Mathematics - I
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Lecture - 22
Functions of Several Variables

Today, we are going to learn about Functions of Several Variable. So, before coming to the things we are going to learn about functions of several variables. Let me tell you why exactly we are interested in studying functions of several variables. Well, you must have noticed that even to draw curve given by a function, we need to know certain analytic facts about functions.

For example, whether that derivative of the function exists or not, to understand where the curve is increasing. Then the convexity, concavity, maxima, minima with all these information, sometime we can draw the curve of a function. So, this is where calculus helps us understanding curve of a function. Now, suppose instead of curves you are interested in surfaces. For example, I am interested in the sphere in three dimensions.

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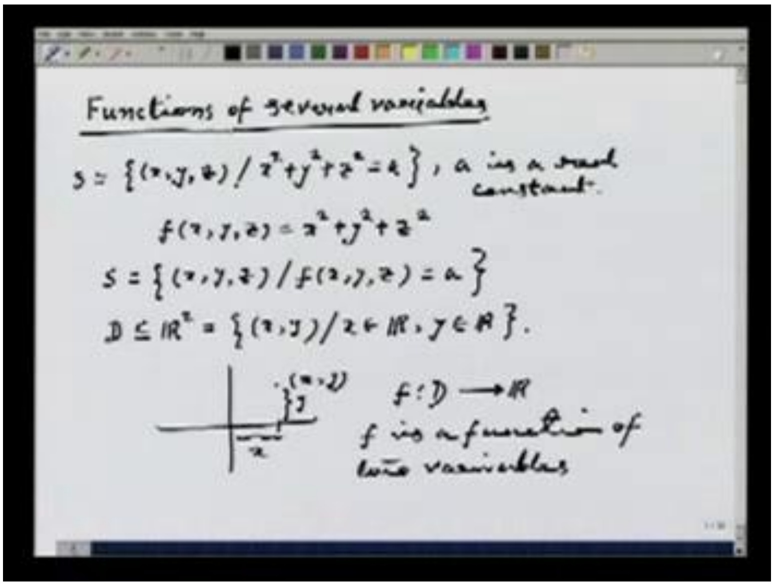

So, what is the equation it is collection of all x, y, z. Such that, x square plus y square plus z square equal to some constant, where a is a constant, is a real constant, that is it is a real number. Now, this is a sphere centered at origin with radius a, now this can actually be understood in terms of certain function. So, if I define f of x y z equal to x square plus y square plus z square. Then if I call this s, then s can also be written as all x, y, z such that, f of x, y, z is equal to a.

So, with this surface there is naturally associated a function, the surface in some sense comes out of the function. So, if you really want to understand this surface properly, the geometric properties of this surface. One way is to look at the function which produces it and try to understand the function. So, this is a big motivation for studying functions of several variable, so what exactly is a function of several variable. So, let me start with functions of two variables.

So, let us say D is a subset of R 2. What is R 2, R 2 is the set of points of the form x, y where x is a real number, y is a real number. Usually, we try to draw R 2 in this form, there are two axis. One is called the x axis, other is the y axis and if this point I want to say is x, y; that means, this distance is x and this distance is y. Now, if I have a function f from D to R, then f is a function of two variables namely x and y. I will tell you more clear if I look at examples.

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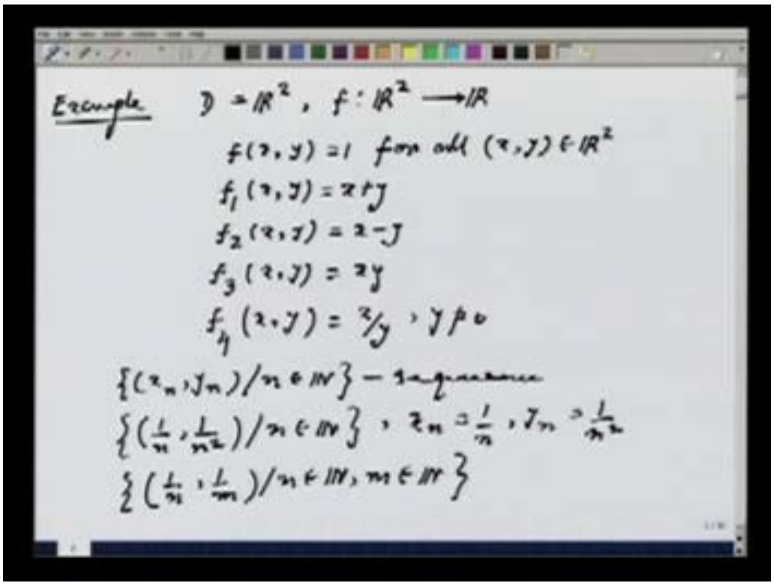

Let me take D to be equal to R 2. So, I am going to talk about a function f defined from R 2 to R. That means, I take an point in R 2 and f of that point should be real number. So, I can define f of x, y equal to 1 for all x, y in R 2, this is certainly a function of two variables a constant function. I can also define f 1 of x, y to be equal to what can you do with two real numbers, well you can add them. So, you define this f 1 of x, y is just x plus y.

