Mathematics-I
Prof. S.K Ray
Indian Institute of Technology, Kanpur

Lecture - 20
Length of a curve

In today's lecture, the main topic of discussion is the Length of a Curve. Intuitively, speaking given any curve, we can or we should able to thing about length of that curve. But, if we analyze mathematically, it turn out, there are might exists curve, which do not possesses the length. So, first we will try to make a rigorous mathematical definition of, what do exactly mean by length of a curve. And then of course, our will be given a curve. If we already know, that it has got a length, then how does one calculate the length of that curve? So, to start with again, we are trying to follow our intuition. And here, our main tool again is Riemann integration.

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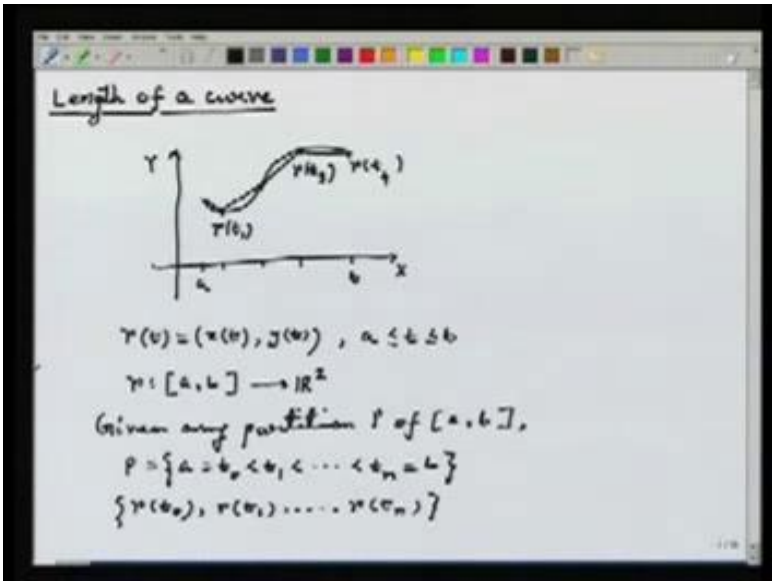

Let us start with a curve. Let us say it look like this, these are the axis, suppose this point is a, and the end point is b. Now, if I view this curve, I can view this curve as a parametric equation. Let us say r t is equal to x t, y t, where t is the parameter, it ranges in between a and b. So, in a sense, I just have a function r defined on the closed interval a b, such that r t is a point in r 2. That is, it is a vector. So, every r t has an x component and then, y component, we view that as x t y t. So, this is x axis, this is my y axis.

Now, how do you go about determining the length of that curve? What we do is, first we take a partition of a b, let us say three points. And then, we plot the corresponding image of those points under r on the curve. They look something like one point here, one point here, one here and other one here. Now, I join these points by polygonal path. So, this is first line, this is second line, then I get the third line and the fourth line.

Now, this polygonal path has got a very well defined length. Because, these are straight lines, so I can calculate the lengths. So, given any partition p of a b, let us say, the partition looks like. What we do is, we plot the corresponding on the curve. So, the corresponding points are r t 0, r t 1 up to r t n. These are points on the curve, which I have drawn in the above picture. For example, this point is r t 1, this point is r t 3 and the last point is r t 4 and so on. Whatever, partition I take, I get this point and then I calculate the corresponding length.

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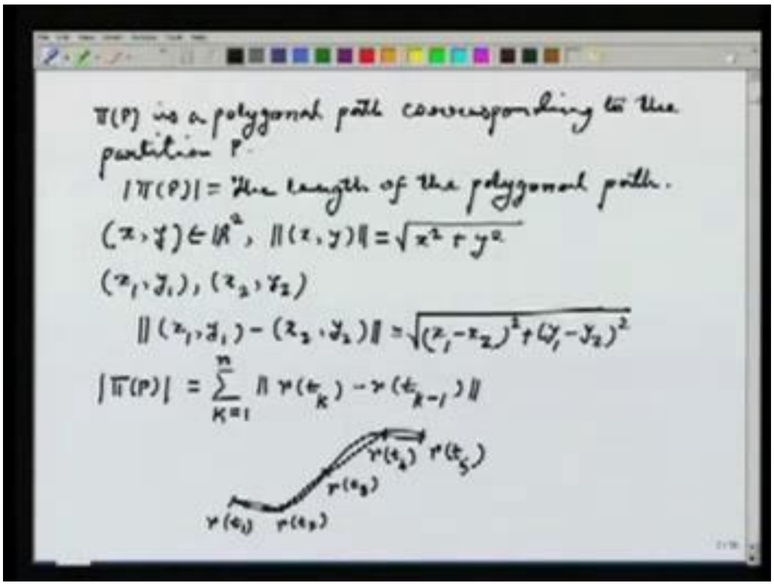

Now, let us denote this polygon pi p. So, pi p is a polygonal path corresponding to the partition p. And by mod pi p, this is just a simple, I am going to tell you, what exactly the meaning of this, it denotes the length of the polygonal path. Now, given any point x y in r 2, I define by norm x, y. The distance of the point from the origin given by the Euclidean distance, which you know from the coordinate geometry is given by x square plus y square.

So, given two points, let us say x 1, y 1 and x 2 and y 2. When, I talk about distance between these two points. That is given by norm of x 1, y 1 minus x 2, y 2 which we again know from coordinate geometry is given by x 1 minus x 2 square plus y 1 minus y 2 square. Using this formula, now I can say, that mod pi p is given by summation k from 1 to n norm of r t k minus r t k minus 1.

So, what exactly is this quantity, I will again draw the picture of the curve. It was something like this; these are the corresponding points coming out of the partition. So, the points are let us say, r t 1, r t 2, r t 3, r t 4 and r t 5 and then, I have joined them by straight lines. Then, the distance between r t 1 and r t 2 is actually given by norm of r t 1 minus r t 2.

