Mathematics-I
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Lecture -19
Applications of Riemann Integral

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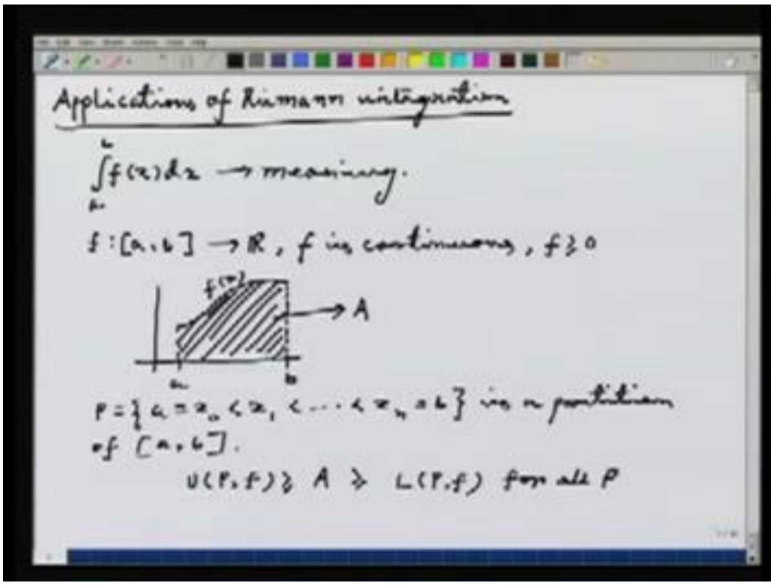

Today, we are going to see some applications of Riemann Integration. So, far we have seen only the theory development of integration. That is given a nice function f, defined on the interval a b. What does it mean to say, integral a to b is f x d x. The meaning of this symbol, we have explained in the previous lectures. Well, that is an abstract mathematical formulation. As per as history is concerned, Riemann Integration was known to people much earlier than Riemann. The sole purpose was to find areas of certain shapes or volumes of certain shapes.

Today, we will see that with the mathematical rigger, which we have injected in the theory of integration. How can it be, use that, again to solve those kinds of problems. So, first of all go to the motivating factor of Riemann integration. That suppose, I have a function f, defined on the closed interval a b to R. To be on the safer side, let us assume, f is continuous, you then already know. Because, of our previous lectures, that every continuous function is Riemann integrable.

Now, the function let know also assume f is non-negative. Well, for the purpose of integration that is not needed. All you need, that your function is a bounded function. Then, you can talk about Riemann integration of the function. It is a different matter, whether the function is really integrable or not. But, the Riemann integrable functions are necessarily bounded functions.

Now, here I am taking a extra assumption f non-negative. Then, let us say, the picture look something like. This is the point a, this is the point b and let us say this is the function f, this is f x. Now, let us concentrate on this area. This is the area, I am interested in. I want to find the area of this region. Call this area A. Intuitively; you know that something called area of this shape also.

So, then if P is a partition, that is the collection of finitely many points. Let us x n; that is equal to b. So, this is a partition of the closed interval a b. Then, form the sums U, P, f, I can also form the sum L, P, f. I know, what the connection between these two is; I know that, this is bigger than or equal to. Now, if A is the area, then the kind of picture, we drawn for the U, P, f and L, P, f. That is U, P, f is always made off the bigger triangle, which goes out of the graph. And L, P, f are made of the triangles, which lower height, which is below the function.

Then, if A is the area of the region, then I also know, that this is true. For all partition P, because U, P, f, again I will repeat made off rectangle, whose heights go above the graph of the function. And L, P, f made off rectangle, which lie below the graph of the function. And hence, it follows U, P, f must be bigger than or equal to the area, between the function and axis. That is the quantity A and this quantity certainly bigger than or equal to L, P, f.

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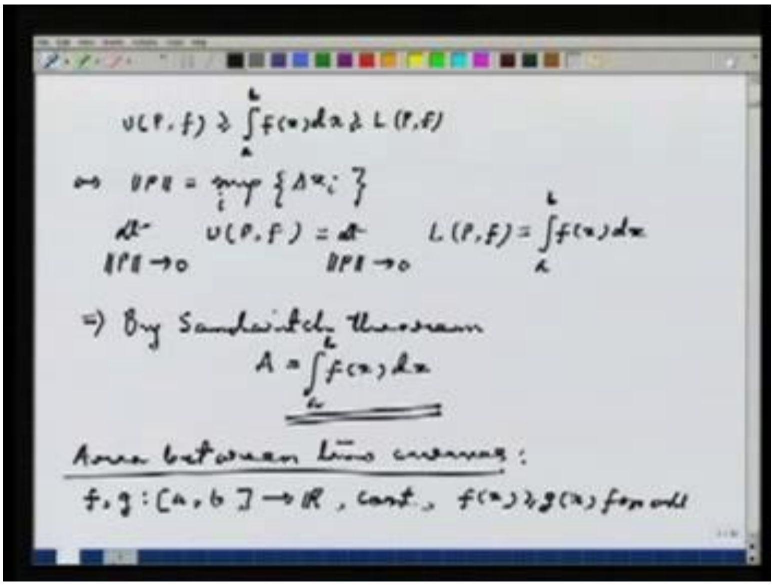

Now, since f is continuous, it is also Riemann integrable. And hence, this is also known to me. That U, P, f is bigger than or equal to integral a to b, f x d x, which is again bigger than or equal to L, P, f. Now, as norm P, if I take to be supremum of delta x i, supremum over i. And then limit norm P going to 0, it can be shown U, P, f is same as limit norm P going to 0, L, P, f. This is not very difficult to show. Trans out to be integral a to b, f x d x.

