Mathematics-I
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Lecture - 21
Line Integrals

In today's lecture, we are going to start discussing about Line Integrals and Surface Integrals.

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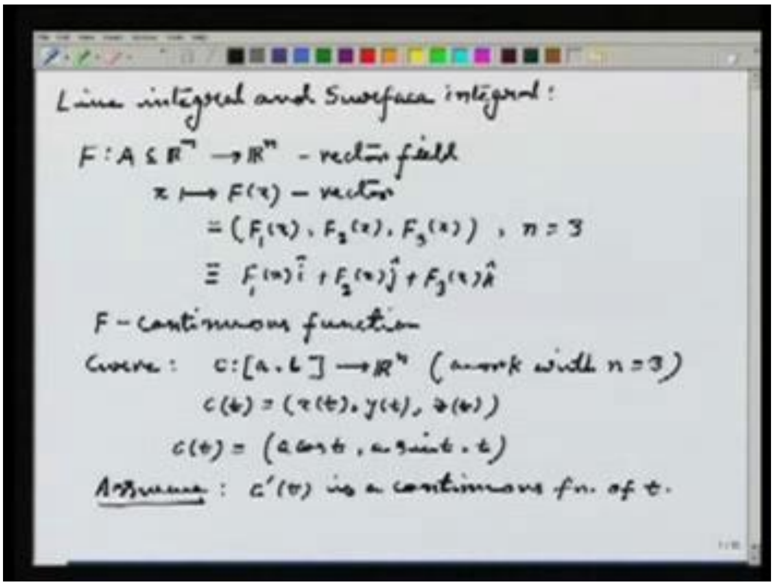

To start with we will first deal with line integrals. It talks about integrals of functions, of course which are defined on r n and taking values also in r n. And suppose on r n I have given a curve, I want to integrate the function over the curve. What exactly does it mean that is what I am going to explain now. So, suppose I have a function F defined on a subset A of R n to R n, where most of the time I am assuming any strictly bigger than one of course.

Usually they are called vector fields. That is given an element of capital A it assigns a vector which is f x. So, x goes to F x which is a vector now. So, I can denote it sometime by F 1 x, F 2 x, F 3 x the case n equals to 3, of course. Or I can also use the vector notation, that is F 1 x i plus F 2 x j plus F 3 x k the standard vector notation of R 3. Now, these are called the vector fields in space. Because, n equals to 3 accordingly we can define vector fields on plane also. That is the case equals to 2.

Now, in whatever I am going to do I will assume that capital F is a continuous function. Now, suppose I have a curve, what do you mean by a curve well it is a map c. Define on some closed interval a, b to R n I will usually work with n equals to 3. That is a space curve if I choose n equal to 2 I will call it a plane curve. Now that means, the curve is usually denoted by this also c t equals to x t, y t, z t. For example, I can look at this kind of a curve, that c of t equals to a cos t, a sin t and then t this is certainly curve.

So, that is a continuous function, so I will demand a property on c that is will assume. That means, I am looking at some specific kind of curve, which has certain analytic property. That would help in defining line integral. The property is that c prime t is a continuous function of t with this data.

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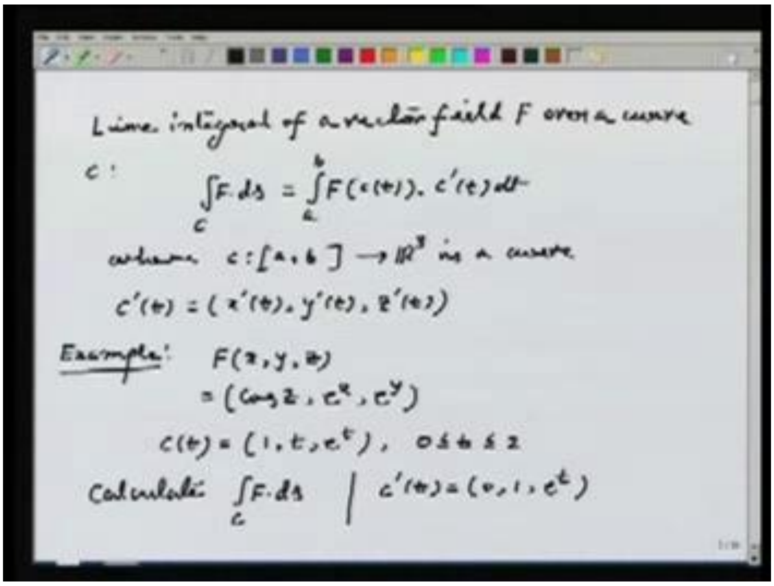

Now, I am going to define the line integral of a vector field f over a curve c. What do I mean by this well the symbol for the line integral is integral over c F dot d s. I will explain what is the meaning of this symbol? The meaning is this following integral, integral a to b F of c t dot c prime t d t, where c from a, b to R 3 is a curve. Of course, it is a curve with the assumption, which I have assumed that c prime is a continuous function.

Now, let us look at the definition again. Now, c t is a vector because, c is a curve in R 3 throughout I am going to work with n equals to 3. So, it is always R 3, but there is no losing generality in saying that. So, c t is a vector it makes sense to apply F on that vector, as a result F of c t is a vector. Now, if I look at c prime t that is also a vector because, what is c prime t. In the parametric notation it is just x prime t y prime t z prime t, so given a value of t this is a vector F of c t is also a vector.