Similarly, if I look at the line joining two points, r t 3 and r t 4, then the distance between those points is given by norm of r t 3 minus r t 4. And then, I sum them up, that gives me the total length of the polygonal path. Now, if believe in your intuition, it seems, that the total length of the polygonal path, which I get should be less than total length of the curve. Because, given any two points, the shortest distance between the two points is given by the length of the straight line, which joins these two points.

Just by that logic, then if you go between two points, through the curved line. Then, the length actually, what ever seen, it some length exists must be bigger than the straight line path. Using that, I would guess, that if that there is something called length of the curve. It should be bigger than this length of the polygonal path.

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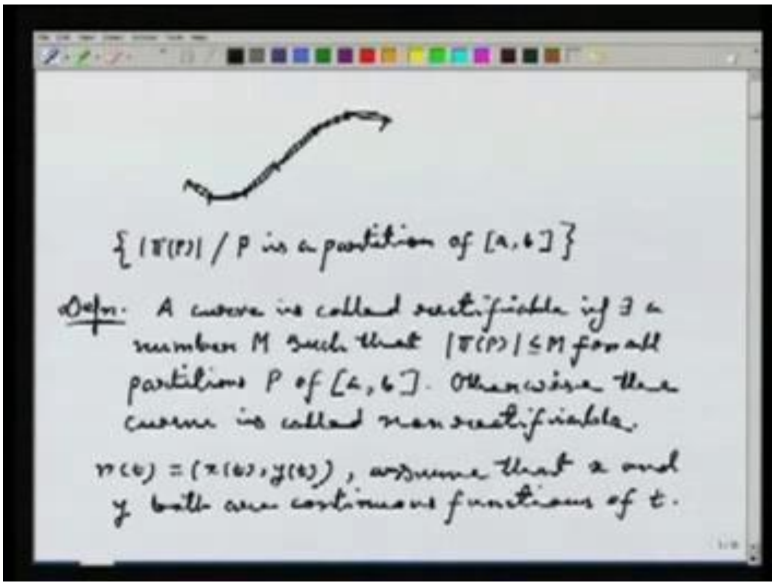

Now, suppose I look at any refinement of the previous partition. That means, I am taking more points now. So, I would take smaller distances, I would get something like this. Because, I am putting more points, I join them by straight lines. You see, now the length of the polygonal path, must have increased from the previous one. And it is getting closer to something, which might be the length of the curve. This suggests that, we should perhaps look at the supremum of mod pi p s.

Now, there is trouble in doing that. The trouble is, how do, I know, some such supremum exists. That is, I am going to look at this set. That is, mod pi p, where p varies over partition, p is a partition of the closed interval a b. I want to look at this set; this is the collection of numbers. Somehow, I feel that, if I can get hold of the supremum of that set, that perhaps should be called the length of the curve. The question is, why the supremum should exist? Now, at this point, I define something.

A curve is called rectifiable. If there exist a number, capital M, certainly bigger than or equal to 0, such that, mod pi p pies less or equal to M, for all partitions p of a b. The idea is, that if I can get a rectifiable curve, perhaps I can go for defining the length of that curve. Well, if know such number M exists, those curves are bad curves. They are called non rectifiable. So, I will just say, otherwise the curve is called non rectifiable.

Now, to make things precise, I would start with curve r, r t is equal to x t, y t. I will put some condition on x and y, these are the good curves, I would like to deal with.

Because, I do not face weared curves, assume that x and y continuous functions of t. So, whenever we talk about curves, we mean this continuous curve. That is, the x component and y component, both are continuous functions. Actually, it would mean, that the function r from r 2, itself is a continuous function. But, I do not want to go into that, because right now we do not know what exactly we mean by continuity of a function, which is defined on r. But, it is values are not in r are in r 2. That is the function of several variables. Let us now talk about that here right now. Let us just assume that the component functions are continuous.

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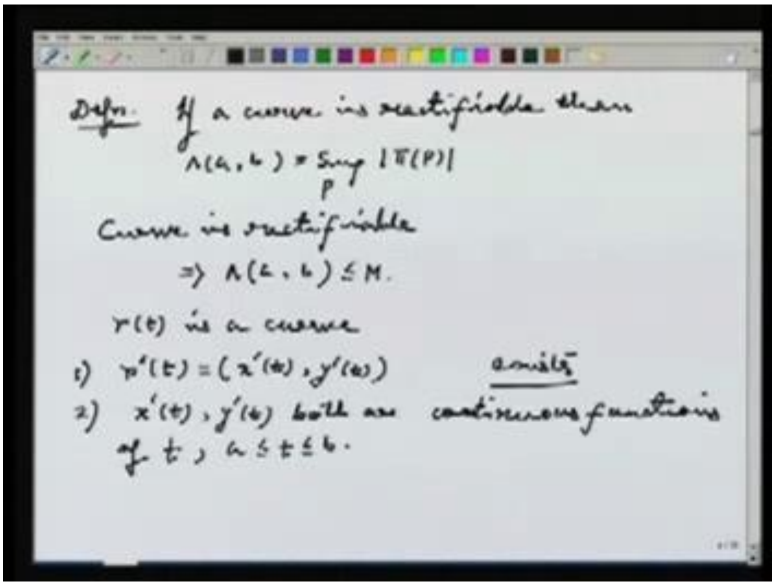

Now, if a curve is rectifiable, I will make another definition. If a curve is rectifiable, then the length of the curve, which I will denote by lambda, is a b. Because, it is parameter space is a b, as supremum over p. That is, it is the supremum over the all the partitions of the intervals of the quantity mod pi. Notice that, if the curve is the rectifiable, this would imply that capital lambda A B, which is the length. We are going to define it to be the length, is certainly lesser equal to capital M.

