the points. Whatever, x, y you choose, if mod x minus y is less than delta, then modulus of f x minus f y is less than Epsilon. Now, what I do is choose a partition P of a, b, let us say P is equal to the set with the following property. That delta x i, that is the length of the sub-intervals, which appears because of the choice of the partition. This is less than delta for all i.

That is I am choosing a partition P of the interval a, b. Such that, the length of given distance of any two consecutive points in that partition is less than delta. That can be done however; small delta may be I choosing large number of points. So, that this happens. I can look at the picture this is a, this is b. I can choose closely, very related points. But, there will be only finitely many, such that the distance between any two points between consecutive points is less than delta.

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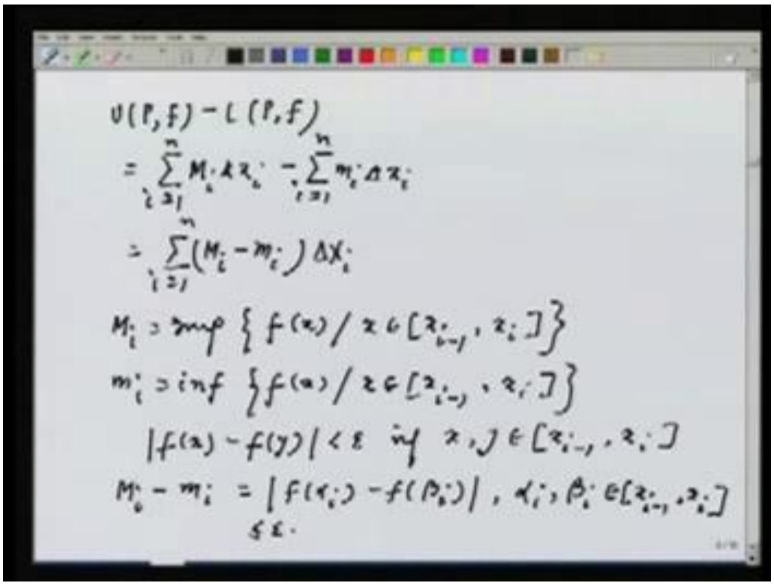

Once I do this. Then, the following things happens, I look at U p f minus L p f. So, this then is equal to summation over i from 1 to n. Capital M i times delta x i minus summation i from 1 to n little m i times delta x i, which then same as summation i from 1 to n capital M i minus little m i times delta x i. Now, this delta x i's I know they are less than Epsilon, but what about this capital M i minus little m i. This is, where I am going to use the uniform continuity actually.

Because, what is capital M i, capital M i is equal to supremum of f x. Where, x belongs x i minus 1 x i and little m i is equal to infimum of all the f x, such that, x belongs x i minus 1 x i. Now, I know that modulus of f x minus f y this quantity is less than Epsilon. If x y belongs to x i minus 1 x i, because the length of the interval x belongs x i minus 1 x i is less than delta. And hence, by uniform continuity if I take two points x and y from that interval. Then, modulus of f x minus f y is less than Epsilon.

Now, the supremum of the function f and infimum of the function f will be attained in the closed interval x i minus 1 x i, because f is continuous. So, capital M i minus little m i, I can certainly say, is certainly mod of f alpha i minus f beta i. Where, alpha i and beta i are points in x i minus 1 x i.

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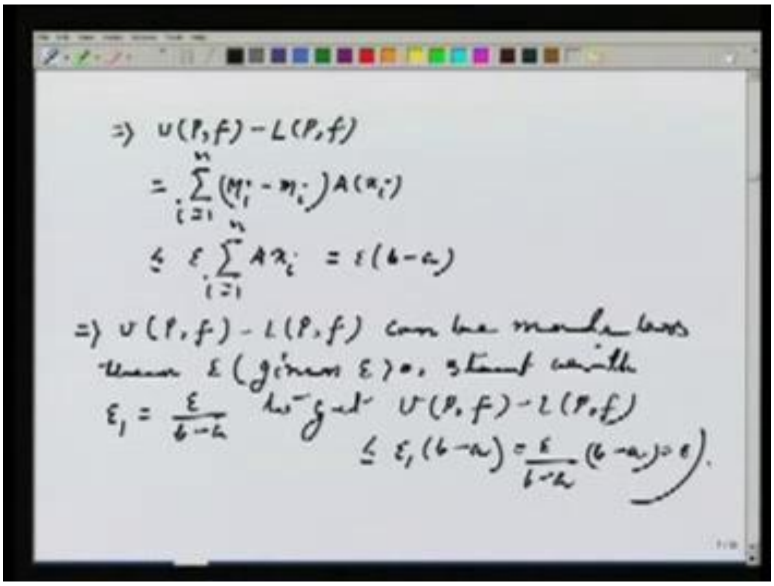

And hence, this quantity is lesser equal to Epsilon. So, this means what this then implies that U p f minus L p f, which is summation i from 1 to n m i minus m i times delta x I, it less or equal to Epsilon times. Summation i from 1 to n delta x i, that is capital M i minus little m i is dominated by Epsilon for all i. So, in the sum, I am replacing those quantities by Epsilon. So, I am getting something bigger.