Intuitively, the meaning is the same, that you go on taking finer and finer partitions. See, norm P means what, it is the maximum possible length, which appear as the sub intervals. Now, if you go on taking finer and finer partitions, then the lengths, maximum length of the sub intervals are going to decrease heavily. Now, I go on choosing the finer and finer partitions means, that norm P are actually going to 0. If that happens, that U, P, f, converges to the integrable of the function, so does L, P, f.

Now, this implies then, by Sand witch theorem, that A must be equal to integral a to b, f x, d x. So, this is in a way, defining the area between the graph of the function and the axis. Because, to start with, it is not been defined, we know only the area of rectangles, triangles and things like that. Now, if I have weared shape, which we had. If I go back at the previous thing, this is weared kind of shape, what do you exactly meant by area of this region.

Well, we have got the meaning of that. That area is actually defined by this integral, integral a to b, f x d x. Now, I want to find area between two curves, if that possible. So, let us say situation is this, that f g, both is continuous function on interval a b. These are continuous and another given condition is f x is bigger than or equal to g x, for all x in the closed interval a b.

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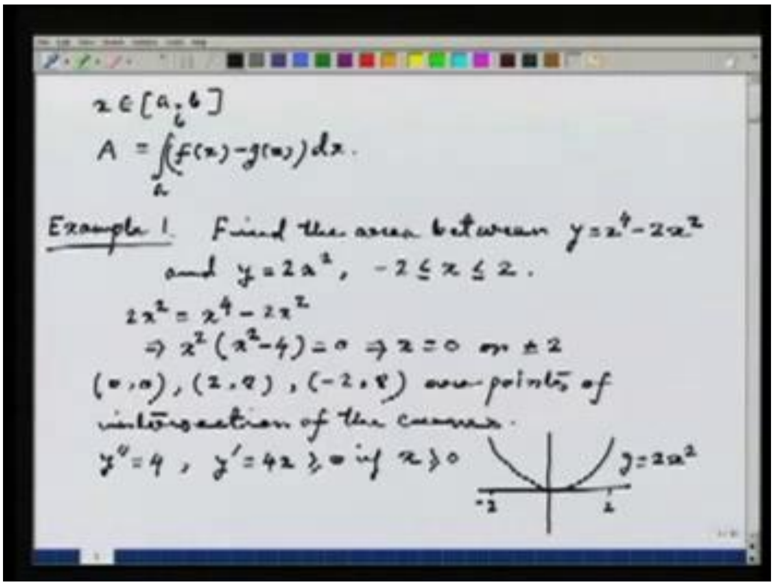

Then the area between the curves, if I call it A; that is defined as integral a to b, f x minus g x, d x. So, in the earlier case, what happened is, if I take the function g x to be the x-axis, if the graph of the function is just the x axis. Then, the previous case actually follows from this case. Now, let us try to see the examples. Let us first find try to find the area between two curves. One is y is equal to x to the power 4 minus twice x square and the other curve is y is equal to twice x square.

What is the variation of x? Well, I will say minus 2 lesser equal to x lesser equal to 2. Let us, first try to have a idea of these curves, how do they look like, then it would become easier for us to calculate. Now, first note that what are the points of intersection of these two curves, that is suppose I put twice x square is equal to x to the power 4 minus twice x square. This would imply x square into x square minus 4 is equal to 0. That would imply, x is equal to 0 or plus minus 2.

Now, if I put x is equal to 0, then I get y is equal to 0. So, 0, 0 is 1 point of the intersection of the two curves and other points are minus 2, 8. These are points of intersection. Now, I look at the curve y is equal to twice x square. What kind of a curve is that, how does it look like? The first of all, if I look at y double prime, that is 4, that means, the curve is always convex. And y prime, that is 4 x, which is bigger than or equal to 0, if x is bigger than or equal to 0.

And since, it is an even function; the curve is actually symmetric about the y axis. And then, it is very clear, how do the curve, look like. So, I will draw it here, this is minus 2, 2, then the curve look exactly like this. This is the curve y is equal to twice x square.