So, it makes sense to talk about the dot product of these two vectors, as a result then I get a scalar valued function. And what is the domain of definition of the function that is F of c t dot c prime t. While t varies in the closed interval a, b, so it defines a function from closed interval a, b to r. And all because of all the continuity assumptions, which I am having, because my vector field is also a continuous function c prime is continuous. That means, obviously, c is continuous it turns out that the integrand actually is a continuous function of t.

And hence it is integrable. So, we do the ordinary Riemann integration of this function. This is called the line integral of the function over the curve c. To understand the definition more clearly let me look at an example now. So, I will give you a particular vector field now, so the example I have in mind is given by. So, let me define the vector field as F of x, y, z, which is defined now as cosine of z. Then e to the power x, then e to the power y this is certainly a vector field.

Now, I define a curve that is c of t. So, the curve is defined as 1, t, e to the power t and what is the variation of the parameter well. Let us say 0 lesser equals to t lesser equals to 2 and I want to calculate integral over c f dot d s. For that what is the first thing I need to do, I need to calculate what is F of c t? Well, the first thing I will calculate first is c prime t, which seems to be easier c prime t is 0, 1, e to the power t. You can see what I am doing, I am just differentiating each component with respect to t. Then I have to calculate what is F of c t?

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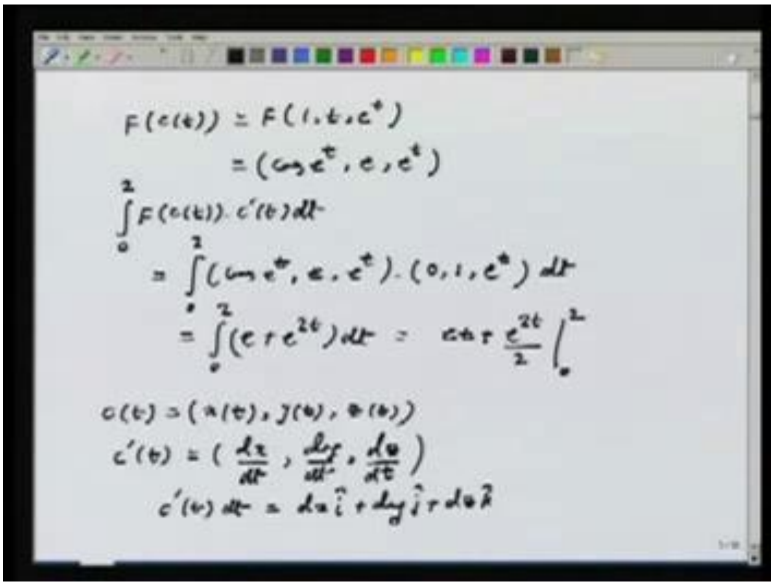

So, F of c t equals to F of 1, t, e to the power t, which then by definition of my vector field is cosine of e to the power t, then e, then e to the power t well that is F of c t. Now, I just look at this integral, integral 0 to 2 F of c t dot c prime t d t equals to integral 0 to 2 cosine of e to the power t, e, e to the power t dot. Now, c prime t, which I know is 0, 1, e to the power t then d t. I can calculate this vector product very easily, it is integral 0 to 2 e plus e to the power 2 t d t.

If I do this integration what I get is e t plus e to the power 2 t divided by 2. And then, I look at the limit 0 to 2 which you can very easily calculate. Just put the values of t, we get the line integral of the given vector field over the given curve. Now, in some of the calculus books you can see an alternative expression on the line integrals. That also I will explain now that given the curve c t, which is written as x t, y t, z t. I can write c prime t as d x d t, d y d t then d z d t. Now, this can be formally written as c prime t d t equals to d x.

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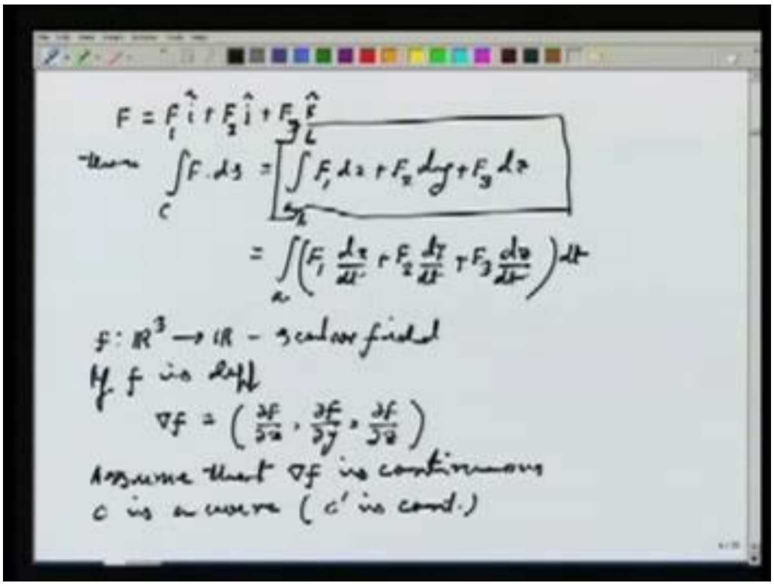

And if the vector field f is given as F 1 i I am suppressing the variables. Of course, that you can see, it should actually be F 1 of x y z i plus F 2 of x y z j and so on. Plus F 3 k then integral over c F dot d s that trans out to be integral a to b F 1 d x plus F 2 d y just taking the dot product plus F 3 d z. What it means is, it is integral a to b F 1 d x d t plus F 2 d y d t plus F 3 d z d t and then d t. The formal expression of this kind you can see in calculus book written of line integrals.