Because, M is an upper bound and lambda A B is the least of the upper bounds. So, it is must be less than or equal to the supremum. Now, what we need is a criteria by which you can determine, when the curve is rectifiable. So, let us make it very clear, that I am going to concentrate only on the rectifiable curves. Because, non rectifiable do exists and rectifiable curves just mean that, it has got well defined length.

And how does define the length. Well, you look at the any partition p of a b form the mod pi p. Then, go on varying the partitions, you get a set of real numbers; you get the supremum of that set. That is the called the length of the curve. Well this is the abstract kind of definition. Most of the times, when you need to do the calculations for specific curves; this definition will not going to work.

So, we have to find some close analytic formula for finding length of a curve. If the length exists, that is what we are going to do now. But, the first thing is, under what conditions we can say a curve is rectifiable, so given a curve r t. So, x and t, both are continuous functions. I define something called r prime t, which is given by x prime t, y prime t, if it exists. That is, I am imposing more conditions on the component function x t and y t. I am demanding that, they are differentiable.

Earlier, I have started just with continuous curves. But, it is something more. I am assuming that the derivative also exists of the component function. Well, now what I am going to put is another condition. So, this is my first assumption. Second assumption is, x prime t, y prime t, both are continuous functions. So, I have taken a curve r, such that, r prime t is equal to x prime t, y prime t, this exists.

And second condition is x prime t y prime t, both are continuous functions of t, where a lesser equal to t lesser equal to b. So, these are the two conditions. So, it is just a curve, such that, it is component functions are differentiable. And they are continuous also on the space of parameter. Let us assume these two conditions, under this conditions, I am going to show the curve is rectifiable.

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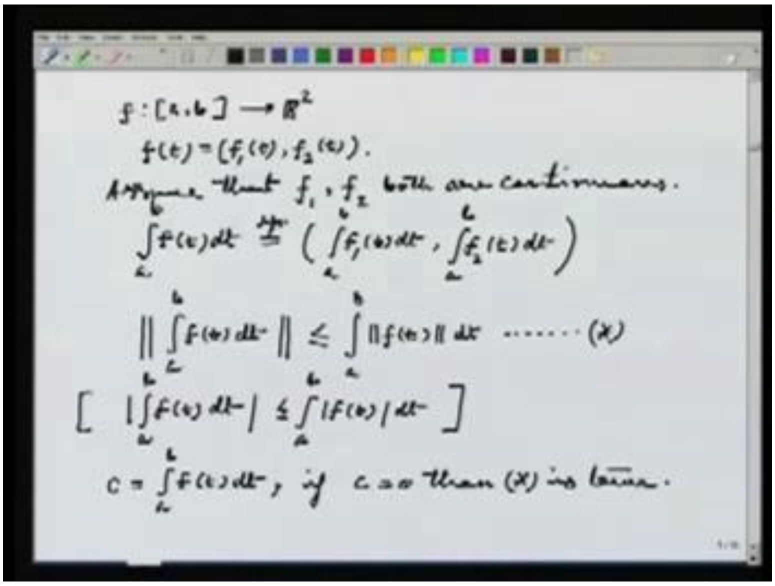

Now, for that I have to build some machinery, it is as follows. Suppose, I have function f a closed interval a b to r 2, then f of t can be written as f 1 t, f 2 t. That is the x component and the y component. Now, assume that f 1 and f 2, both are continuous functions. And then, I am going to define something called integral a to b, f t, d t. This requires a definition, because note that, f is no longer real valued functions. It is a function, whose values around r 2. So, it is a vector valued function and I am going to talk about it is integral.

Now, our natural intuition suggests, this integral should define vectors only. Since, f is a vector valued, the integral of f should be a vector. Well that vector, we define by, so this definition as this vector integral a to b, f 1 t d t. That is the x component, second one is integral a to b, f 2 t d t. Once, I do this and then, I can talk about an analogue of the something, which we done already.

So, what I do is, I look at the norm of this vector, integral a to b, f t d t. Notice that, integral a to b, f t d t is a vector by my previous definition of integral. Thus, it has got a norm. That is distance from 0. At the same time, I can also talk about integral of a to b norm f t d t, because f t is a vector. So, this has got a norm and that norm, now defines the functions. Well, if you assume that f 1 and f 2, both are continuous functions. Then, it can actually proved, that the function norm f t, which is the integrand of the right hand integral is actually continuous function of t. And hence both the integral actually exists, no problem with that. With the assumption of continuity on f 1 and f 2 1, one can prove that, both the integral exists. Now, a question is what is the connection, between these real numbers?

The left hand side is norm of a vector, so it is a real number. The right hand side is the integral of, now real valued functions. T going to norm f t is a continuous real valued function on the closed interval a b. So, I integrate it, I get a real number, what is the connection between these two real numbers. Well, the connections turns out to be lesser equal to this. And you know, whose analogue is this, it is the analogues of the fact, mod integral a to b, f t d t is lesser equal to integral a to b, mod f t d t.

So, is the analogue of this, this well known result, which we have done earlier? Now, the questions is, how do you go about proving it. Well, what do is, I define a vector c by this left hand formula, that it is integral a to b, f t d t, this is the vector. If c is equal to 0, let me given name to this inequality, call it star. If c is equal to 0, then star is true. We careful about the meaning of c is equal to 0, it is not the number 0, it is the zero vector. That is it actually is the origin 0 0. If c is equal to 0, then what happen is left hand side of star is 0, because length of the origin from itself is anyway 0. Now, if you look at the right hand side integral, that any way is nonnegative. Because the given function is norm f t; that is also a nonnegative function, hence the inequality follows.