But, then I can sum it up to get Epsilon times b minus a. Because total length of the sub-intervals in the partitions P is certainly is equal to b minus a. Now, b minus a is constant and Epsilon is arbitrary. So, it can be made arbitrary small, this implies U p f minus L p f can be made less than Epsilon. You know this is the standard step, I mean given Epsilon. You start with, I will write it here, given Epsilon bigger than 0.

Start with Epsilon 1, which is defined as Epsilon by b minus a to get U p f minus L p f lesser equal to Epsilon 1 into b minus a. But, Epsilon 1 is Epsilon by b minus a multiplied by b minus a, you get Epsilon, that is it. So, this shows that any term continuous function defined on the closed interval a, b is actually a Riemann integrable.

And see, how easy it is by using the necessary sufficient criteria of Riemann integrability. That is precisely, what I have done, given Epsilon bigger than 0.

I have just shown that, there exists a partition P such that, U p f minus L p f is less than Epsilon. But, that is enough to show Riemann integrability of a function. Now, the question is, can I give examples of a large class of functions, which are not continuous. But, Riemann integrable, well that can also be done, continuity is not necessary.

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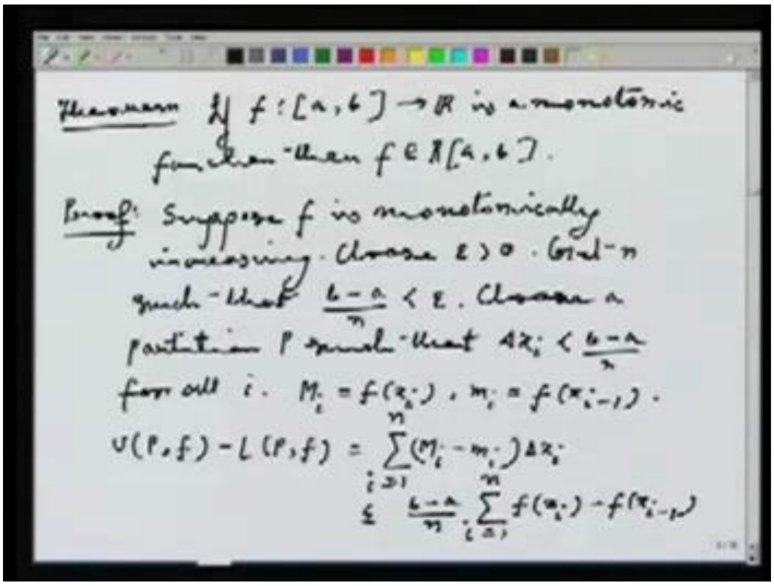

So, the next theorem proves that, there are enough discontinuous functions, which is also Riemann integrable. So, the theorem says if f from a, b to R is a monotonic function, then f is again Riemann integrable. See, there is no question of continuity here. I can just choose any monotonic function, it might be monotonically increasing. It might monotonically decrease. But, then, still it is Riemann integrable.

So, we will prove this assuming that f is monotonically increasing. The proof for monotonically decreasing is the analogous, so let us go to the proof of this. Suppose, f is monotonically increasing, I want to again try to see, whether I can prove the necessary sufficient criteria for Riemann integrability. That is given Epsilon bigger than 0, I will try to find a partition P, such that U p f minus L p f is less than Epsilon. So, choose Epsilon bigger than 0.

Get n, such that, b minus a divided by n is less than Epsilon, choose a partition P. Now, such that, delta x i is less than b minus a by n for all i. In the partition, what we do is, choose points in such a fashions that the difference between the two consecutive points always less than b minus a by n that can be done. Now, notice one more thing that since f is monotonically increasing capital M i is actually is equal to f of x i and little m i is f of x i minus 1.

This is simply because; f is monotonically increasing at the right end points. The value of the function is highest and at the left end points the value of the functions is lowest. Then, I look at U p f minus L p f and if I calculate this, I know what is going to come it is summation i from 1 to n M i minus m i times delta x i, which any way lesser equal to. Since, each delta x i is less than b minus a by n I can take it out of the sum and what remains is i equal to 1 to n. I just write it in the fashion f x i minus f x i minus 1.