It actually means the previous definition, which can be alternatively expressed as the last equality, which we have written in terms of the components of the function capital F that is the vector field. Now, for the as far as the calculation of line integration are concerned. You can see that, it can be very difficult if the integral ultimately which you are getting as F of c t times c prime t if that is complicated. But, for certain functions the line integrals actually become much easier, now well go to those cases.

So, let us start with some simpler kind of line integrals. Suppose, I have a function f which is a function from R 3 to R, so this is a scalar field no longer a vector field, this is a scalar field, because the value of the function is a real number. Now, out of this scalar field, if f is differentiable I can manufacture a vector field out of this f the following way. That is the gradient of the function, which you are already accustomed with that is. So, the gradient of the function is del f del x, del f del y and then del f del z.

Now, assume that grad f is continuous. So, I am putting two conditions on f that first of all it is differentiable. So, that I can talk about the gradient, second the gradient is a continuous function. So, it is a continuous vector field now, and suppose c is a curve. That means, c prime is continuous I am always using this property on the curve. Then I can always think about integral over c grad f dot d s.

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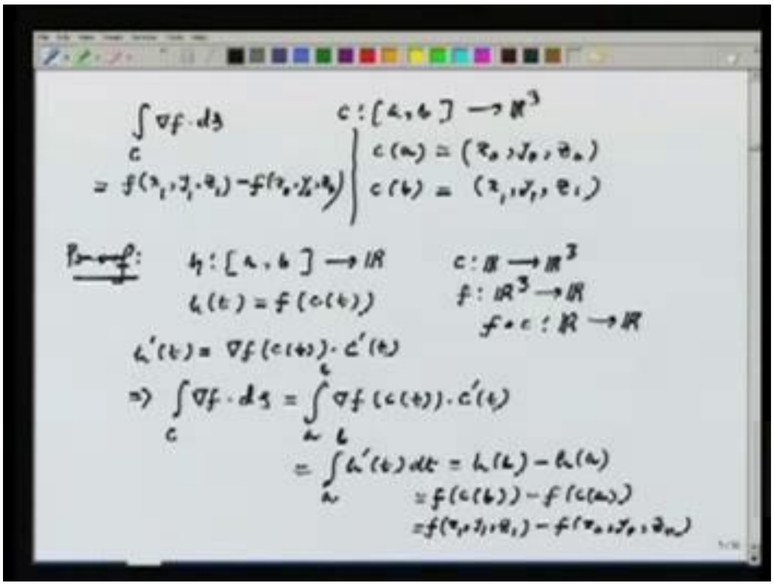

Let us say c is from a, b to R 3. So, what is this integral I want to say it become something simpler, if c of a equals to x 0, y 0, z 0 and c of b equals to x 1, y 1, z 1. Then it turns out that this is equal to f of x 1, y 1, z 1 minus f of x 0, y 0, z 0. So, this you can actually see is a very neat analog of the second fundamental theorem of calculus. That integral of f prime is f b minus f a, it is more less an analog of that. And that is not just an accident. This result can actually be proved using the second fundamental theorem of calculus.

So, let us see how to prove this what I do is, I define a function h from a, b to R. It is given by the formula h of t equals to f of c t. Notice, that c being a curve is a map from R to R 3 f being in a scalar field is a map from R 3 to R. So, as a result f compose c is a map from R to R, h is essentially f compose c. It is only the domain of c impose there that is why it is a map from a, b to r. Now, h turns out to be differentiable function because of the assumptions which we have. And now I want to calculate what is h prime at t?

Now, if I use that chain rule of several variable calculus, which you have learned already what I get is this is grad f at c t dot c prime t. This implies that integral over c grad f dot d s is nothing but, integral from a to b by definition grad f at c t dot c prime t equals to integral a to b h prime t d t. And now I can use fundamental theorem of calculus to get that this is h b minus h a. Apply this to the definition of h to get that this is f of c b minus f of c a, that is f of x 1, y 1, z 1 minus f of x 0, y 0, z 0. This is precisely what we wanted to prove.

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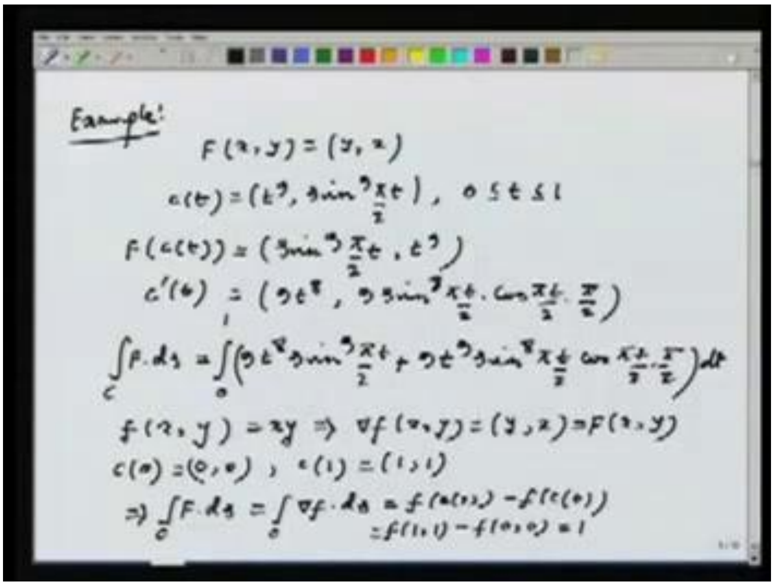

Now, before observing some more things about this result. Let us see how does it make our life easier. So, I will try to calculate another line integral, in this case to make things simple. But, I will do is I will concentrate in the two dimensional case. So, let us take this force field that F of x y equals to y x, so this is s vector field. And now I take a curve, so let us take this curve c t equals to t to the power 9 sin 9 pi t by 2, where 0 lesser equals to t lesser equals to 1.