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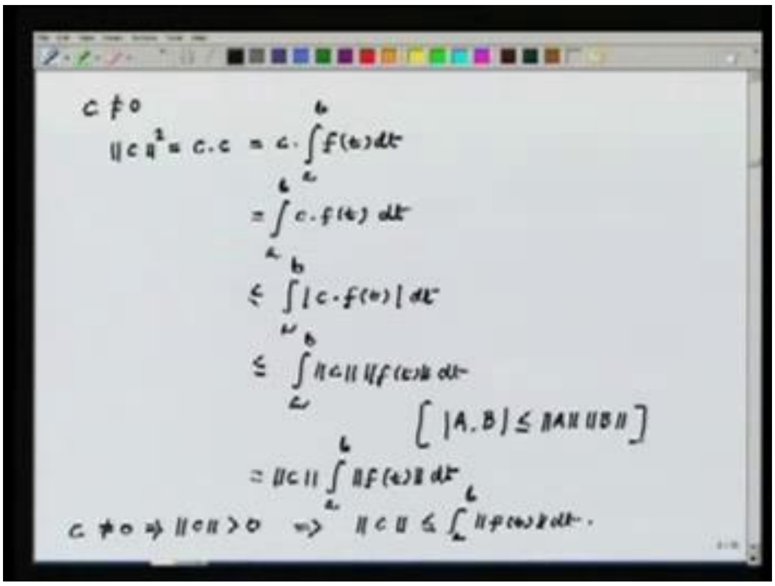

So, the genuine thing to prove, when c is not equal to 0, so it is a non zero point in r 2. Well, then I start with norm c square, you know, your usual knowledge of vectors, this is actually the dot product c with itself. I am talking about the vector dot product in two dimensions. This now, I can write as c dot integral a to b, f t d t by my definition of c. This now I can write as integral a to b, c dot f t d t. This you can check yourself very easily, just by writing down the definitions of integrals, writing f as f 1, f 2 and c as c 1, c 2.

Now, this quantity, any way is lesser equal to, since this is a real number, c dot f t is a real number, do not forget that. Because, it is dot product of two vectors, it gives you a scalar. Since t is a function, t is real number, f t is a vector, c is a vector. So, the dot product is the scalar. So, this is lesser equal to modulus of c dot f t d t. Now, this anyway lesser equal to integral a to b norm c times norm f t, which is an elementary inequality about vectors, I will write it separately, what do I mean by this.

I mean, if you have two vectors a and b, you look at the dot product. Then, this is the number, then the modulus of this dot product is lesser equal to norm a times norm of b. This is called Cauchy Schwartz inequality. It is not very hard to prove, you can look at any book, which talk about vectors. This result will be proved there or you can try to prove it yourself.

Anyway, that would then mean, that this is equal to norm c times integral a to b norm f t d t. Since, I know that c is non zero; that means norm c is strictly positive, c is not equal to 0, implies norm of c is positive, it is bigger than 0. That means, I can cancel this norm c from both sides of the inequality. Because, it is a positive number, if I do that. That implies from the above inequality. That norm c is lesser equal to integral a to b, norm f t d t.

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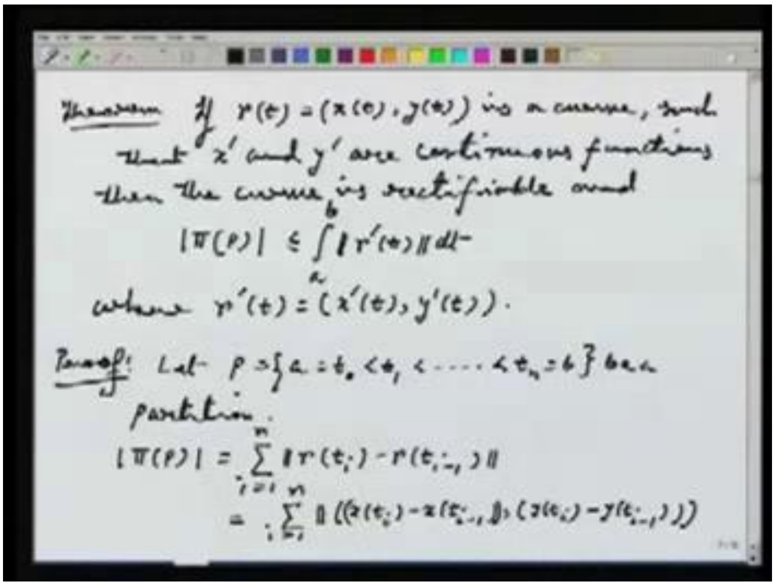

Now, using this inequality, I will go to my next result. So, I call it as a theorem. If r t is equal to x t, y t is a curve, such that, x prime and y prime are continuous functions. Then, the curve is rectifiable and mod pi p is lesser equal to integral a to b, norm of r prime d t. Where, r prime t is equal to x prime t and y prime t. Well, this mean then, that mod pi p is, the set of pi p is, if p varies over the set of partition of a b; that is a bounded set.

And one upper bound, at least, we know that is integral a to b, norm r prime t d t, which certainly shows, that the curve is rectifiable. That means to prove this theorem, the only thing, I need to prove is to prove this inequality. Well, that is very easy. So, start with the proof of this, start with any partition. So, let p is equal to, this be a partition and then I, form mod pi p. That is, summation I from 1 to n, norm r t i minus r t i minus 1. Now, this is the quantity, which I have to play around with. Well, I write that as now, summation i from 1 to n, norm x t i minus x t i minus 1 comma norm y t i minus y t i minus 1.