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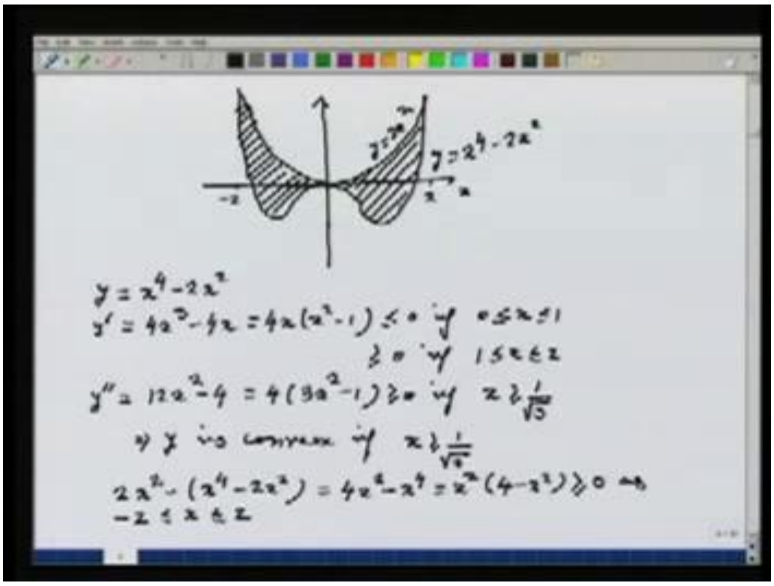

Now, comes the second one. So, I will try to draw the both curves here. So, the difference I can make out. One is x-axis, the other 1 y axis. I already know y is equal to twice x square looks like this. This is minus 2, this 2. Now, I concentrate on the next curve, which is y is equal to x to the power 4 minus twice x square. Notice, that this is also an even function, so this curve is also symmetric about the y axis.

First the derivative, that is y prime. That is 4 x cube minus 4 x, that is 4 x into x square minus 1, which is less than 0, if 0 lesser equal to x lesser equal to 1 and bigger than or equal to 0, if 1 lesser equal to x, lesser equal to 2. So, the function first up to 0 to 1, it is a decreasing function and then, it increases. Now, I look at y double prime, that is 12 x square minus 4, that is 4 into 3 x square minus 1, this is bigger than or equal to 0, if x is bigger than or equal to 1 by root 3.

Notice, I am just concentrating on the right hand side, that is positive x. Because, it is symmetric about the y axis, so the negative side of the x-axis, I can take care of the positive side. Now, from this information it implies, that the curve y is convex, if x is bigger than or equal to 1 by root 3. Before that point, it is concave. And then, the drawing of the curve is very easy. I would say that, it would look something like; that is one side, second side look like, something like this.

Well, how do, I know, one curve is always less than the other. That is very easy to check. Let us just check it here, that I want to look at twice x square minus x to the power minus 2 x square. That is 4 x square minus x to the power 4, that is taking x common x square into 4 minus x square, which is certainly less or equal to 0. Well, this is certainly bigger than or equal to 0 as minus 2 lesser x lesser equal to 2. That means, the curve y is equal to 2 x square lies above the curve y is equal to x to the power 4 minus twice x square. Now, if I want to calculate the area between these two curves, which is this area. This is the area, we are trying to find out, which is easy by our previous formula.

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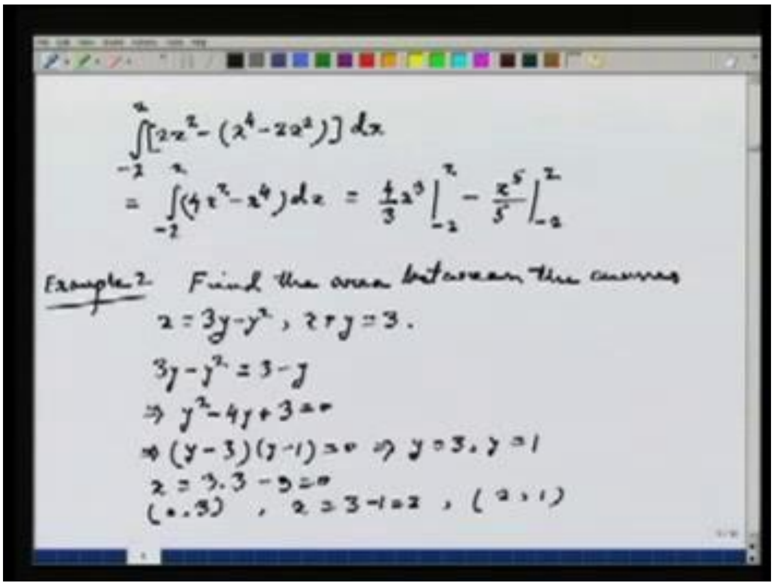

This is integral minus 2 to 2, twice x square minus x to the power 4 minus twice x square d x. Now, from the standard formulas of integration, you can very easily see that, this integral can be done. I mean, it just a polynomial and integral of x to the power n is x to the power n plus 1 by n plus 1. So, using that, what I will get is this. So, this is minus 2 to 2, 4 x square minus x to the power 4 d x, that is 4 by 3 x cube from minus 2 to 2 minus x to the power 5 by 5 from minus 2 to 2.

Now, just put the values, the answer will come out. We do not want to the exact calculation, but this is the way, you can find area between two curves, if the curves are given to you. Now, let us go to another example. This is slightly differing from the previous one. So, this is example 2. The problem is to find the area, between the curves x is equal to 3 y minus y square and x plus y is equal to 3.