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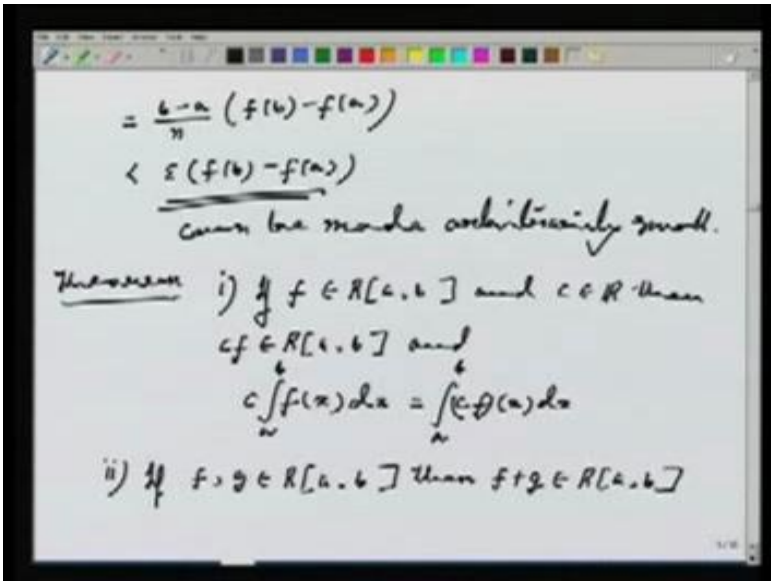

Now, if sum them up, what I get is b minus a by n times. Because, of the cancellation, I get that, this is b minus a by n times f b minus f a. So, this is certainly less than Epsilon times f b minus f a. Again using the trick, because f b minus f a is constant and Epsilon is arbitrarily small I can make the quantity. This quantity can be made arbitrarily small. That is, I can always do by choosing Epsilon smaller and smaller one.

This is precisely, what we have done in the last result also. So, that shows that the function satisfies the necessary sufficient condition criteria of the Riemann integrability as a result f is Riemann integrable. So, now we have seen that there are enough examples of functions which are Riemann integrable. Next, we need some elementary properties of Riemann integration. So, that, I will put as a theorem.

So, first one is if f belongs to R a, b and c is a real number. Then, look at the new function c f that is the scalar multiple of f, that also belongs to R a, b not only that integral a to b f x d x, is same as integral a to b c f x d x. That means, the scalar, which I multiplied with the function, can be taken out of the integral. Second property is, if f and g both are Riemann integrable that is they belong to R a, b. Then, the new function f plus g that also belongs to R a, b.

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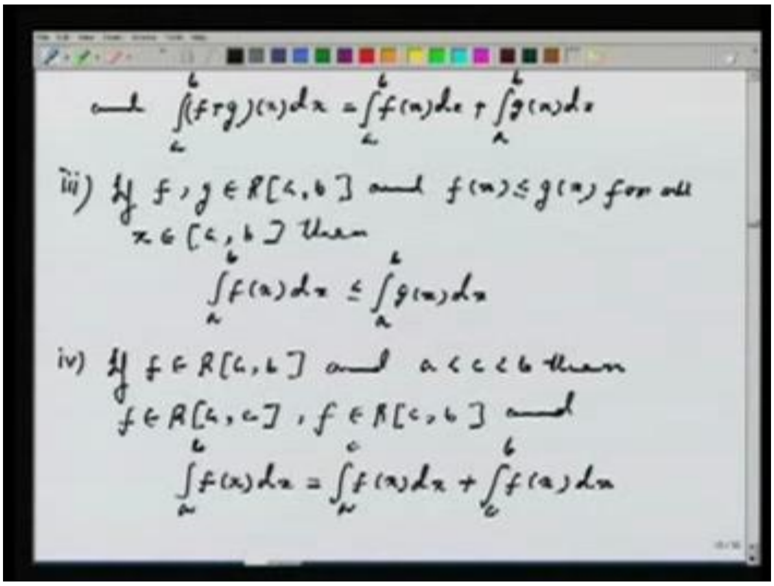

And integral a to b f plus g x d x is same as integral a to b f x d x plus integral a to b g x d x. Third property is, if f and g both are Riemann integrable and f x is less or equal to g x for all x in a, b. Then, their integral also maintain the same relation that is integral a to b f x d x is lesser equal to integral a to b g x d x. Fourth property, if f belongs to R a, b and c is a point which lies between a and b.

Then f belongs to R a, c well here what I mean is if I restrict f on the interval a, c. Then, f belongs to R a, c that is what means by this, if I restrict f on a, b, then also f belongs R a c b. That is what I need and integral a to b f x d x is equal to integral a to c f x d x plus integral c to b f x d x.