Similarly, I can define f 2 of x, y equal to x minus y. I can also define f 3 of x, y equal to x times y, what I mean is the multiplication of two real numbers. I can define f 4 of x, y x by y, if I like with y not equal to 0 and so on. So, given x and y there must be some rule which tells me what real number I get out of x, y, that is a function of several variables. Now, what do you really want to study about function of several variables. So, we will try to have analogs of what we have done for one variable functions.

So, what did you do, we first looked at sequences. So, x n converges to x that was the very fundamental notion, using that we could talk about limit continuity and such things, and then finally differentiability maxima, minima, integrals. Now, I want to do all these things with functions of two variables. So, here the most fundamental thing to start with is, like real variable case, the convergence of sequences. So, what is a sequence here, well it is exactly analogous to one variable I look at x n, y n where n belongs to the set of natural numbers. This is the sequence, but it is a sequence of two variables.

So, to give an example I can define 1 by n, 1 by n square, this is a sequence of 2 variables anyway. Here x n equal to 1 by n, y n equal to 1 by n square. I could have as well defined it as 1 by n, then 1 by m where n and m both are natural numbers. You know. So, it is just a mix of two sequences of real variables, in the first coordinate you have one sequence. In the second coordinate I have another sequence, coupled together that gives me sequence of two variables. And now I want to talk about the convergence of the sequences.

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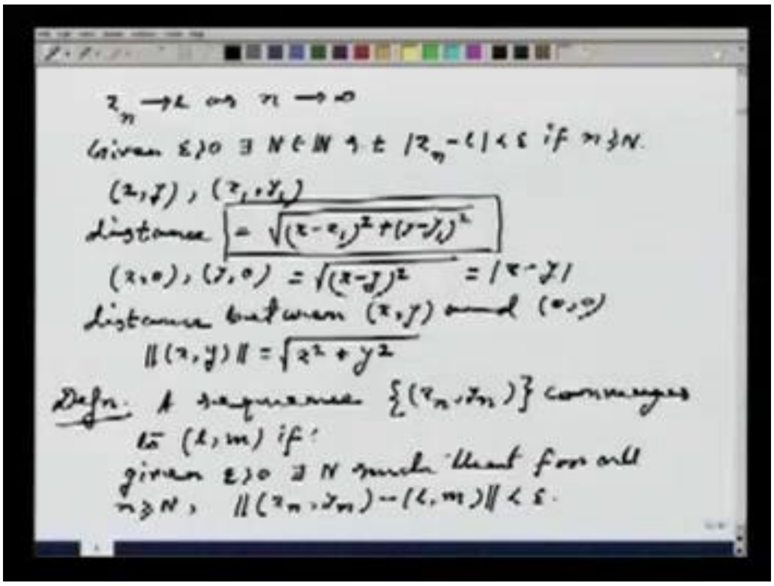

So, let us go back to the definition in the one variable case what was it. I have a sequence x n, what does it mean to say x n converges to L as n goes to infinity. It means, so given Epsilon bigger than 0, there exists a natural number n which belongs to the set of natural numbers. Such that, modulus of x n minus l strictly less than Epsilon if n is bigger than or equal to capital N.

So; that means, if you fix Epsilon bigger than 0. Then if you take larger and larger ends, then the distance between x n and l becomes less than Epsilon. Modulus of x n minus l just stands for the distance between x n and l. Here the key point is to generalize this notion of distance in the two variable case.

Now, in the two variable case if I take a point x, y and another point x 1, y 1, then there is a naturally defined distance between this two the Cartesian distance, which we always use in the coordinate geometry. So, what is the distance between this? So, distance between the above points is square root of x minus x 1 square plus y minus y 1 square. It is a generalization of the notion of modulus becomes very clear, if I look at the point x 0 and y 0.

Then, the distance according to the above formula is square root of x minus y square. But, since I am taking the positive square root, it is just modulus of x minus y. So, in that sense this distance called Euclidean distance between two points is the generalization of modulus. So, I am going to use this to define convergence of sequences. So, I am going to use some notion for this, that the distance between x, y and 0 0 is denoted by it is called norm. You know double modulus kind of this is by the above definition is square root of x square plus y square.

That means, in this definition you just take x 1 to be equal to 0 and y 1 to be equal to 0, then what you get is this. Now, the definition a sequence x n, y n, since I am in R 2 I have to talk about sequences where each element is a tuple. This converges to element l, m in R 2 if you will see it is exactly analogous to the one dimensional case. That given Epsilon bigger than 0, there exits capital N. Such that for all n bigger than or equals to capital N, norm of x n, y n minus l, m is less than Epsilon. The only difference in the one dimensional case here is, that the modulus has been replaced by the norm.