Now, let us try to see, which curve is above the other one. So, for that, well, first I need to find the point of intersection also, let us try to do that first. So, 3 y minus y square is equal to 3 minus y. This implies, y square minus 4 y plus 3, that is equal to 0. This implies, y minus 3 into y minus 1. That is equal to 0. Implies, y is equal to 3, y is equal to 1, but if y is equal to 3, then I get x to be equal to 3 into 3 minus 9, that is 0. So, one point is 0, 3 and the other point is, if I put y is equal to 1, then x is equal to 3 minus 1, that is 2. So, another point is 2, 1. So, these are the point of intersection of the two curves.

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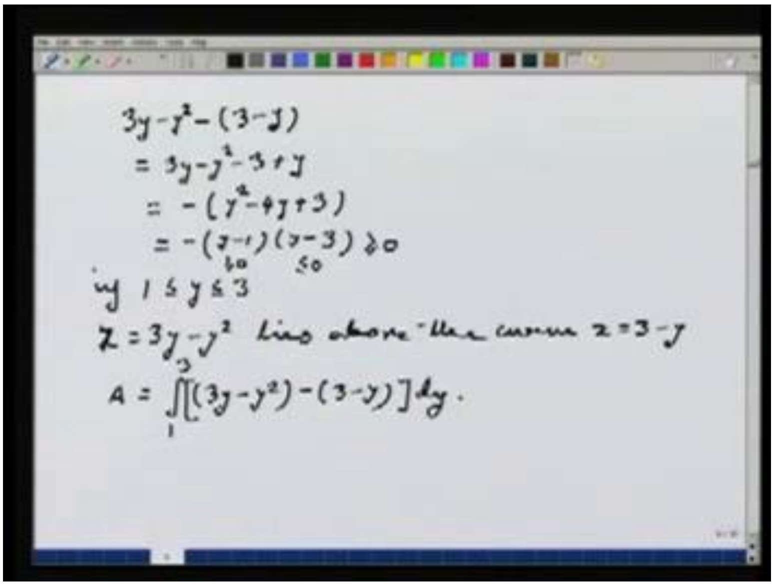

Now, to see, which curve lies above? Let us just compare, that is 3 y minus y square minus 3 minus y. That is 3 y minus y square minus 3 plus y. That is minus of y square minus 4 y plus 3. That we know, minus of y minus 1 into y minus 3. Now, if y, now question is, why do y lie, if y lies between 1 and 3. Then, what do I get, I get y minus 1 is nonnegative, this is bigger than or equal to 0. But, y minus 3 is less than or equal to 0.

But, already minus sign is there, so the whole quantity is bigger than or equal to 0. So, this implies the curve x is equal to 3 y minus y square, lies above the curve x is equal to 3 minus y. Now, to calculate the area between these two curves, what I do is, I just look at A is equal to, I look at the variation of I here instead. It is 1 to 3, 3 y minus y square minus 3 minus y, d y. This is the area between the given two curves. Here, instead of y is equal to f x, I am rather am looking at the curve x is equal to g y, that can also be done.

This where, what I wanted to talk about the area between two curves, you can find out by integration, which is certainly expected, because integration was device to find area, between axis and a function. Here, the axis, you are representing by another curve and trying to find the area, between these two curves. So, we have seen examples of two such curves.

Now, let us go to the different topic, again applications of Riemann integration. Now, as another application, we want to find certain volumes using Riemann integration. Well, the basic idea is as follows. We start with the simplest possible the three dimensional object, what is that cube certainly.

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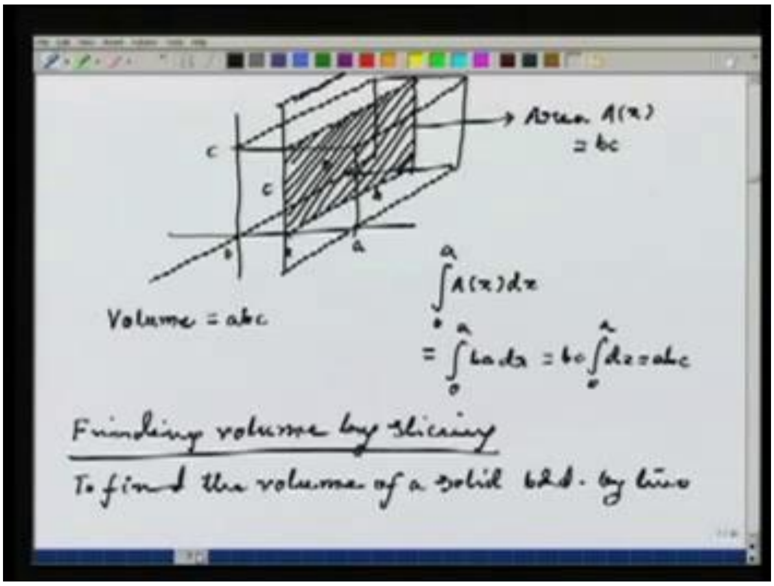

Well, I take sides of different length, that is not exactly a cube. So, these are the axis and I am trying to demonstrate, what the idea is. So, let us say this is one side, whose length is a, this is the origin. This is one side, whose length is b, I form the rectangle, which is not certainly the three dimensional and it is two dimensional. I take this height, this c and now got something three dimensional. Let us look at this and I want to find the volume of this.