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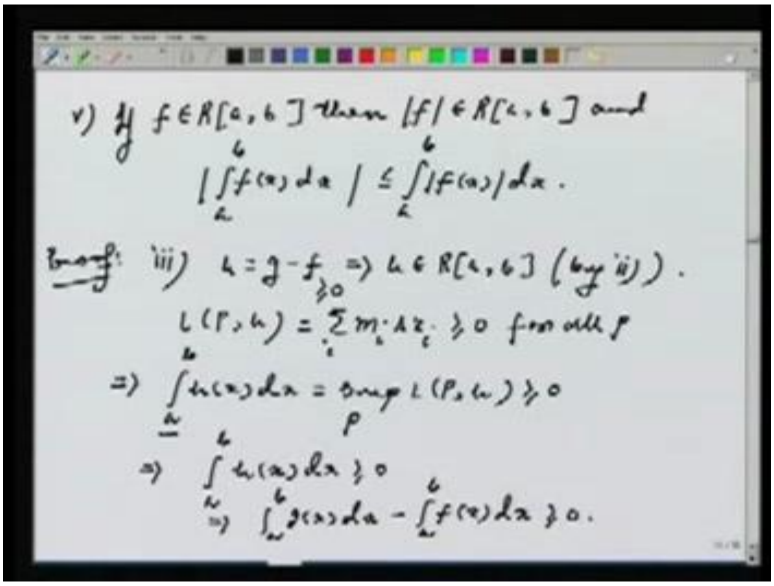

Then, the last property the most fundamental one, that if f belongs to R a, b. Then, the function mod f also belongs to R a, b and modulus of integral a to b f x d x is less or equal to integral a to b mod f x d x. We will see that this inequality has fantastic application most of the important theorems will need this inequality. Now, here most of the result we will not prove. Because, there is very standard and follows in the standard way from the definition of the Riemann integration, you can try it out yourself.

So, what I will do is, I will try to prove only 3 and 5, because that is what I need now. So, I start with the proof of 3, what I do is I define h is equal to g minus f, f and g both are Riemann integrable this then implies that h is Riemann integrable by 2, this is by 2. Now, I look at well according to my notation and then it should be L p h, well I look at L p h. That is summation over i little m i times delta x i, here the little mi is the infremum of the function h over all the sub-intervals.

Now, since h is always a non-negative function. Because, f is lesser equal to g implies g minus f is bigger than or equal to 0. So, h is always bigger than or equal to 0. That means, this m i is always bigger than or equal to 0, that means this is bigger than or equal to 0, this is true for all P. This then imply that, the lower Riemann sum, that is a bar to b h x d x, which is supremum of all the L p f in this case L p h. That is bigger than or equal to 0.

But then, again by 2, this would imply integral a to b h x d x. Well, I had integral a bar to b h x d x, how can write as integral a to b h x d x. Well, by the previous property, since f and g is Riemann integrable I know h is Riemann integrable. So, the lower Riemann sum upper Riemann sum is same that is the Riemann integral. So, integral a bar to b h x d x is same as integral a to b h x d x i. That is, why I could write this and this is bigger than or equal to 0. But then this would imply that integral a to b g x d x minus integral a to b f x d x. This is again by 2 is bigger than or equal to 0. That means integral a to b f x d x is less or equal to integral a to b g x d x.

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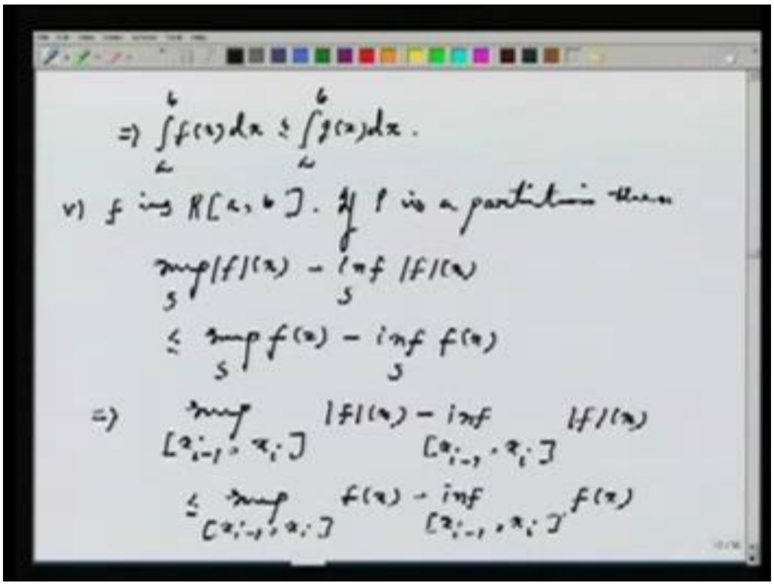

This is the conclusion; we wanted to prove, so this proves 3. Now, I want to prove 5, so here the only trouble is to prove that if f is Riemann integrable. Then, so is mod f, suppose I know this then the result is actually obvious that will see very soon. So, first I assume f is Riemann integrable. I have to prove that; mod f is Riemann integrable, it follows very easily.