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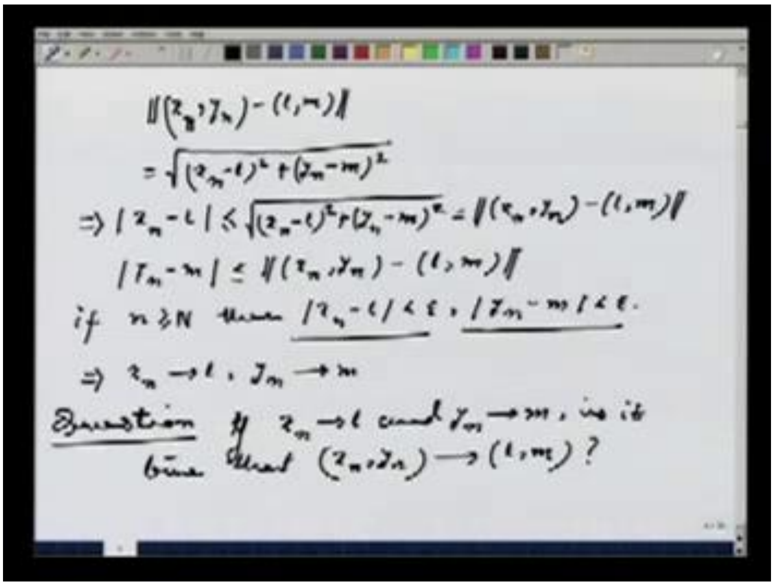

Now, let us look at this definition closely again. I write down the quantity norm of x n minus l, I just write down this quantity x n, y n minus l, n. I know what this means, it is square root of x n minus l whole square plus y n minus m whole square. Because of this I immediately get that modulus of x n minus l, which is less than or equal to square root of x n minus l square. Plus y n minus m square, which is equal to norm of x n, y n minus l, n.

And the similar thing happens with modulus of y n minus m also that this is also less or equal to norm of x n, y n minus l, m. Now, if little n is bigger than or equals to capital N, since norm of x n, y n minus l, n is less than Epsilon. It then follows that modulus of x n minus l is less than Epsilon and modulus of y n minus n less than Epsilon. So, what I got is that given any Epsilon bigger than 0, there exists a capital N, such that if n is bigger than or equals to capital N.

Then modulus of x n minus l is less than Epsilon. Modulus of y n minus l is also less than Epsilon, it means that x n converges to l and y n converges to m. So, x n, y n converges to l, n that means, if you concentrate on the first coordinate. That is if you get that x sequence x n, then x n converges to l. If you concentrate on the second coordinate, that is if you look at y n's then y n's converts to m. What about the converse, that suppose x n converges to l and y n converges to m. So, I will put it as question, if x n converges to l and y n converges to m, is it true. That x n, y n converges to l, m fortunately the answer is yes, and it can be proved as follows.

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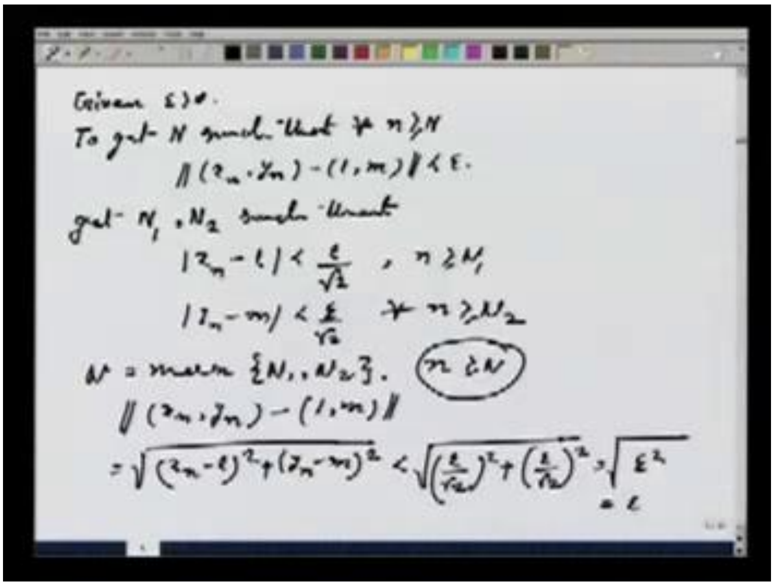

So, start with an Epsilon. What is the job, the job is to get capital N. Such that, for all n bigger than or equals to capital N, this should happen, this should be less than Epsilon. Now, I already know that x n converged to l and y n converges to m. So, what I do is get N 1 and N 2 such that, x n minus l strictly less than y less than Epsilon y 2 root 2 for all n bigger than or equals to n 1.