So, what exactly is the method? Well, what I do is I come to a distance x here and I just pass a plane, which is perpendicular to x-axis through this body. You know, just slice it, the way; you will cut the bread, exactly the same way. So, I put a plane passing through this point. So, this plane now going to intersect the body, it will intersect the body in a rectangle, you can see the rectangle, this is the rectangle, I am getting.

So, it is a plane, which intersects the body. This rectangle, which I have formed by cutting the body through a plane at the point x, has got an area, which I call A x. So, this shaded portion A x, can I calculate this A x. Well, I just need to know, how to find an area of a rectangle. Well, one side of a rectangle is b, the height is c. So, area of A x is b c. Well, you can say something special, because the area does not depend on x, whatever x you take, you get the same area, which is A x, well that good for us.

Now, I want to find the volume of the total body. And I know the total body actually, all of you know that, I do not need integration for that; total volume is a b c. I want to see, that how this total volume comes from integration. So, what I do is, now, I know the area of the slice A x. This A x, which is the area of this plane called a slice. Now, I want to sum up the area of the slice together by varying x.

But, now would say, how do sum up. To sum up, I need certain quantity 1, 2, 3, 4, 5, 6, but here I do not have it. Because, x is varying continuously, but Riemann integration is an analogue of sum. So, what I do is, instead, I just look at integration 0 to a A x d x, just think of this integration as a sum. It is short of a, I would say continuous sum.

So, I am just continuously summing up all these A x s together. You know, integrating from 0 to a, what do I get, since A x is constant integration 0 to a b c d x. But, b c is a constant. So, it comes out of the integral, it is b c times integration 0 to a d x. But, that formula of our integration, which we all know a b c. See, what I got back is the total volume.

Well, this is the idea behind the volume by slicing. That means, given the body, I take a plane, which is perpendicular to x-axis. Through that plane, I cut the body and I get some section of the body which is two dimensional. Suppose, I find it is area, you know call that A x, then just integrate, that A x over a limit of x from where the body starts and where the body ends. So, the general method called finding volume by slicing.

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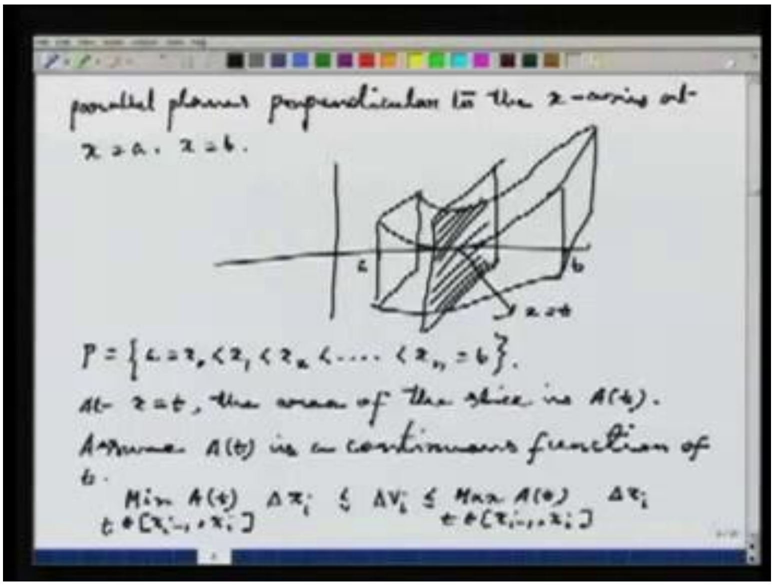

So, suppose a body given to you, to find the volume of the solid. Bounded by two parallel planes, perpendicular the x axis, at let us say x is equal to a and x is equal to b. So, the picture of the body, let us say, look something like, let us say, this is the body, I have, which is not as symmetric as the other one and I want to find a volume of this body. So, these are the plane a and b. This is bounded by two planes. I want to find the volume of this body.

So, what I do is, I take a partition P first, P is equal to x 0 less x 1 less x 2 and so on. And let us say, at x is equal to t, the area of the slice is A t, it would look something like this. So, this would be the area of the slice inside the body, it is at the point x is equal to t, I am going to continuously sum them up, but why should that come. Well, you can observe one thing, that minimum of A t, well to make things correct mathematically.

Let me also use some convention, some assumption, I need about A t, I do not want my A t to a very bad function. So, assume A t is a continuous function of t in most of the cases, you will have this assumption satisfied. Because, it some weared body is given to you, this A t can be quite a funky function, you do not want that. And then, notice the minimum of A t, where t belongs to x i minus 1 x i times delta x i is certainly lesser equal to the volume of the body. When x t varies from x i minus 1 to x i. You know, I call that delta v i, which certainly lesser equal to maximum of A t. When t belongs to x i minus 1 x i times delta x i.