Now, suppose I want to calculate this line integral. So, first I need to know what is F of c t I will calculate that, this turns out to be easy this is sin 9 pi by 2 t. Then t to the power 9, then I need to calculate what is c prime t that is 9 t to the power 8. Then 9 sin 8 pi t by 2 times cosine pi t by 2 times pi by 2. And then I need to look at the line integral that is f dot d s that is integral 0 to 1 F of c t times c prime t. If I calculate this what is get is 9 t to the power 8 sin 9 pi t by 2 plus 9 t to the power 9 sin 8 pi t by 2 cos pi t by 2 times pi by 2 this whole thing d t.

Which you can see is quite a complicated integral, it might be very difficult to solve this calculate this integral explicitly. So, instead of that what I will do is, what we have observed in the previous case we will try to do that. That is ill try to realize this capital F as gradient of some function and that is very easy. I will just look at this function f of x y equals to x y, this implies. Then that grad f at x y that is del f del x, which is y comma del f del y which is x that is capital F at x y.

And then, I notice what is c 0 from the definition of curve it is clear that c 0 is 0, 0. And what is c 1 that is again very clear, that this is 1 then sin pi by 2 to the power 9 that is again 1. So, by the previous formula it turns out that integral over c F dot d s is nothing but, integral over c grad f dot d s. That means, it is f of c 1 minus f of c 0, which then is f of 1, 1 minus f of 0, 0. But, I know my definition of f you just multiply the x coordinate and y coordinate. That means, f of 1, 1 is 1 and f of 0, 0 is 0, that means the line integral is actually equal to 1.

See, how complicated the previous integral was, but that does not really mean that the line integral is complicated to calculate. Once, we have used the fact that this given vector field capital F can be thought of as gradient of a scalar valued function. It becomes easy using the analog of the second fundamental theorem of calculus.

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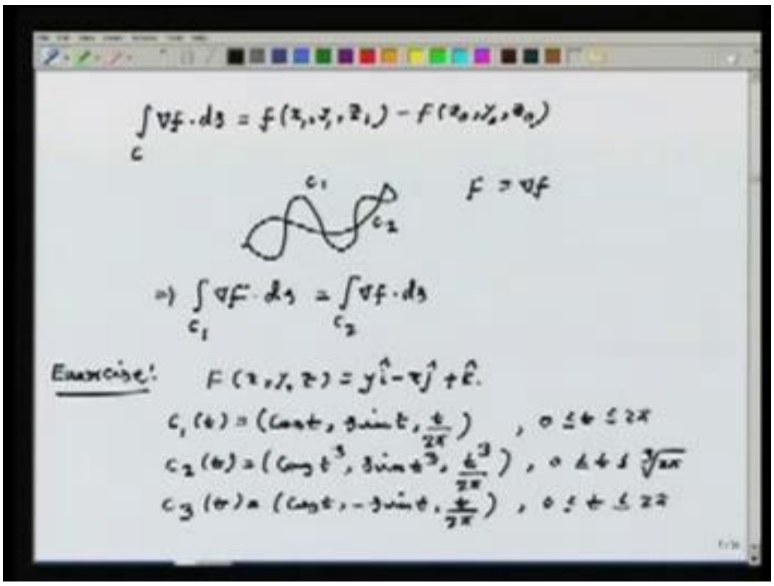

Now, let us go back to the observation. What I have proved is that integral over c grad f dot d s is f of x 1, y 1, z 1 minus f of x 0, y 0, z 0. It means that if I am integrating a gradient over a curve. Then the line integral does not really depend on the trajectory of the curve. It just depends on the initial point of the curve and the final point of the curve. Whatever way the curve joins those two points, the integrals are the same if the integrand is a gradient.

That is what follows from here, that if I have a point here and have a point here I can join it this way I can also join it this way you know. So, this is c 1, this is c 2 and my capital F is some grad f. Then from the previous formula it follows that integral over c 1 grad f dot d s equals integral over c 2 grad f dot d s, because the result depends on just the final point and the end point, final point and initial point.

But, in general that is not the case, if you have given a line integral of an arbitrary function, which is not really a gradient. Then the integral can depend on the trajectory of the curve that is it might matter how those two points are joined by a curve. So, I will give an exercise here, you can check it yourself consider the vector field F of x, y, z equals to y i minus x j plus k.

And let us take this curves, first one is c 1 t I define it as cosine t sin t, t by 2 pi. Then I define c 2 t as cosine t cube sin t cube, then t cube by 2 pi here 0 lesser equals to t lesser equals to 2 pi. Here 0 lesser equals to t lesser equals to cube root of 2 pi. And then, I define c 3 another curve that is cosine t minus sin t, t by 2 pi where 0 lesser equals to t lesser equals to 2 pi.