That if P is a partition, then supremum of mod f x over any set, let us say s minus infimum of mod f x over the set s is certainly less than or equal to supremum of f x over s minus infimum of f x over s. It is, because of the oscillated nature of the function, f can take the positive value as well as the negative value. Then, the infimum will be negative, supremum will be positive. But, for mod f both the quantities are positive.

So, the difference between the supremum and infimum of mod f is always less than or equal to the difference between the supremum and infimum of f. Now, instead of s I actually concentrated on x i minus 1 x i, so this then implies that supremum over the set x i minus 1 x i mod f x minus infimum over the same set x i minus 1 x i mod f x is less than or equal to supremum over the set x i minus 1 x i of f x minus infimum over x i minus 1 x f x. What does this actually show, it shows if I concentrate on mod f. Then, the corresponding capital M i minus little m i is less than the corresponding quantity of f.

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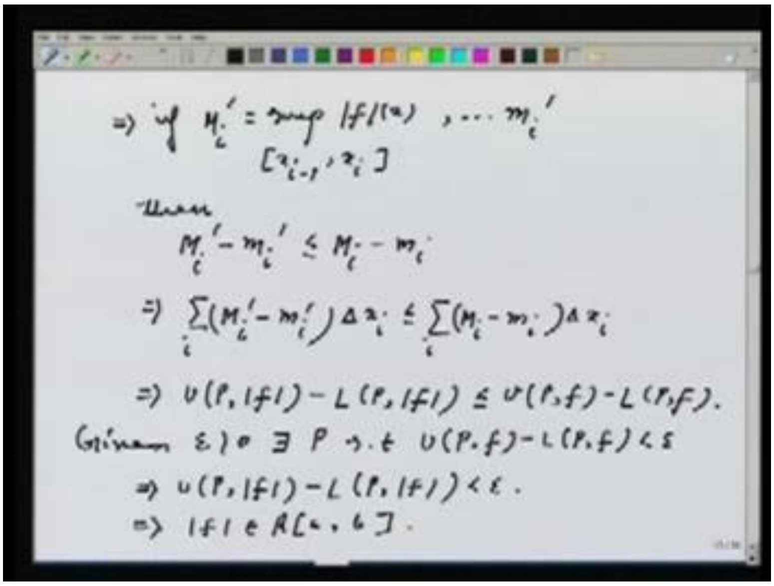

So, this then implies if M i prime is equal to supremum of mod f x over the set x i minus 1 x i and correspondingly m i prime. Then, M i prime minus little m i prime lesser is equal to capital M i minus m i. This then implies that summation over i capital M i prime minus little m i prime times delta x i is less or equal to summation over i capital M i minus little m i times delta x i.

Which in turn imply U p mod f minus L p mod f, U p f minus L p f. Now, since f is Riemann integrable I know that given Epsilon bigger than 0 there exists a partition P such that, U p f minus L p f is less than Epsilon. This because of the previous inequality then implies that U p mod f minus L b p mod f is less than Epsilon. But, that means mod f is Riemann integrable, because of our theorem. That this implies mod f belongs to R a, b. so at least, this much we have proved. But, now we have to prove the inequality of the integral well that follows quite easily.

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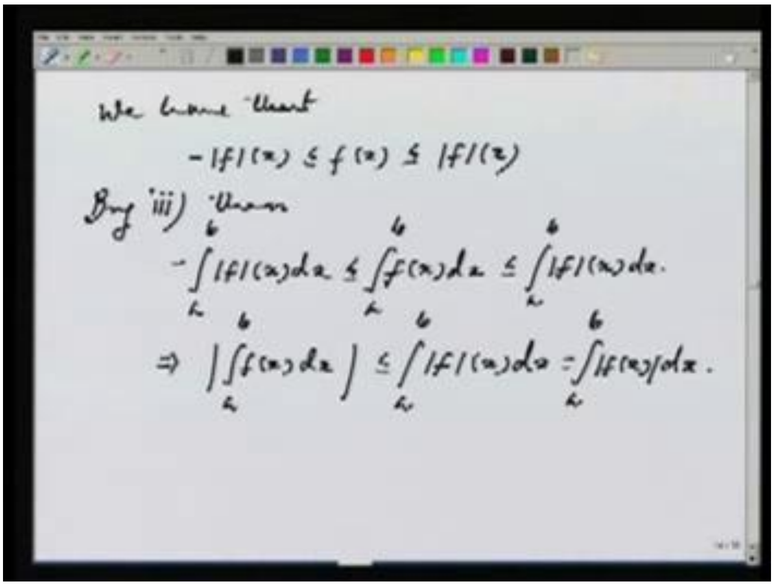

We have any way, that minus mod f x is less or equal to f x, is less or equal to mod f x this is by definition of modulus. Then, look at the theorem 3 ((Refer Time: 45:28)). Now, you want to use 3, by 3 then we have that minus integral a to b mod f x is less or equal to integral a to b, which is less or equal to. But, this precisely means that modulus of integral a to b f x d x is less or equal to integral a to b mod f x d x by the definition of mod f. This is same as integral a to b modulus of f x d x, which is precisely what we wanted to prove.