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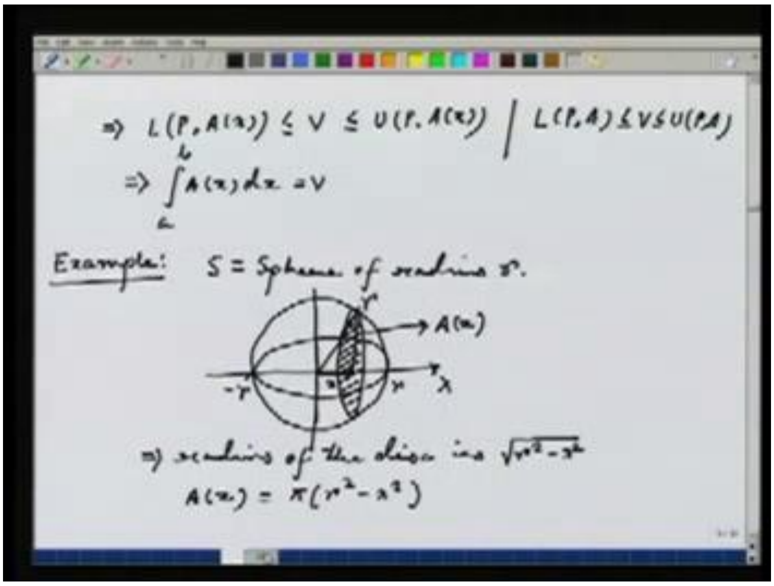

So, this in particular it would imply that L, P, A, x. That is the function lesser equal to the volume lesser equal to U, P, A, x. But, this precisely imply integral a to b A x, d x is equal to V. Well, here I have written A x to show to just mean that A is a function of x, the correct notation would have been L, P, A is lesser equal to V lesser equal to U, P, A just function of x. I have written x there, I hope discussion creating confusion, but ultimately this would imply then that integral a to b A x d x is equal to V.

Now, let us this leads to something correct and which we already know. So, the next example, we take is sphere. Notice that to find the area of the cube, we need to know the area of the rectangles. Using that, by the method of slicing, we can find the volume of the cube. Similarly, if you want to find the volume of the sphere, it is reasonable to expect. That if we know volume of a disk using perhaps that you can manage.

Because, if I slice the sphere by a plane, what I will get is a disk. And we know the area of the disk, we started with that any disk of radius r has the area pi r square; I am going to use this to calculate the area of the sphere. Anyway, given a sphere of radius r, we know that it is volume is 4 by 3 pi r cubes. I want to test, whether I get it to this method of slicing.

So, let S be my sphere of radius r. Now, it looks like this is the sphere of radius r, what I do is this is x-axis, I come to distance x, I pass a plane through this, which is orthogonal to the x-axis these are perpendicular to x-axis. That would produce a disk here. I need to know the area of this disk, so this is my A x in this case. So, what is this area, I need to know the area of this disk, that is easy to find out. Because, this distance is r, I just need to know this height.

Now, the base is x, I can apply Pythagoras theorem. This would imply the radius of the disk is square root of r square minus x square. And hence, A x is equal to I know the area of the disk is pi r square. So, it is pi of square root of r square minus x square whole square. That is pi times r square minus x square.

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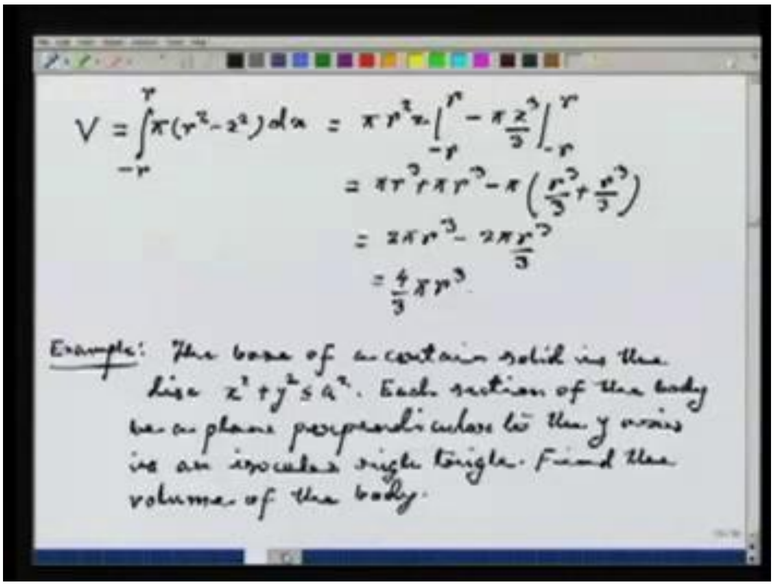

Now, much x varies, it varies certainly from minus r to r. This is minus r here goes up to r. So, look at this integral, then integral minus r to r pi r square minus x square d x. This should be the volume of the sphere. Well, this turns out to be pi r square x from minus r to r minus pi x cube by 3 from minus r to r, what do I get; I get pi r plus pi r cube. Then, minus pi into r cube by 3 plus r cube by 3, what I get is, 2 pi r cube minus 2 pi r cube by 3, that is 4 by 3 pi r cube, which matches what we already know…

Now, our method shows that to find the volume of the body. You do not exactly always need to know that, exact shape of the body. Certain, information about the body is actually good enough. For example, suppose we know area of each of the slice, then by the method of integration. You can find the volume of the body without knowing exactly what the body look like, that is demonstrate in the next example.