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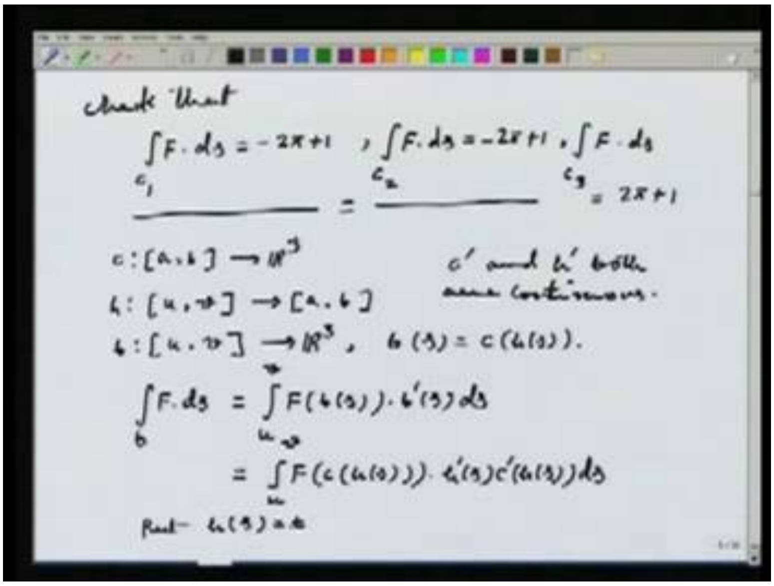

Now, just check the following that integral over c 1 F dot d s that turns out to be equals to minus 2 pi plus 1 integral over c 2 F dot d s. That is minus 2 pi plus 1 and integral over c 3 F dot d s that turns out to be 2 pi plus 1. Just calculate these integrals, it is essentially easy well observe this notice, that the first integral and the second integral they have different values. But, the final point is 2 pi that end the initial point is 0, they are joined by two different curves actually. And then it turns out that the integrals are different.

That means, in general it can happen that the line integral of a function over two different curves. But, joining the same points can be different. But, here something has also happened that integral of the function over c 1. And integral of the function over c 2 has become equal, well this is not just a happy accident this happens. Because of the following fact, let me explain to it. So, suppose I have a curve c which is from a, b to R 3 and suppose I have a map h, which is from u, v to a, b.

Assume that h and c has the property, that c prime and h prime both are continuous. Now, I can form a new curve call it b. So, b is a curve which is defined on u, v goes to R 3 it is defined as b of s equals to c of h s. Then b defines a new curve I want to know what is integral over b F dot d s, F is a given vector field I want to integrate f over b. Well, usually this new curve b which I have defined it is actually not new it is the same curve. Because, the image is the same as the image of c, but the space of parameters now are coming from u, v. So, this is usually called a parameterization of the curve.

Now, I want to calculate what is the integral over b F dot d s, so what I do is I just write down the definition. So, this is integral from u to v F of b s dot b prime s d s. Now, I write it down it is integral u to v F of c of h s times I have to calculate what is b the prime, that is h prime s times c prime at h s by chain rule times d s. Now, I do a change of variable to put h s equals to t. And then let us see where do the integral change, well if h s equals to t. Then the limits of integration will certainly change from h u to h v.

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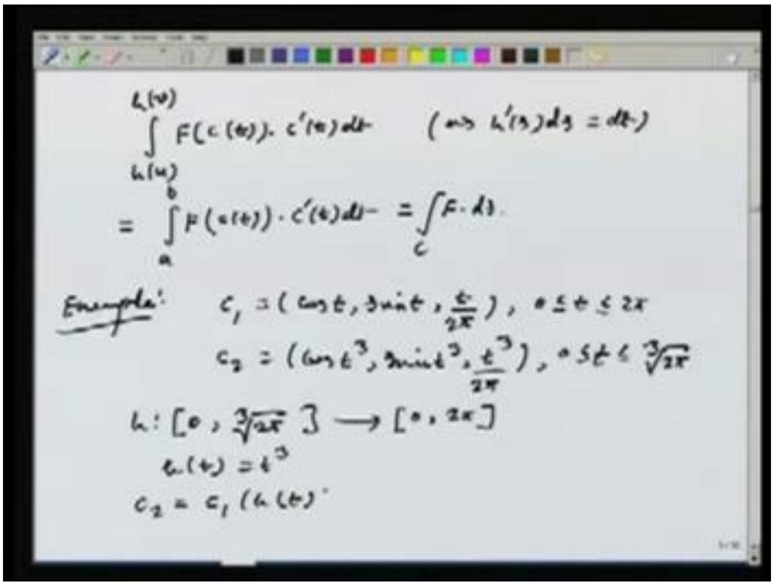

And then F of c of t, because earlier I had F of c of h s times. Now, comes look at the integral again ((Refer Time: 32:48)) it comes to h prime s then c prime h s, but h s equals to t I am using the change of variable. It just changes to c prime t d t as h prime s d s is equals to d t, but now observe that h u is equals to a and h v equals to b. So, what I get is F of c t dot c prime t d t which is nothing but, the line integral of f over the curve c. Now, if you look at the previous example what was my c 1, c 1 was cos t sin t, t by 2 pi where 0 lesser equals to t lesser equals to 2 pi.

And c 2 t was cos t cube sin t cube, then t cube by 2 pi where 0 lesser equals to t lesser equals to cube root of 2 pi. So, in this case what actually has happened is I have defined a map h from 0 and cube root of 2 pi to 0 to 2 pi given by h t equals to t cube. And then, my new curve, if I look at the relation c 2 is actually c 1 of h t.