And now we are going to use these things for the most fundamental result of Riemann integral. That is the connection between Riemann integration and the differentiation of function. Now, our first theorem concerning integration and differentiation of functions is the first fundamental theorem of calculus.

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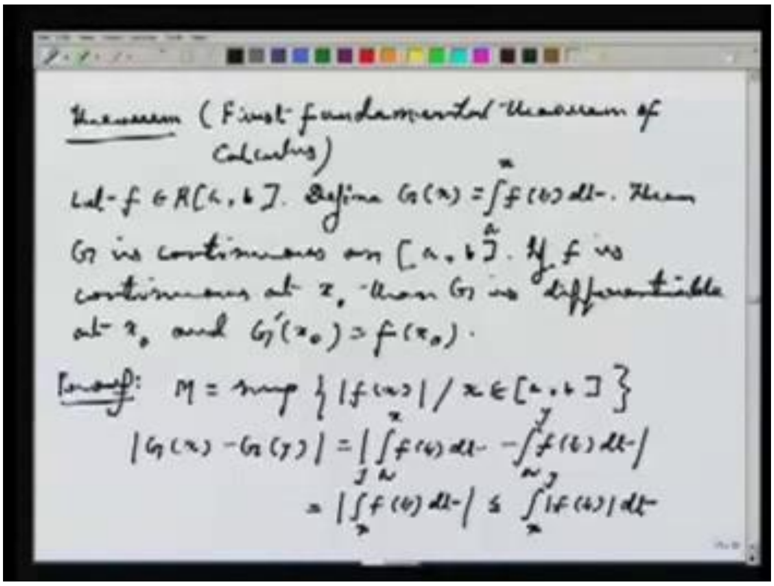

It says that suppose f belongs to R a, b I define g x as the given integral. Then, this integrated function g is a continuous function f, further if I assume that f is continuous at a point mind you to start with f is assumed to be Riemann integrable not continuous at all. If I further assume that f is continuous at x naught, then the integrated function g trans out to be better. In fact, it is differentiable at x naught and the derivative of g at x naught is given by f x naught. So, let us come to the proof of this.

Since, f is continuous on the closed interval a, b well I am just assumed to be a bounded function. So, certainly I can say capital M i is equal to supremum of mod f x in a, b that will be a finite quantity, because f is a bounded function, so the supremum exists. Now, I start with modulus of g x minus g y. If write it down modulus integral a to x f t d t minus integral a to y f t d t. Using a previous result, this is same as modulus integral x to y f t d t. Then I use the last inequality, which I have proved. Then this quantity is lesser equal to integral x to y mod f t d t.

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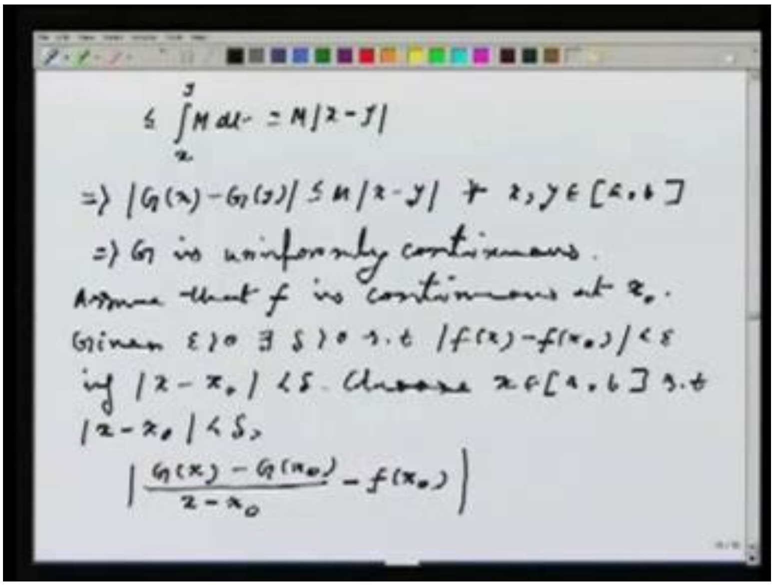

Then is less or equal to integral x to y M d t which is nothing but, M into mod x minus y. So, it follows then modulus of G x minus G y is lesser equal to M times mod x minus y, for all x y in the closed interval a, b. What does this mean; this implies G is actually uniformly continuous, something more than continuous. This is the first part of the result. Now, assume that f is continuous at x naught.