So, let us say, the base of certain solid is the disk x square plus y square lesser equal to a square. Suppose, I also tell you that, each section of the body, by a plane perpendicular to the y axis is an isosceles right triangle. Means, what it means, this triangle has two same side and the including angle is 90 degree. Then, the point is finding the volume of the body. First, well will try to have some idea of the body, we do not know the exact shape certain things, I know certainly.

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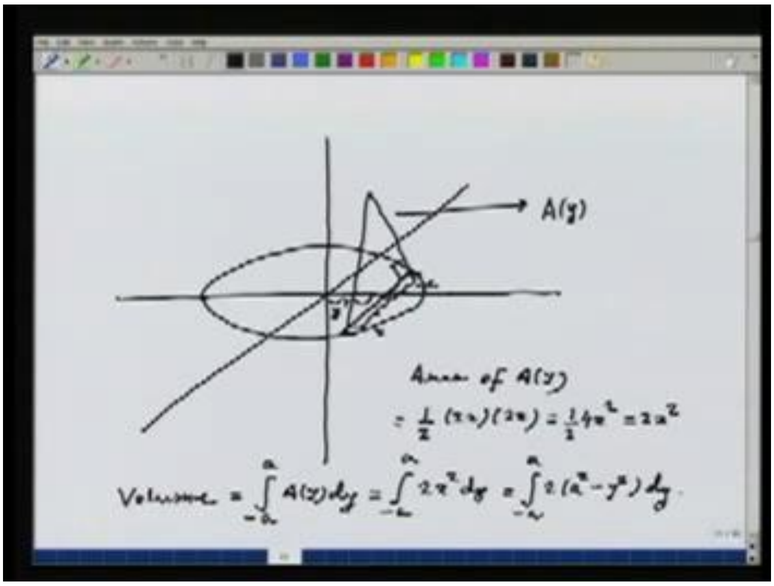

So, let us these are the axis, I will rather say, now this is the x-axis, this is the y-axis. This is the base, which has been told that is the disk, fine. Now, what has been given is, that if I come to the distance y, then this is the right triangle. It means, it looks something like this. So, the triangle, what I am getting, it looks I will write a bigger picture here, then this one, this angle is 90 and then this one.

So, this is actually instead of A x, this is my A y. I need to know the area of this slice, this slice in the body, I do not know the exact body, all I know that this is a slice. So, I want to find this area. Now, this is y, the total distance, this is the centre from centre to this distance is y and other thing. What I need to know is that height and the base of this triangle. So, the question is what else I know about this. Let us say, this portion, I do not know exactly, what is this that this portion, this portion is x.

Now, we have the following picture of the sections of the body. We do not have the idea of the full shape of the body, but what we know is that the base of the body is a disk. And I have this triangle drawn here, which is a section at a distance y. So, this distance is y. That is why, I call as A y, obviously my intention is finally, to integrate over y s.

Now, this length, I call it x. That means, the other portion also x, one side is 2 x. Since, it is an isosceles triangle, the other side is also 2 x. And hence, the area of A y, I can easily find out, it is half times 2 x, what is the height of this triangle. Since, one angle is the right angle; I know the height is also 2 x, since 1 angle is 90 degree. What I get is, half times 4 x square, that is twice x square.

Then, what I want to know is the volume, which my formula by slicing would be given by integral from minus a to a. Because, y is certainly varying from minus a to a, A y d y, that is integral minus a to a twice x square d y. Now, the variable is x square and I am integrating with respect y, it cannot be done. But, if you check where is x, x is actually point on the disk. Now, I have the equation of the disk, that is x square plus y square is equal to a square.

So, through that equation given x square, I can find it in terms of y. So, it is integral from minus a to a 2 into x square minus y square d y and you can see that this equation. This integral can also be very easily calculated, one is x square times y, other is y cube by 3 and so on. And the limit is also known and then, gets the full volume of the body. But, the point is, I do not know anything about the shape of the body, it can be very shape. All I know is, how do, each section of the body looks like. And from that, I could calculate area of those sections. And then, integrate over those sections, that are the method of slicing, that gives the volume of the body.