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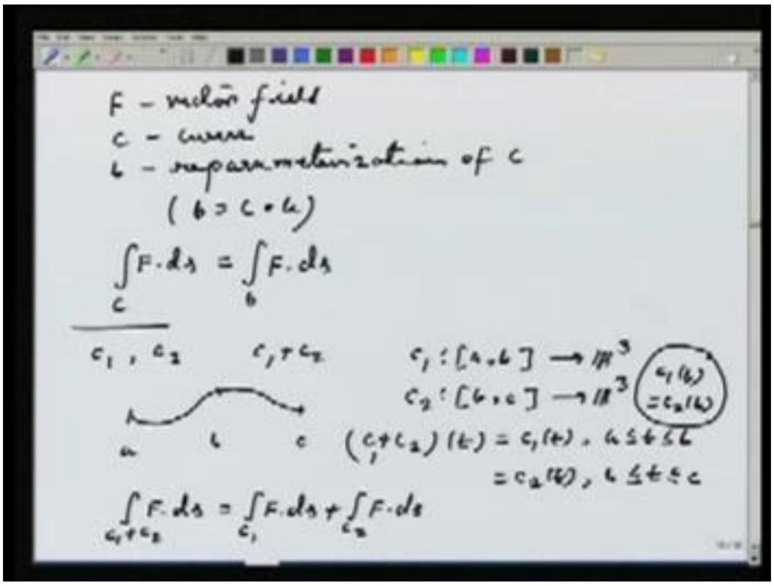

So, I will repeat again what we have observed here suppose, f is a vector field given, c is a given curve. B is another curve which is a reparametrization of c I have just defined, what does it mean to say that b is a parametrization of c. That means, the parameters spaces are given by some differentiable function. They are connected by some differentiable function h that is b equals to c compose h. Then integral over c F dot d s that is integral over b f dot d s, that is the line integral does not change even if you change the parameterization of the curve.

Now, we will go to a different meaning of line integrals, but before that there are some elementary facts of line integrals, which can be easily proved. One is suppose we are given two curves c 1 and c 2. Then you can talk about c 1 plus c 2, what does it mean to say c 1 plus c 2, so will just explain it by picture. Suppose this is c 1 it starts from a goes to b, then from b it goes by another curve let us say which is up to c.

So, the domain of c 1 is a, b and domain of c 2 is from b, c. Then from c 1 and c 2 I can talk about the curve c 1 plus c 2, which is defined as this is equals to c 1 t, if a lesser equals to t lesser equals to b and equals to c 2 t. If b is lesser equals to t lesser equals to little c only care has to be taken that c 1 b equal to c 2 b only then you can join the curve. In that case one can show that integral over c 1 plus c 2 F dot d s that is equals to integral over c 1 F dot d s plus integral over c 2 F dot d s. It follows just from the definition of the line integral.

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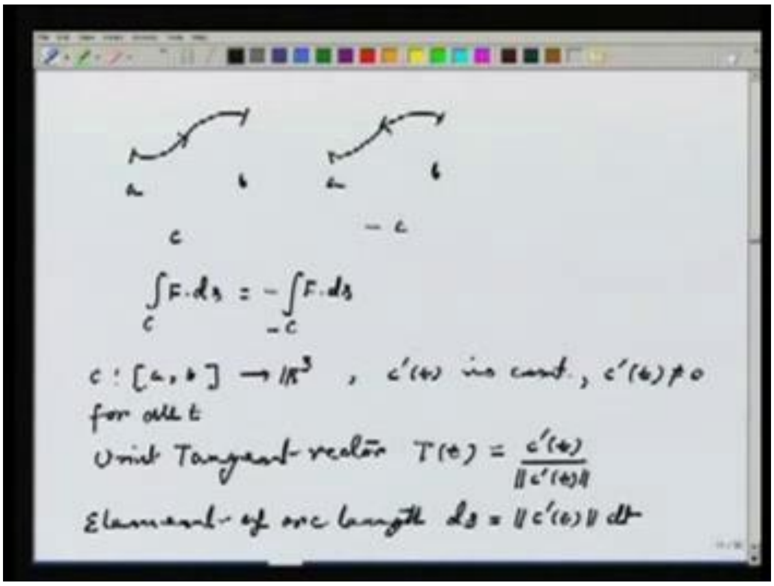

Similarly, one can take the negative of a curve also just by changing the direction. Suppose this is my curve c, this is the initial point, this is the final point. Then, I can look at the same curve, but now in a reverse direction it goes this way let us say I can come back this way, see if this curve is c this is called minus c. And then using the definition of line integrals again one can check that integral over c F dot d s that is minus integral of F dot d s, which is integrating the function over the curve minus c these two facts I will be using later they can be proved just by the defining properties of line integrals, it is nothing serious.

Now, let us shift to another geometric way of understanding the line integrals, so what we observe first is the following thing. Let us say c from a, b to R 3 is the curve with the property of course, that c prime t is continuous, now I also assume that c prime t is non 0 for all t. So, it is an additional property I am throwing in, in most of the cases you will have this property satisfied, that the derivative of c is always non 0, then I can form the unit tangent vector I will call it T t capital T of t, this we know is defined by c prime t divided by norm of c prime t, so that it has unit length. Now, the element of arc length d s that is I know is norm c prime t d t.

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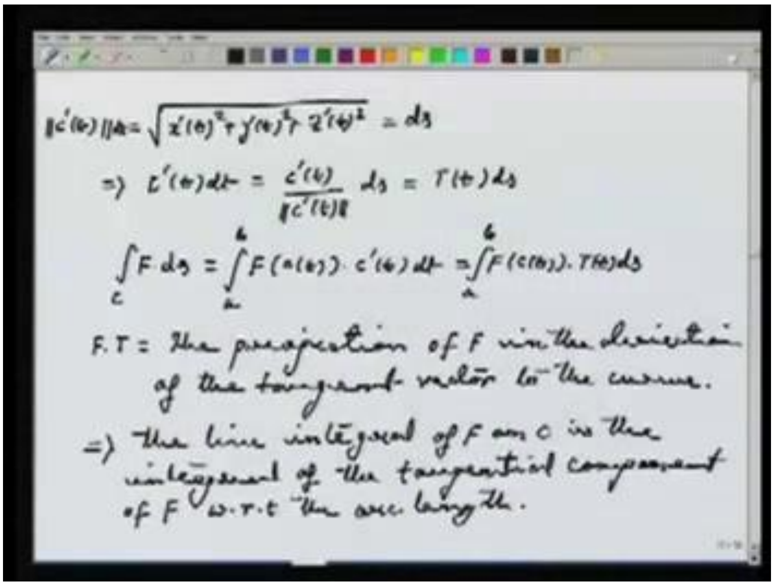