And y n minus m less than Epsilon by 2 root 2 for all n bigger than or equals to n 2. That mean, here instead of Epsilon I am using Epsilon by 2 root 2 in the definition of convergence x n converges to l. Now, I look at norm of x n, y n minus l, m I have the definition of this. So, it is square root of x n minus l whole square plus y n minus m whole square. If I assume my n is bigger than or equal to maximum of n 1, n 2.

Then, what I get is that this is less than square root of half Epsilon by 2 square plus half Epsilon by 2 square. What I get is square root of Epsilon by 2 whole square. So, that is Epsilon by 2 which is any way strictly less than Epsilon. This is then less than Epsilon by root 2 square plus Epsilon by root 2 square, which is square root of Epsilon square by 2 plus Epsilon square by 2. That is Epsilon square which is equal to Epsilon as Epsilon is positive.

That means, if I choose my n to be bigger than or equals to this capital N. Then norm of x n, y n minus l n is less than Epsilon, that precisely means that x n, y n converges to l, n. So, you understood something important here about convergence of sequences. What you got is I will put it as proposition.

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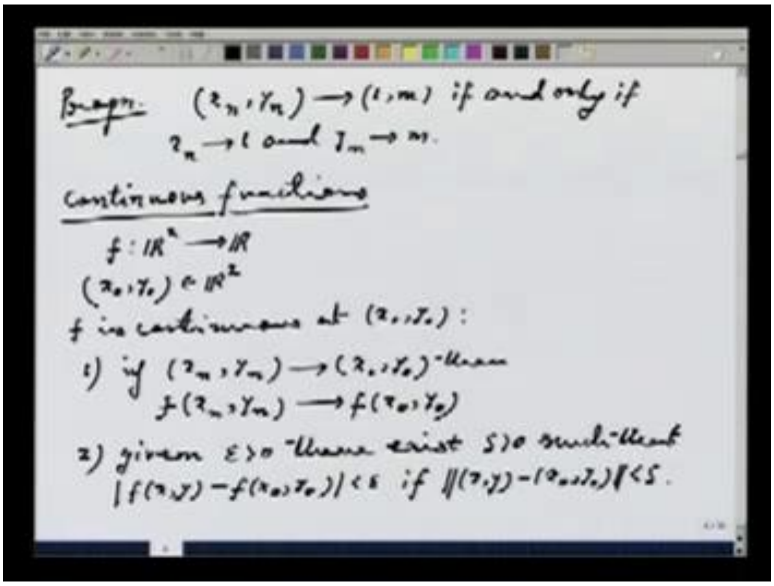

All it says is that is sequence x n, y n converges to an element in R 2 which I call it l, m. If and only if x n converges to l and y n converges to m. Now, because of this proposition it becomes real easy to give examples of sequences in R 2 which converges. All you have to do is we pick two convergence sequences x n and y n from real line and put them together as x n, y n.

And that is the only thing which you have to do to give examples of a sequence in R 2, which converges to some l, m. Now, given this I can go to definition of continuous functions. So, I start with continuous functions, here we will not prove everything, because things are very analogous to the one dimensional case. But, let me just go to the definitions and try to convince you that this is an analog of the one dimensional case.

So, let us say that f is a function. Let us start with from R 2 to R. Suppose, x naught to y naught is a point in R 2, what do you mean by f is continuous at x naught y naught. Well, analogous to the one dimensional case, I say one way of defining this is. It means if a sequence x n, y n converges to x naught y naught. Then the sequence of real numbers f of x n, y n converges to the real number f of x naught y naught.

This is exactly analogous to the definition of continuity on real line. That f is continuous at a point x naught, if for every sequence x n, x n converges to x naught implies f of x n converges to f of x naught. So, this is one way of defining. But, there is an alternative way of defining continuity using Epsilon and delta exactly as it is on real line. So, that is this that given any Epsilon bigger than 0, there exists delta bigger than 0. Such that, modulus of f x, y minus f of x naught y naught is less than Epsilon.

If now instead of modulus I have to use norm, because in the case of real line I used modulus of x minus x naught, because there the variables on which the function is defined a real numbers. The only reasonable notion of distance that is modulus, but here the variables on which the function is defined they are elements of R 2, so the only natural way to define the distance to use the Euclidean distance. So, here if the distance between x, y and x naught y naught is less than delta. So, these are two ways. Now, what I want to do is I will try to see some simple examples of continuous functions by these two criteria's.