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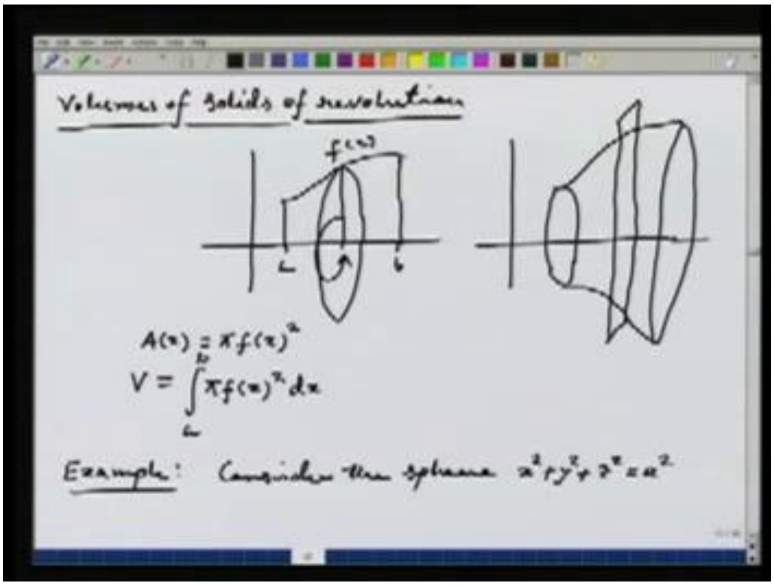

So, the next topic is volumes of solids of revolution. Well, here again, we will use the method of slicing, but slightly different way, the set up is this. That, suppose I have graph of the function again a b and this is my function f x. Then, I get the plane region which is this. Now, what I do is, I rotate this region about x-axis, you know in this form, if I do this, it will form a solid, which will look like this. So, you want to find the volume of this solid.

Now, the idea is again the method of slicing, that if I look at the slicing here. What would be the area of this? Well, then slice actually is a disk, this whole volume is rotated. So, it forms a disk here, whose radius is f x, this height is f x. So, the A x in this case, trans out to be pi times f x square and then, by the method of slicing, which we have already done. The total volume of the solid of revolution is integral from a to b pi f x square d x. Again, let us try to check how to use this. So, I look at the examples, again I will take the examples of the sphere. Because, the volume of sphere, already know, so consider the sphere. Let us say the radius r, x square plus y square plus z square is equal to a square.

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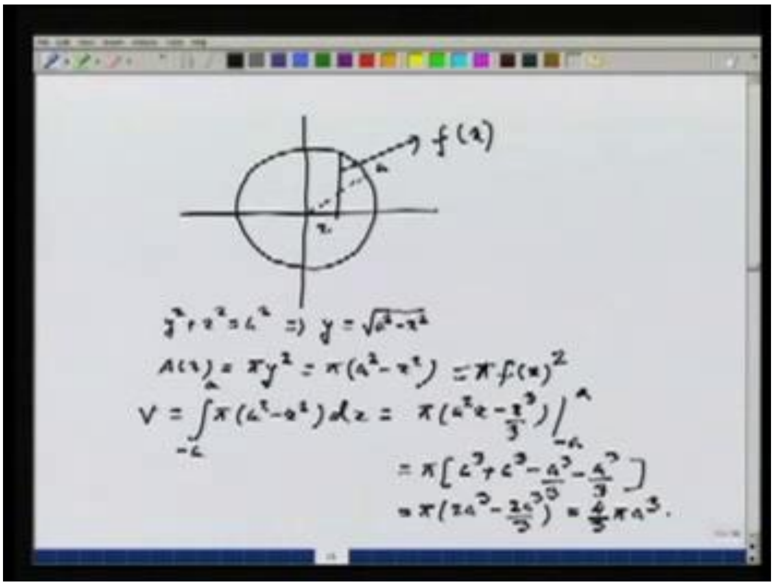

Then what I do is, I view it in the following fashion, that I look at the disk of radius a. This is the disk of radius a and I rotate this chap around the x-axis. And try to see, what do I get? Well, the slice again would be a disk, but what is this height, if this distance is x, this height, which I call f x. That y satisfies y square plus x square is equal to a square. So, this would imply, that y is equal to square root of a square minus x square.

In this case A x, according to my formula pi y square, that is pi times a square minus x square. And then, the total volume is integration is minus a to plus pi of a square minus x square d x. So, we calculate this, this is pi times a square x minus x cube by 3 from minus a to a, if I put it, I get pi times a cube plus a cube minus a cube by 3 minus a cube by 3. That is, pi times twice a cube minus twice a cube by 3, which 4 by 3 pi a cube. We know that, this is the volume of the sphere.

So, there are actually two ways, I have found out the volume of the sphere. One is the method of slicing, the slide variant of it, that is by using it as the surface of the evolution. So, the idea is, you just take upper half of the disk, rotate it about the x-axis, what you get is sphere and you want to find the volume of that. Then, the method of finding the volume of solid of revolution can be used; this is precisely what we have done. This height y actually playing the role f x, then this is my pi f x square.

And then, I have used the integration of this, with the variation of x. What I get is the volume of the sphere. So, we have seen some application of Riemann integration, we will some more, one most important, one would to find the length of a curve. For example, if you have a circle of radius r, then we know the length of the circumference is 2 pi r. We will see how by integration one comes to such conclusions, that we will continue with.