Because, I know that this is square root of x prime t square plus y prime t square plus z prime t square. This is the result which we have proved while talking about length of curves. We have done the thing only on R 2, but the analogous analysis can be done on R 3 and you can show that the length of the curve is actually given by norm c prime t d t. So, all I am saying is that this formula we have seen and actually proved in two dimension, I am saying stating the same formula in three dimension.

So, I will write again what we have proved that this is norm c prime t d t, which is the element of arc length d s. So, this implies then that c prime t d t, if I look at that, that is c prime t by norm c prime t d s correct, how did I get this because, d s is norm c prime t d t. So, just formally look at it this way that d t equals to d s by norm c prime t, so c prime t d t is c prime t by norm c prime t d s, this is just a formal way of saying this, but this then by my definition of the unit tangent vector is T t d s.

Now, this has some implication, the implication is integral over c F dot d s, now can be written as integral a to b F of c t dot c prime t d t, which is our usual definition is same as integral a to b I am not changing F of c t that stays dot, now c prime t d t I can write as T t d s. So, but what is F dot t what does this mean, it means actually the projection of the vector field F in the direction of the tangent vector to the curve, so this implies then. So, I will write it the meaning as the line integral now of F on c is the integral of the tangential component of F with respect to the arc length.

So, I will repeat again what I have shown is that integral of F over the curve c is integral a to b F of c t dot T t d s. Now, F dot t is the tangential component of the function in the direction of the tangent t on the curve c, so what I am doing is I am looking at the tangential component of the function this tangent is of course, drawn on the curve c that component is now getting integrated with the arc length of the curve, that is another meaning a geometrically related fact about the line. Because you can view the line integrals in this way also, it is interesting to observe that line integrals can actually be connected with the work done by a force field also that is what we are going to do now.

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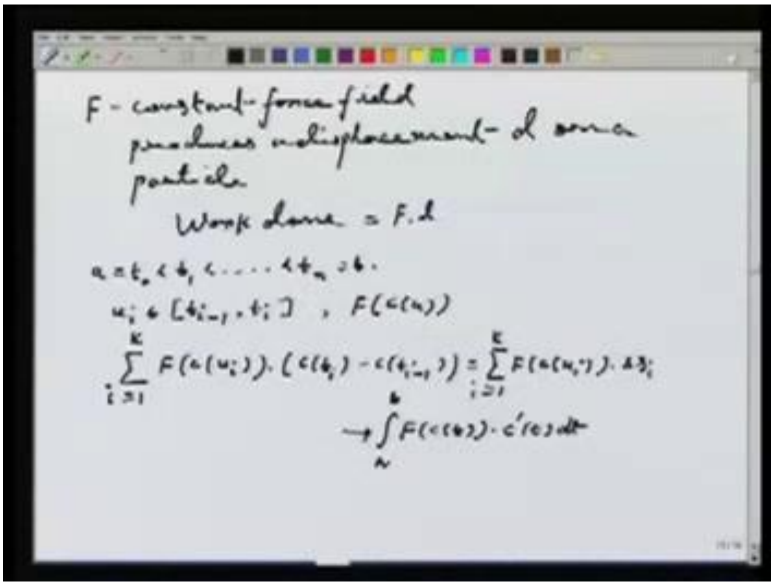

So, suppose F is a constant force field it acts on a particle and produces a displacement d on a particle, then we can actually calculate what is the work done, work done in this case is F dot d. Now, if f is not constant then you want to know how does one calculate the work done by this force field, this is where you are going to bring in the theory of line integrals. For that first I need an analog of the Riemann sums, which we have done in the one variable case, in this case the situation is as follows.

So, I take a partition of the parameter space a, b by t 0, t 1, to t n let us say, let us take some sample points u i in the integral t i minus 1 t i and I look at the sample values that is F of c u, where f is a given force field. And then I can look at this Riemann sum summation i from 1 to k F of c of u i times then the vector dot product c t i minus c t i minus 1, which is equals to summation i from 1 to k F c u i dot product delta s i. Now, it can be shown that if I take finer and finer partitions, that is the norm of this partition going to 0; that means, the maximum length of this partition, which is the norm of a partition if that goes to 0.

One can show that this delta s i is actually better and better approximated by c prime t d t and this sum actually converges to the line integral, that is a to b F of c t dot c prime t d t, it can be shown, but the proof of that is beyond the scope of this lecture. So, you are not going to prove it let us just assume this that this line integral can also be viewed as Riemann sums, the limits of the Riemann sums.