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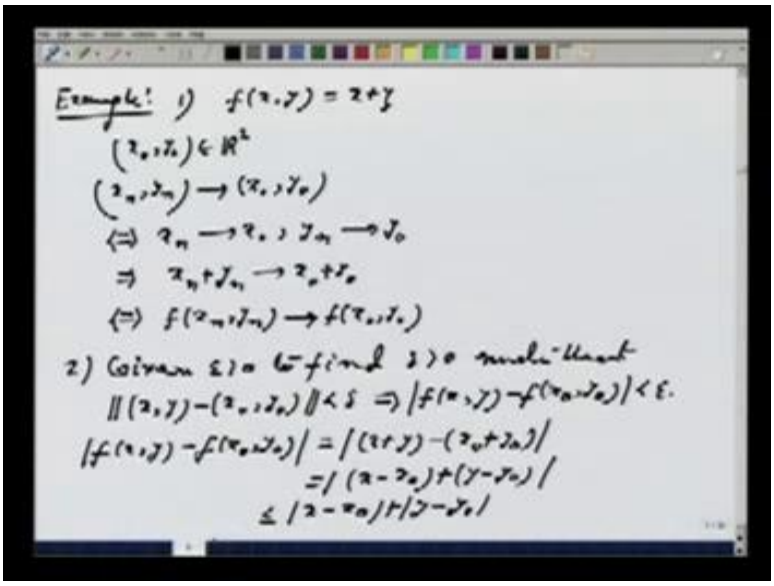

So, we start with my example, suppose I look at the function f of x, y equal to x plus y. I want to see whether it is continuous at any point x naught, y naught, choose a point arbitrarily. I want to check whether my function f is continuous at x naught, y naught by the first criteria of continuity that is through sequences. So, I choose a sequence x n, y n which converges to x naught, y naught. Then by my proposition I know that x n converges to x naught, y n converges to y naught.

Now, apply the result about sequences which we have proved long back. That if you have 2 sequences x n and y n such that x n converges to l, y n converges to m. Then x n plus y n converges to l plus m. Now, by that result this implies that x n plus y n converges to x naught plus y naught. But then, by the definition of the function it just means that f of x n, y n which is defined as x n plus y n it converges to f of x naught, y naught.

So, this is by one criteria. Now, the second one given Epsilon bigger than 0 to find delta bigger than 0. Such that, norm of x, y minus x naught, y naught less than delta should imply that modulus of f x, y minus f of x naught, y naught is less than Epsilon. So, given Epsilon I have to find a delta. So, I start experimenting with modulus of f x, y minus f of x naught, y naught. This simply turns out to be modulus of x plus y minus x naught plus y naught, which is modulus of x minus x naught plus y minus y naught, which is any way lesser equals to modulus of x minus x naught plus modulus of y minus y naught.

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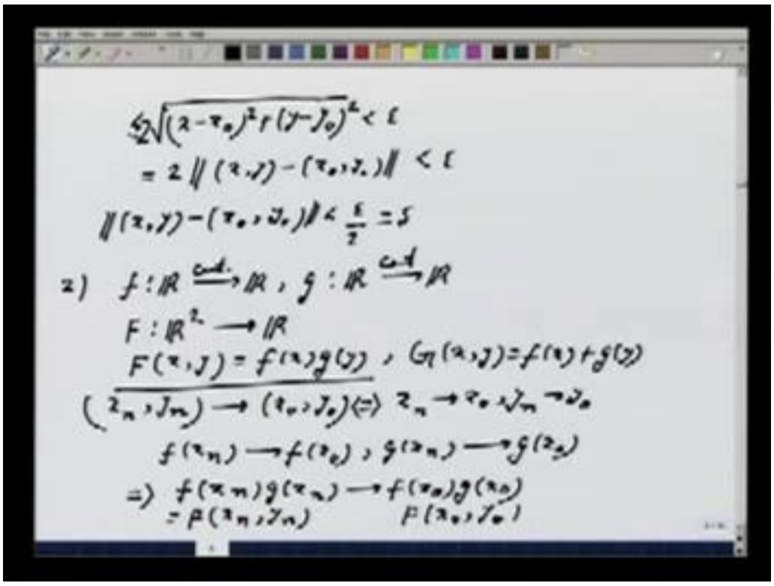

Now, I have observed then, that this is lesser equal to square root of x minus x naught square plus y minus y naught square into 2. Because, look at this quantity ((Refer Time: 27:15)) I have observed that this is lesser equal to square root of x minus x naught square plus y minus y naught square. Similar is the second quantity. So, both of them are actually less than square root of x minus x naught square plus y minus y naught square.

So, as a whole the whole things is less than this. So, I want to prove this to be less than Epsilon for which delta it should happen. Now, the choice is obvious because, this quantity is nothing but, 2 times norm of x, y minus x naught, y naught. I want this to be less than Epsilon. That means, norm of x, y minus x naught, y naught should be less than Epsilon by 2 choose this as delta. That gives me continuity of the function by the second criteria. But, in many cases you will see it turns out that the first definition works.