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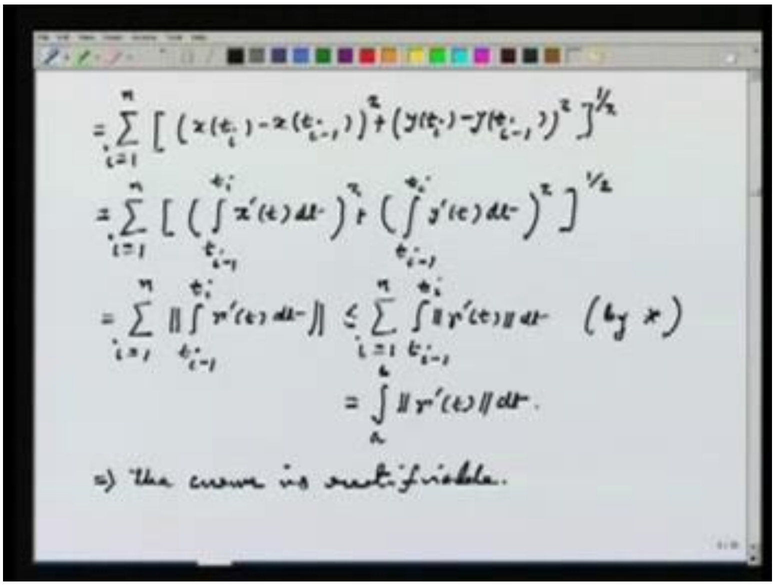

So, these are the quantities I get. And now, I can calculate the norm, which I will write as summation i from 1 to n, norm x t i minus x t i minus 1. I am just writing down the expression of the norm plus y t i minus y t i minus 1, whole to the power half. This is the expression of the norm. Now, at this point, I am going to use the fact x and y are differentiable. And the derivatives are continuous, because now I need second fundamental theorem of calculus.

So, this I will write as summation i from 1 to n, integral t i minus 1 to t i, x prime t d t square plus integral t i minus 1to t i, y prime t d t square whole to the power half. This is by second fundamental theorem of calculus. Now, what does this mean, if I view this x prime t y prime t, t going to x prime t, y prime as a vector valued function? Whatever, is written here is norm of integral of r prime t. So, this I can write as summation i from 1 to n, norm integral t i minus 1to t i, r prime t d t.

Now, I can use the previous inequality, which would suggest this is lesser or equal to by the inequality star. This is, what I get, this is by star. Now, by the elementary formula of integral, that sum up, then to integral a to b, norm r prime t d t, that the right hand become independent of the partition p. Whatever p, you take mod pi p is lesser equal to this integral, this implies the curve is rectifiable. Thus, what we have proved. So, far is that, if x t and y t is differentiable functions and are continuous, then the curve certainly a rectifiable.

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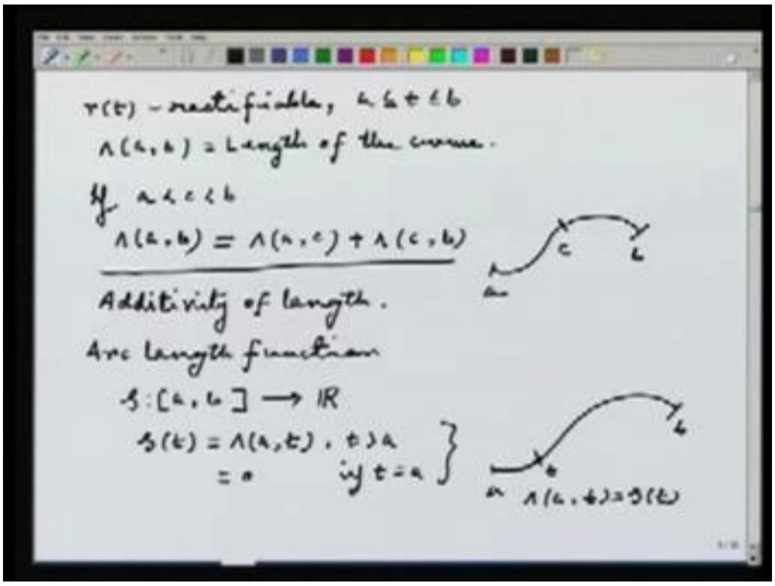

Now, under these assumptions, I will try to find a analytical formula for the length of the curve. Now, let r t be a rectifiable curve, in that case, I can define f of course, the parameter varies on the set a and b, lambda a b is the length of the curve. That we could define, since the curve is rectifiable. Now, the following thing can be proved not very easy, but it is a provable fact. And which we will need is the additivity of the length, that is c lies between a and b.

Then, I can talk about length lambda a b, I can talk about lambda a c and lambda c b. That is the parameter ranges from a to b, then the length of the curve is lambda a b, when the parameter ranges from a to c. Then the length of the curve is lambda a c and if the length of the curve is and the parameter varying from c to b. The length of the curve is lambda c b.

Well the relation is, that this is equal to this plus this. That means, on a curve, if I go up to this point, whatever is the length, then I go up to this point. Whatever is the length, then the total sum actually adds up to the total length of the curve, which is believable. And, it can be proved not very trivial, but if we use the definition of lambda a b using supremum. And using the property of the supremum, one can prove that this is true; this is called additivity of the length. So, this relation is called additivity of the length.

The next is the Arc length function. How does one define the arc length function? It is just, I call this function s. It is a function a b to r defined as follows, s of t is equal to lambda a t, if t is bigger than a, if this is equal to 0. If t is equal to a, what we are doing is that, if the curve is this, let us a and b, then from here I start moving. Suppose, this is the point t, then the length, which I am travel that is lambda a t, that is called s t. As t varying, the way I move on the curve, whatever length I travel along the curve is called s t.