I want to show G is differentiable at x naught and the derivative is f at x naught. So, by continuity I can say that given Epsilon bigger than 0, there exists delta bigger than 0, such that, modulus of f x minus f x naught is less than Epsilon, if modulus of x minus x not is less than delta. So, I choose some x a may b such that, modulus of x minus x naught is less than delta.

And then, look at the corresponding modulus like this, G x minus G x naught divided by x minus x naught minus f x naught. The idea here it shows this modulus is less than Epsilon. Because, then it would prove that x converges to x naught, then the quotient G x minus G x naught by x minus x naught converges to f x naught. But, that would precisely mean that G prime at x naught is f x naught.

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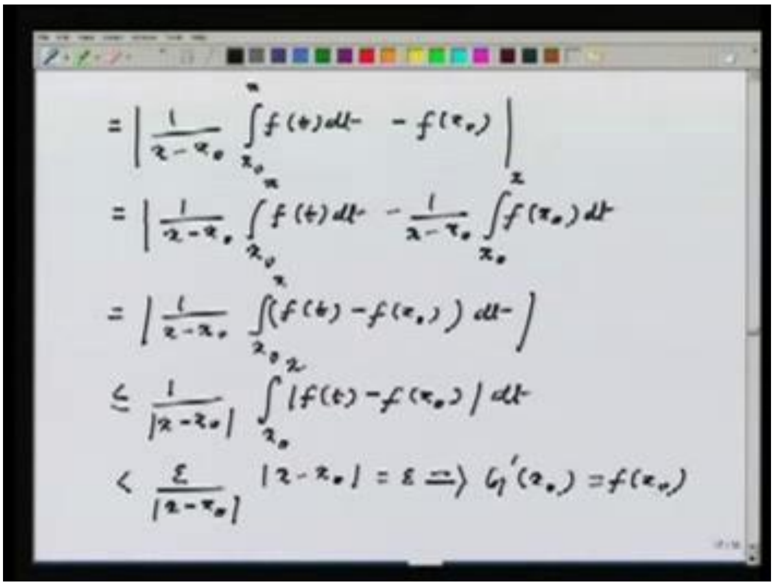

Well, now I will just write the definitions this would imply then modulus of 1 by x minus x naught times integral x naught to x f t d t minus f of x naught. That is, this just follows from the definition of G. But, now I use the trick, I write it as 1 by x minus x naught times x naught to x f t d t, no change here. The next quantity, I write it as 1 by x minus x naught times integral x naught to x f x naught d t.

Notice that, in the integrand there is no function of t f x naught is constant. That means, it will come out of the integral, f of x naught will come out of the integral and the integral becomes just x minus x naught. But, in the denominator there is a x minus x naught, they will cancel each other. So, the end result is f x naught, which I already had in my test. Now, I write this as modulus of 1 by x minus x naught times integral x naught to x f t minus f x naught of d t modulus.

And then, I use the inequality which I already proved it is 1 by modulus x minus x naught times integral x naught to x modulus of f t minus f x naught d t. Notice that, t varies now in between x naught and x. So, by continuity criteria modulus of f t minus f x naught is less than Epsilon for all such t. So, what I get is that this is less than Epsilon by modulus of x minus x naught times modulus of x minus x naught. They cancel each other to produce Epsilon.

So, this would then imply that G prime at x naught is equal to f x naught. So, that is the first fundamental theorem of calculus, that you look at the function G, which now a function of x which appears as the upper limit of the integral. The integrand is f, wherever little f is continuous at those points. The function G is differentiable and the derivative of G is f.

If I do not have continuity, I just have Riemann integrability of the function little f. Then, it trans out the function G is just a continuous function. In fact, it is a uniformly continuous function. Now, we come to the second fundamental theorem of calculus. It says that, integration of a derivative is the original function.