But, in many other cases it is the second one which works. That is why I have given both the definitions. Depending on the problem you use one of the criteria to check continuity of a function at a point. Now, using this you can actually create more complicated continuous functions. For example, let us say f from R to R is continuous and g from R to R is continuous. Now, I define a function capital F from R 2 to R by the following rule that F of x, y equals to f x into g, y. I can also define G of x, y equal to f x plus g, y I say both the function F and G are continuous function.

Now, I will try to prove that only capital F is continuous function. The proof of continuity of capital G I will leave as exercise. So, how do we prove that capital F is continuous function, I try to use the sequential criteria. So, take a sequence x n, y n converging to x naught, y naught. This, then implies that x n converges to x naught, y n converges to y naught. In fact, it is implies and implied by my proposition. Since, little f and little g are continuous functions I have that f x n converges to f x naught. G x n converges to g x naught.

But, then again by a result on sequences I get that f x n times g x n converges to f x naught times g x naught. But, look at the definition of the function, this implies this is just f of x n, y n and this is f of x naught, y naught. So, by the sequential criteria capital F is a continuous function. That exactly a similar kind of proof will tell you, that capital G is also a continuous function. So, this way you can create more complicated continuous functions of two variable using one dimensional continuous functions.

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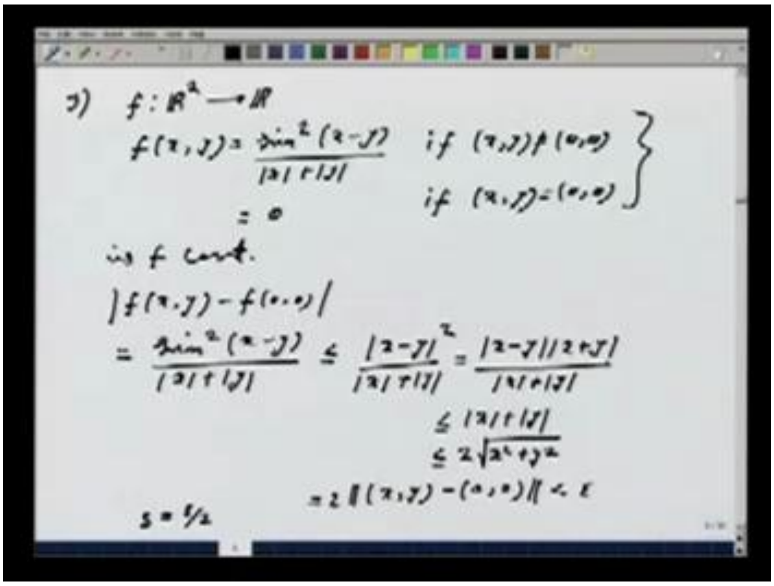

Let us see some more examples. Now, let us look at another example. I define f from R 2 to R given by f of x, y equal to sin square of x minus y divided by mod x plus mod y. If x y not equal to 0, 0, that is for the non-zero points of function is this. I define it to be 0, if x, y equal to 0, 0 question is, f continuous. What I will do is, I will just check continuity of function at 0. For other points the function is certainly continuous and that I leave as an exercise using the previous results.

Now, why is this function continuous at 0, 0. For that what I do is I try to estimate this f of x, y minus f of 0, 0. This by the definitions of the functions is sin square x minus y divided by mod x plus mod y. Because, x y is not equal to 0, 0 here. Now, using the fact that sin x is lesser equal to x I get that this is lesser equal to mod x minus y whole square divided by mod x plus mod y. That means, which is same as mod which is obviously, lesser equals to mod x plus mod y, which then is lesser equal to which is same as.

So, if you start with an Epsilon bigger than 0 you want to make this less than Epsilon for some choice of delta. Then the obvious choice of delta is delta equal to Epsilon by 2. So, given any Epsilon bigger than 0, I can find a delta. So, that mod of f x minus f 0, 0 is less than Epsilon. If norm of x y minus 0, 0 is less than delta. So, that shows f is continuous at 0, 0 at our points which are not the point 0, 0 you can apply our example 2 ((Refer Time: 34:40)) to show that the function is continuous, because you can write then in terms of one dimensional continuous functions.

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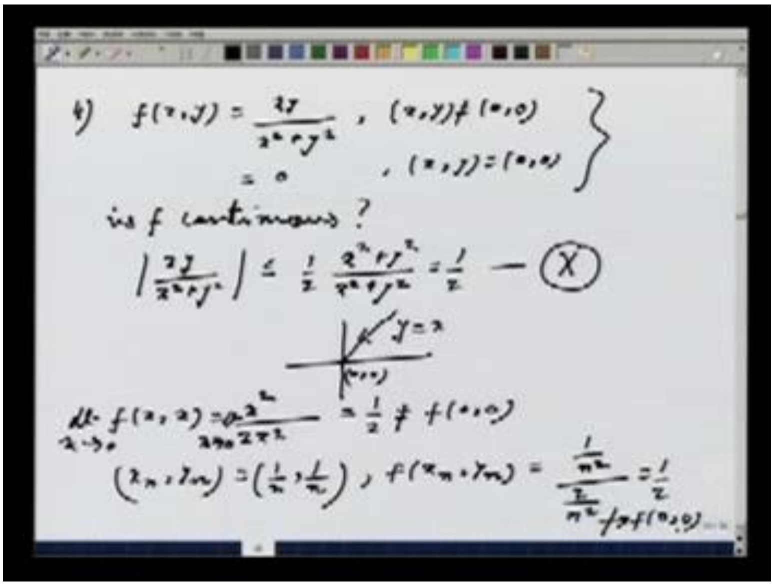