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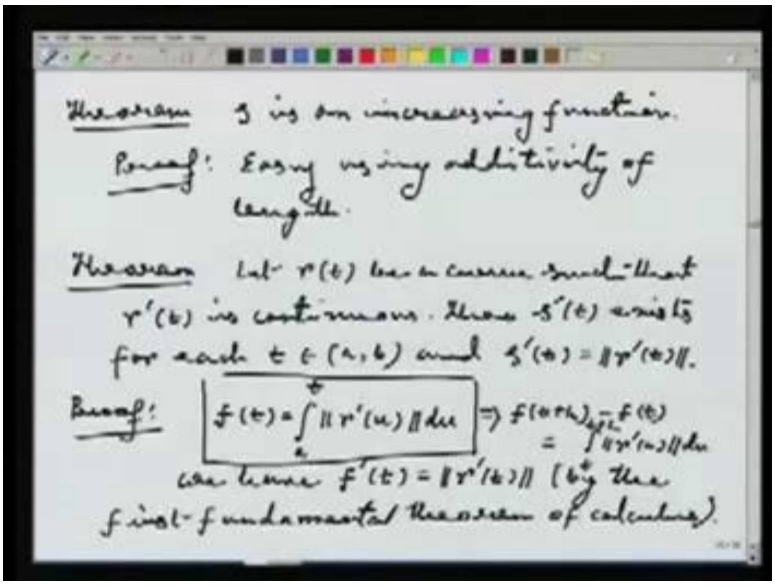

Now, this function using the additivity property of length, I will write it as a theorem. I get some property of this function. That s is an increasing function; it follows very easily using the additivity of the length, which we have assumed, so proof, I will not give, proof is trivial, proof is easy using additivity of length. Now, there are some more important properties of this function s. We are interested in and that is given in the following theorem noise.

So, the theorem is this, let r t be a curve, such that, r prime t is continuous. Then, s prime t exists, that is the function s, which we have defined turning out to be a differentiable function for each t in the open interval a b and s prime t is equal norm r prime t. Not only differentiable, I can explicitly calculate the derivative also in terms of r prime, remember r prime t means x prime t comma y prime. That is the derivative of the component function, that new vector valued function is called r prime t.

Let us try to prove this result, what I do is, first I define f of t is equal to integral a to t norm r prime t d t. Notice that, this function f well defined, because r prime by my assumption is a continuous function. If, I take the norm of that, as a function of t it is still continuous hence Riemann integrable. So, the function f t would make sense. Well, I will make a slide change, because what I mean here is r prime u d u. Because, t is the limit, well then by the fundamental theorem. We have f prime t is equal to norm of r prime t, this by the first fundamental theorem of calculus.

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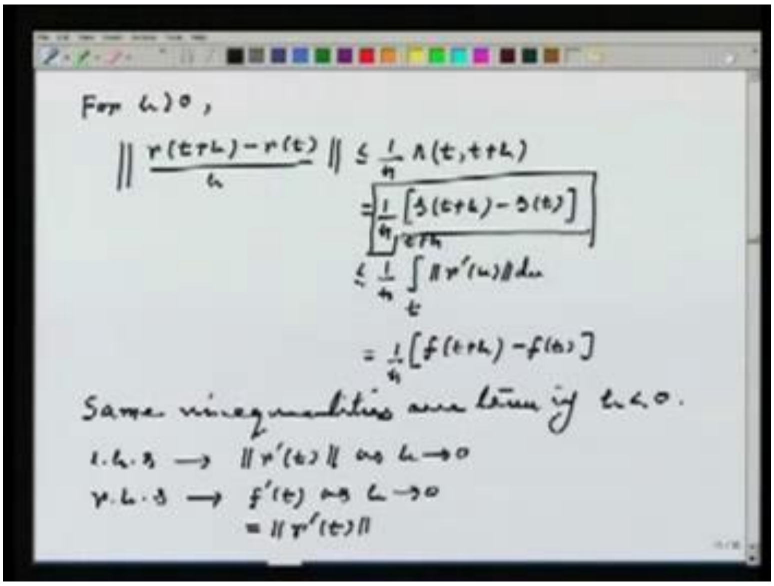

Now, start looking at different quotient, the following different quotient interested in. So, for h bigger than 0, I look at norm of r t plus h minus r t divided by h. Then, this is certainly lesser equal to 1 by h lambda t, t plus h. Because, norm r t plus h minus, r t is just is the length of a line, which is certainly lesser equal to curved length on the curve, which is lambda t t plus h.

Now, this is 1 by h times, I can write as s t plus h minus s t by the definition of the arc length function. Now, already proved that, this lambda t, t plus h is lesser equal to 1 by h is just a factor integral t to t plus h norm r prime u d u. Remember, this was the result, which I proved, while showing a curve is rectifiable. If, r prime is continuous, remember that, I have proved that, lambda a b is lesser equal to integral a to b norm r prime t d t.

Instead of a and b for t and t plus h, I am using the same result to get the inequality on the right hand side. Now, this is again by my definition, 1 by h, f t plus h minus f t. This comes, just from the definition of little f, if I go back the definition of little f, it is given here from here. It implies that f t plus h minus f t is equal to integral t to t plus h, norm r prime u, d u, this precisely, what I have applied.

Now, what I want to do is, take limit. Well, another observation is which you can prove easily, that the same inequalities are true. If h is less than 0. That can be very easily shown, it is not very difficult just by the same reasoning. Now, I want to take limit h going to 0 on the left hand side and on the right hand side. Now, the left hand side goes to norm r prime t as h going to 0. And I look at the right hand side, that goes to f prime t as h goes to 0. But my first observation, after defining f from that, it follows, that f prime t is also is equal to norm r prime t.