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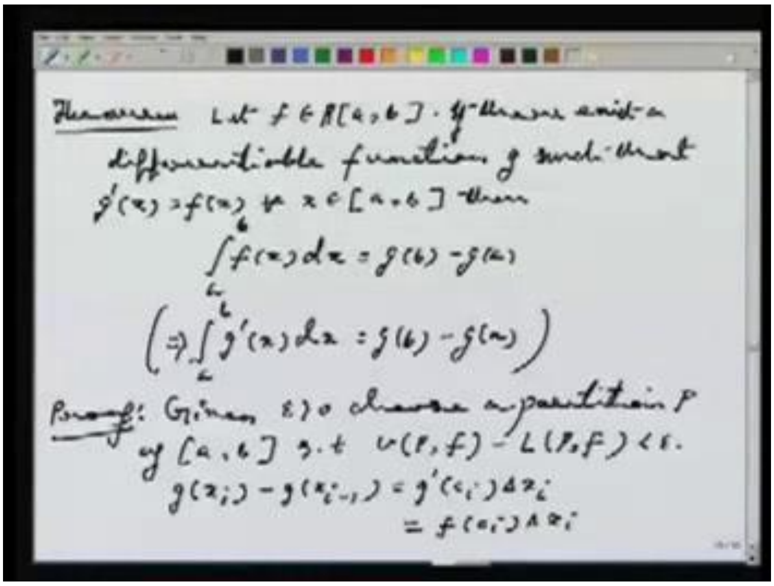

So, the precise statement is this, that suppose f belongs to R a, b. If there exist a differentiable function g such that, g prime exists for all x in a, b, then integral a to b f x d x is g b minus g a. Another way of writing would be, that integral a to b g prime at x d x is equal to g b minus g a. So, to prove this, we proceed as follows, given Epsilon bigger than 0, choose a partition P of a, b such that, U p f minus L p f is less than Epsilon, I can do that. Because, f is known to be Riemann integrable, now I look at the quantity g of x i minus g x i minus 1. Since, g is known to differentiable, I can apply mean value theorem that would tell me that this is g prime c i for some c i times delta x i, which is same as f of c i times delta x i.

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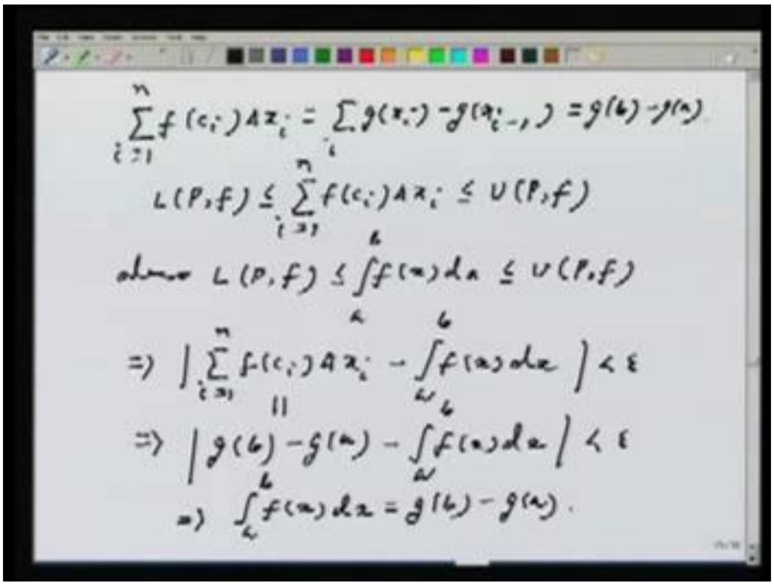

Well, then it follows that summation i is equal to 1 to n f of c i times delta x i. That is equal to summation over i g x i minus g x i minus 1, which will certainly turn out to be g b minus g a. Now, I also know that L p f is less than or equal to this summation i from 1 to n f c i times delta x i and less or equal to U p f, because U p f comes from the maximum of f and L p f comes from the minimum. But, c i is a arbitrary point, so it lies in between, so this is ok.

Also, I know that L p f since f is Riemann integrable is less or equal to integral a to b f x d x which is lesser equal to U p f. Because, integral a to b f x d x can be thought of as the upper Riemann sum, which is the infimum of the U p f's. So, it is certainly lesser equal to U p f. It can also be thought of lower Riemann sum the supremum of the L p f, so it certainly bigger than or equal to L p f. This would then imply that modulus of summation i from 1 to n f c i delta x i minus integral a to b f x d x is less than Epsilon.

But, this then implies from the previous inequality that modulus of g b minus g a. Because, I already proved using mean value theorem that summation 1 from i to n f c i delta x i is same as g b minus g a. So, this minus integral a to b f x d x is less than Epsilon for any Epsilon bigger than 0. But, this would then imply, since Epsilon is arbitrary, that a to b f x d x is equal to g b minus g a.

So, this is the end of proof of the second fundamental theorem of calculus and this is the result most of the time. We have used to find integrals of certain functions. If we can view it as derivative of something then calculating the integral is becoming very easy. So, second fundamental theorem of calculus is the most important one to evaluate the integrals. So, this is more or less all about the Riemann integrable, which we wanted to prove most fundamentals one. Now, you will go to higher dimensions outside real line to R n to do differentiation and integrations. All those results should be analogues of the result, which we have proved so far.