Let us go to another example, f of x y equals to x y by x square plus y square. If x, y is not the point 0, 0 at 0, 0 I define it to be 0 is f continuous. Now, unless you have any guess about continuity of a function, it becomes little difficult to check whether the function is continuous at a point. So, since I do not know I will try to estimate x y by x square plus y square minus 0. So, mod I want to make this less than Epsilon for some choice of delta.

But, this only thing what I can do is this x y lesser equal to less than half x square plus y square. With a and bigger than or equal to g m then it is x square plus y square. So, I just get that it is half I could not really make it small. So, that since there exists some problem here, perhaps this function is not continuous. So, what I do is since I have to converge to 0 to check continuity. The point 0, 0 is here, I can converge to 0, 0 through any line if I want. So, I choose to converge to 0, 0 to come this way through the line y equal to x.

So, if I put y equal to x in the function. So, f x, x that is x square by twice x square, which is equal to half. So, if I take limit as it is going to 0 I am again going to get half which is not equal to f of 0, 0. So that means, if I choose the sequence x n, y n equal to 1 by n, 1 by n. Then, what happens is f of x n, y n I can easily calculate it, it is 1 by n square divided by 2 by n square. That is half it does not converge to the value of the function f of 0, 0. So, f is not continuous at 0, but that is not end of the story.

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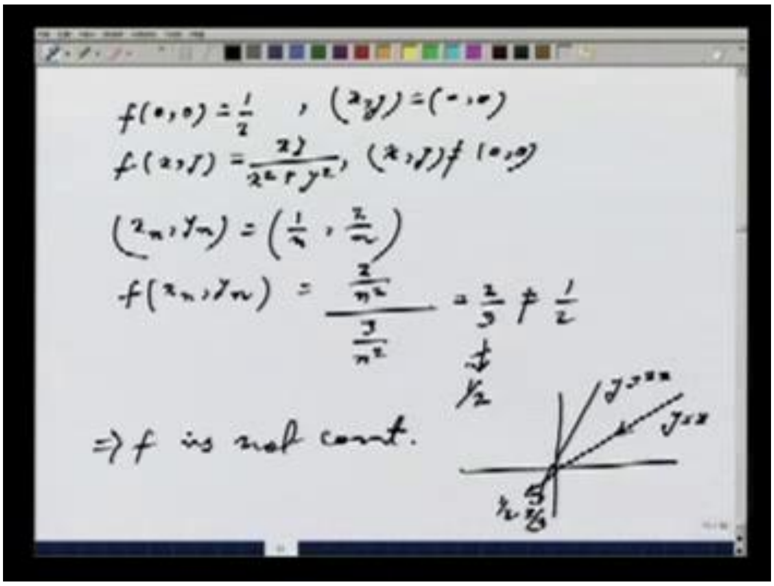

Suppose, I change my value of the function at 0, 0 to be f of 0, 0 equal to half and f of x, y remains same. This is x y by x square plus y square. This is if x, y naught equal to 0, 0 and this is x y equal to 0, 0. What happens then, well in this case I can try to approach 0, 0 through 2 by n. For example, I choose this sequence x n, y n equals to let us say 1 by n 2 by n. Then what is f of x n, y n that is 2 by n square divided by 3 by n square, which is 2 3'rd it is not equal to half again. And hence, this does not converge to half.

So, in this case whatever way you define f of 0, 0 you cannot make it continuous. This implies f is not continuous. The point is if f is continuous at 0, 0 through whatever path you go to 0, 0 f of the values should converge to f of 0, 0. Now, here what happens is I got two paths pictorially if I look at, this is y equal to x. And then, y equal to 2 x, if I go through this path, the function is going to half and if I go through this path this goes to half. But, this is going to 2 3'rd those values are not same hence, the function cannot be continuous.

So, we have seen some examples of continuous functions as well as some examples of discontinuous functions also. And now we know how to create more complicated continuous functions starting from functions of one variable. Now, with all these concept now we want to go towards differentiability of functions. Now, we come to differentiation of functions, so how to define derivative of the function. Again we start trying with analogues for the one dimensional case.