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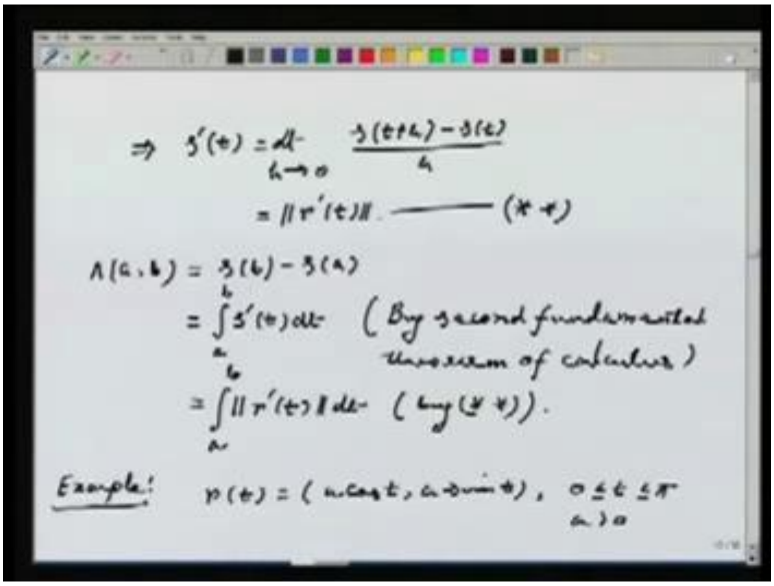

That means as h goes to 0, the left hand side and right hand side both converges to the same quantity, which is norm of r prime t. That implies then, I look at, what is the middle quantity, that is this 1, this is the middle quantity. So, this implies s prime t is equal to limit h going to 0, s t plus h minus s t divided by h, which is norm r prime t. Because, that is the middle quantity and this is precisely, what I we wanted to prove.

Now, this has got some interesting consequences, which is precisely, what we want look into that. If I look at lambda a b, what is lambda a b, that according to my notation is s b minus s a. But, this then by the second fundamental theorem is integral a to b s prime t d t. This is by second fundamental theorem of calculus, but this then, which I have proved.

So, far is norm of r prime t d t, if I call this double star, the last line follows by double star.

Now, this is the concrete formula by which, we can calculate length of certain curves. So, as an example, let me start with the upper half portion of the circle. So, let us say r t is equal to a cos t a sin t, where 0 lesser equal to t lesser equal to pi and a something bigger than 0.

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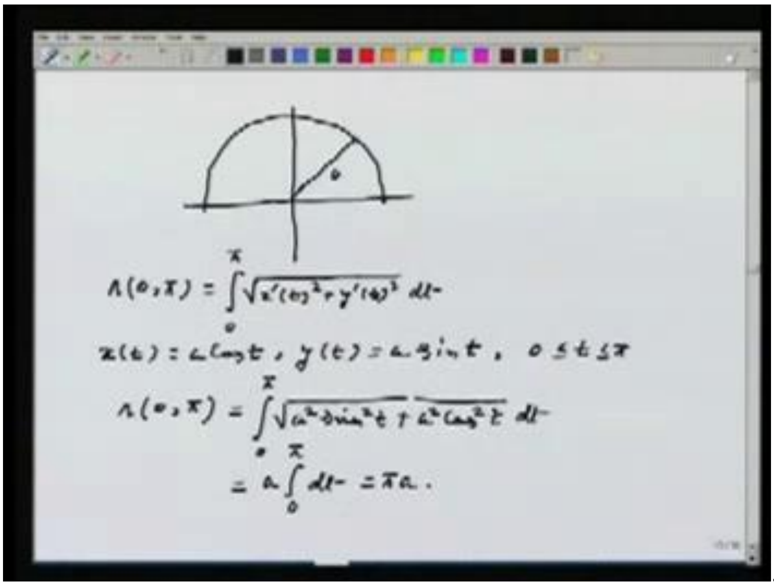

So, in the picture, it would look like, if this length is a, I am looking at this portion of the circle and you know from your elementary mathematics knowledge. That this is actually, pi a, we will try to see, whether I get pi a. By my formula, then lambda 0 pi is integral 0 to pi square root of x prime t square plus y prime t square d t. For me x t is a cosine t and y t is equal to a sin t, t lies between 0 and pi.

You can put all these quantities to get lambda 0 pi. That is equal to integral 0 to pi square root of a square sin square t plus a square cos square t d t. That is a times integral 0 to pi d t, that is pi a. So, it matches with, what we know from the elementary geometry, that length of the upper half of the circle is pi a. Then, using symmetric, you can certainly say that the total length and the additivity. That the total length of the circumference of the circle would be 2 pi a, if a is the radius. So, this is the radius, which is a.

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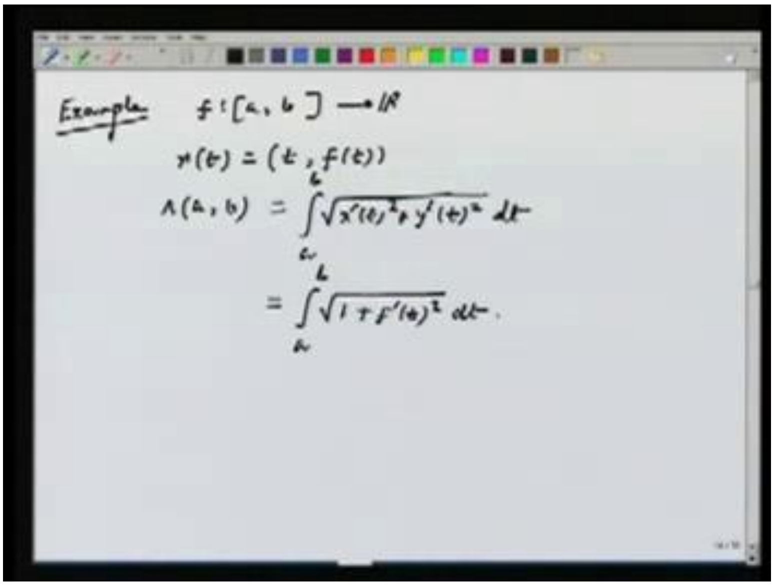

Another example would be, suppose the curve not given in the parametric form, so just f is given from a b to r, then I can look at the curve r t is equal to t f t, that also represents the curve, if I want to know the length of this, then what is the length of a b. This is integral a to b square root of x prime t square plus y prime t square d t, which is integral a to b. Because, x t is t; that means, x prime t is 1, so it is 1 plus y t is f t. So, f prime t square d t. So, when the curve is given in the form as graph. Then, the length of the curve trans out to be integral a to b square root of 1 plus f prime t whole square d t.