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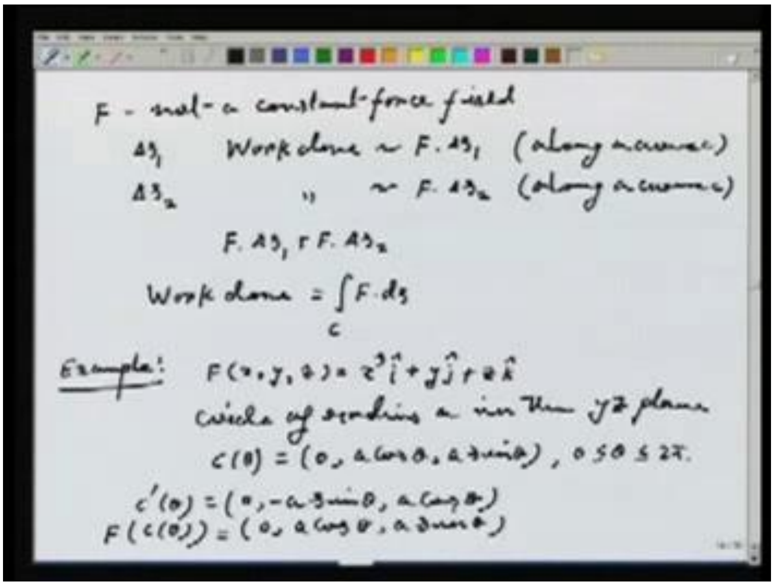

Now, using this let us go back to the question of work done again, suppose F is not a constant vector field in general, not a constant force field it acts on a particle and produces a displacement let us say delta s 1. Then, the work done is approximately equal to F dot delta s 1, if in the subsequent time it produces a displacement of delta s 2, then the work done is approximately equal to f dot delta s 2, so the total work done is F dot delta s 1 plus F dot delta s 2 and so on.

Now, this subsequent times which we are using is actually a partition of the parameter space and then using the Riemann sum one can actually prove that the total work done by a force field F, here we are displacing along a curve c let us say along a curve c, otherwise delta s 1 would not make sense along a given curve c. In that case the total work done using the theorem of Riemann sum turns out to be integral over c F dot d s.

Now, I have not given the proof of all this, but let us illustrate it by an example, so first I will give you a force field then a curve. Well the force field which I give is F of x, y, z equals to x cube i plus y j plus z k and the curve which I give is the circle of radius a in the y z plane. So, then I can write down the equation of the curve also, so this is c t since it is in the y z plane, it is first component is always 0, then a cosine theta, a sin theta 0 lesser equals to theta lesser equals to 2 pi.

So, the question I am asking is suppose I have a particle at the point 0 let us say and then I apply this force field, this particle moves around the whole circle comes back to 2 pi if that happens I will try to calculate what is the total work done. Well, what happens in this case well it is c theta first I have to calculate c prime theta that is 0 minus a sin theta then a cosine theta and then I have to calculate what is F of c t. So, in this case it is F of c theta that is I; obviously, get 0 here, then a cosine theta then a sin theta.

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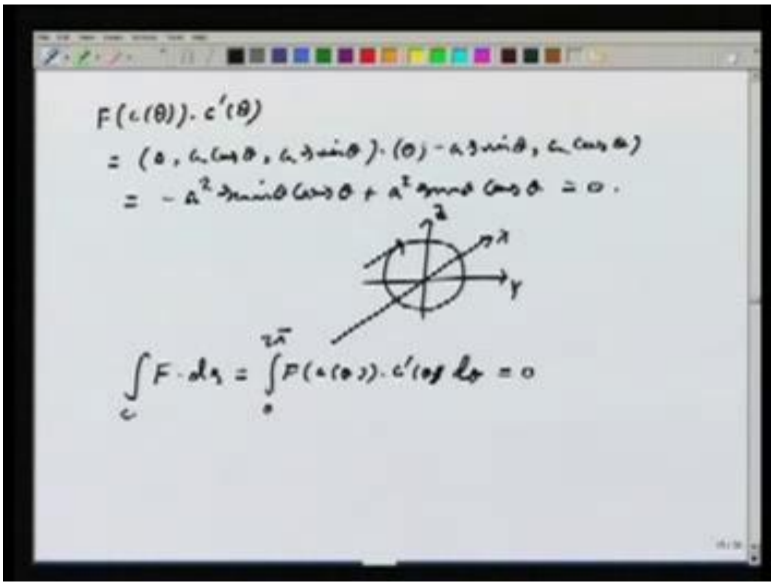

Then what is F of c theta dot c prime theta that is 0 a cosine theta, a sin theta dot 0 minus a sin theta a cosine theta. If I do the dot product what I get is minus a square sin theta cos theta plus a square sin theta cos theta, that is equals to 0, what does it mean, it means the force field is actually in the normal direction on the circle. So, if this is the circle in the y, z plane this is my x axis, then the force field is actually acting in the normal direction.

That means, it cannot move a particle along this curve; that means, the total work done by this force should be 0. That means, integral of F over c dot d s should be equals to 0, but this by my definition is 0 to 2 pi F of c theta dot c prime theta d theta, but I have already calculated that F of c theta dot c prime theta is actually equals to 0, so this is equals to 0, so it matches with the physical observation, so now, let us sum up what we have learnt about line integrals.

First we have defined line integration of a vector field over a curve c of course, under certain conditions on the vector field and the conditions on the curve, the condition on the vector field is that it should be continuous; the condition on the curve is that c should be a differentiable function.

Next, we have shown that there exist an analog of the second fundamental theorem of calculus, when the vector field in the question is gradient of a scalar valued function. In that case many line integrals can be easily computed and we have given one such example. Then we have noted some elementary properties about line integrals one of them being, that if I reparametrize the curve, then the line integral does not change.

Then, we had a geometric view point towards the line integral, where we have proved that if a vector field F is given, then its line integral over a curve c is nothing but, the integral of the tangential component of the function with respect to the arc length. And then comes the last topic, we have shown that line integral actually represents physically speaking, the work done by the force field F in displacing a particle over a given curve c, in the next lecture we will start with surfaces and integrals of function over given surfaces.