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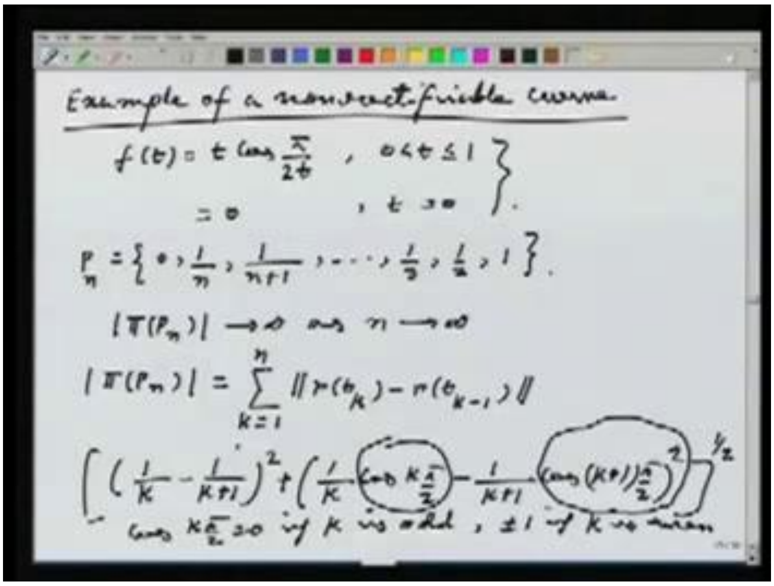

Now, I give you an example of non rectifiable curve. The curve is written here, f t is equal to t cos t pi by 2 t, if t lies between 0 and 1 and 0 otherwise. To show that, it is not rectifiable, what I do is I choose a collection of partitions, one such partition written here. For each n, I can define p n to be this; I can show mod pi p n. That goes to infinity as n goes to infinity. That means, all the pi p has the p varies over the partitions and not bounded above by some number and hence the curve is not rectifiable.

To check this, what I do is mod pi p n is anyway summation k from 1 to n norm of r t k minus r t k minus 1, if I write it in this form. All I have to do is, I have to look at quantity like 1 by k minus 1 by k plus 1 whole square plus 1 by k cosine k pi by 2, minus 1 by k plus 1 cosine k plus 1 pi by 2 square whole to the power half. And then, sum over k, for one partition p, k will vary from 1 to n or some such thing. Now, looking at cosine, I know that cos k pi by 2 is equal to 0, if k is odd and if k is even, it is plus minus 1, if k is even.

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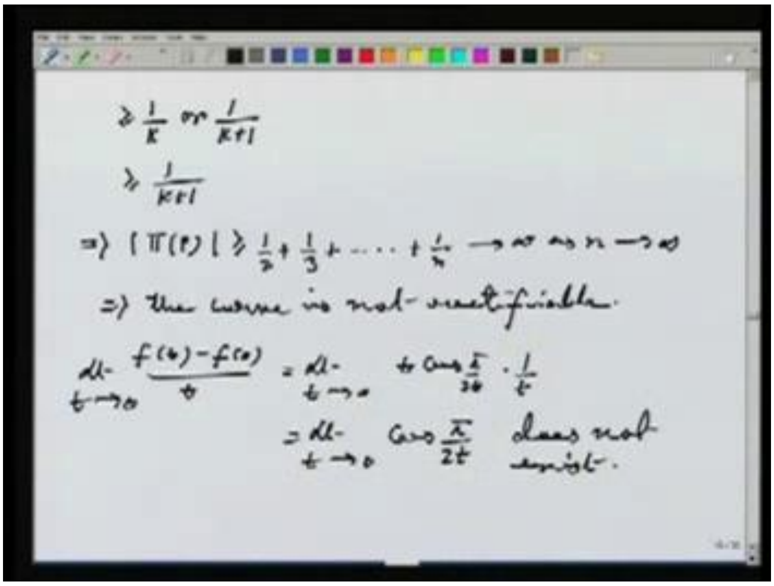

So, whether k is even or odd 1 quantity here or here would be 0. If k is even, then cos k plus 1 pi by 2 is 0, if k is odd, then cos k pi by 2 is 0. Using that, I can say that the above quantity is bigger than or equal to 1 by k or 1 by k plus 1. In any case, it is bigger than or equal to 1 by k plus 1, and hence this would imply that mod pi p. Since, I am summing over the k s is bigger than or equal to half plus 1 third plus, so on up to 1 nth.

But, I know that the right hand side is the nth. Almost, the nth partial sum except the first term is the nth partial sum of the divergent infinite series 1 by n. So, it goes to infinity as n goes to infinity. This implies the curve is not rectifiable, well this is happening, because if I look at the differentiation or the derivative of the function f t minus f 0 divided by t limit t going to 0. This is limit t going to 0 t cos t pi by 2 t into 1 by t, that is limit t going to 0 cosine pi by 2 t.

But, this limit does not exist, it is not a differentiable curve and as a result, I could see that is not rectifiable. Well, it is not really if and only if, but since it is not rectifiable, there must be wrong with the derivatives; that is what I wanted to check in turns out to be the case. So, there do exist the curves, which do not possess the length, they are some curve. But rectifiable curves are those for which the length exists and if the curve is rectifiable. That case, I could show you, that using the theorem of theory of Riemann integration. It is possible to find the length of the curve using some integral expression, which is needs and given the data, one can use it to calculate the length of the curve.