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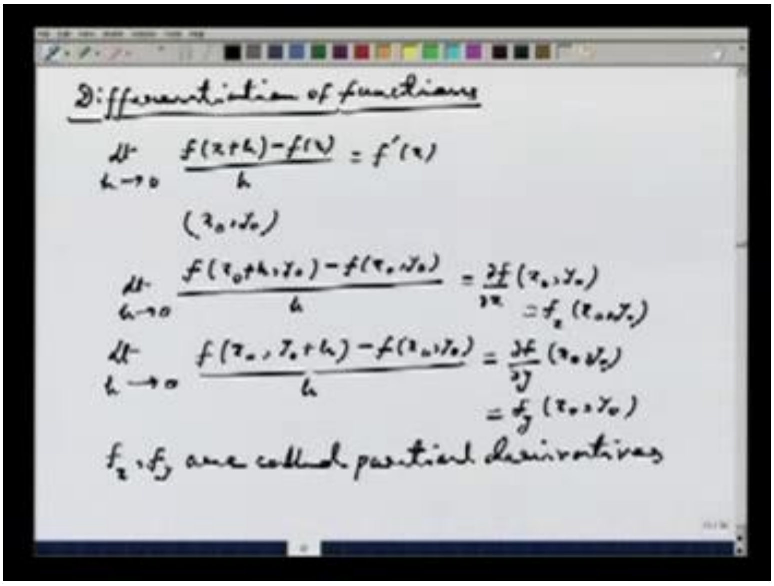

So, in the one dimensional case given a function f we usually look at this quotient limit h going to 0 f of x plus h minus f x by h. If this limit exists we call it f prime at x, the derivative of f at the point x. Now, here if I start looking at the analog of this, I take a point x naught, y naught. And then try to have an analog of the previous one. So, I can write this in this form that f of x naught plus h I do not do anything with y naught, I keep it fixed minus f of x naught, y naught I divide this by h I take limit as h going to 0.

If this limit exists, I will like to say that f is differentiable at x naught, y naught. It is not really doing justice to y naught that we can see immediately, but to still this limit might exist we call it del f del x at x naught, y naught. Similarly, I can define limit h going to 0 f of x naught. Now, I am going to play around with y naught. If this limit exist I call it del f del y x naught, y naught I will also denote this some time as f suffix x at x naught, y naught. And the next one I will denote by f suffix y at x naught, y naught, these are called partial derivatives. So, f x, f y are called partial derivatives.

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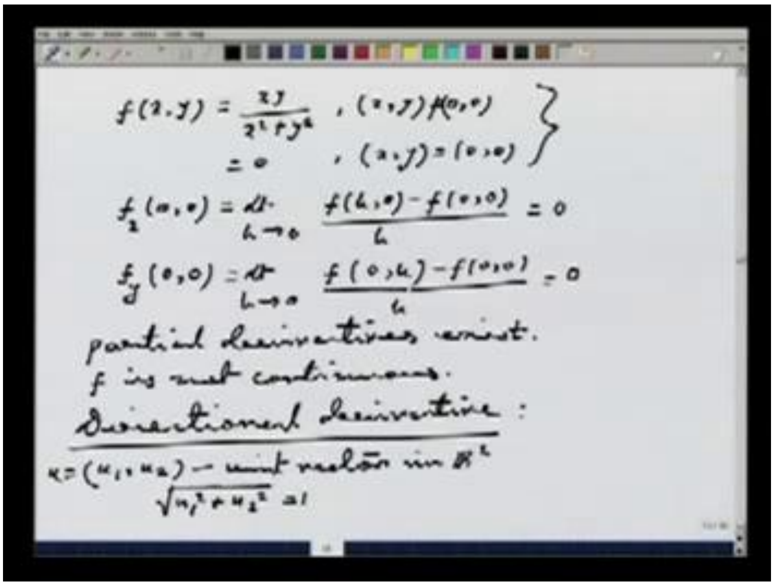

Now, to have an example of this, let me look at the function which I looked at before f x y is equal to x y by x square plus y square. When x y is not equal to 0 equal to 0 if x, y equal to 0, 0. And let me try to check what is f x at 0, 0 by the definition of partial derivative this is limit h going to 0 f of h 0 minus f of 0, 0 divided by h. Now, what is f of h 0 look at the definition of the function, if either x or y is 0 then f is 0. So, f of h 0 is 0, f of 0, 0 by definition is 0, so this is 0 the limit exists and the limit is 0.

Suppose, now I look at f y at 0, 0 I write this as limit h going to 0. Now, f of 0 h minus f of 0, 0 divided by h, this again if you look at the definition of the function. This also is 0, so the partial derivatives exist. So, can this be an alternative of derivative that is the question. The answer is no simply because, we have already seen that the function which we are looking at is not continuous. We expect that if a function is differentiable at a point. Then the function should be at least continuous at that point. But, here I see that the partial derivatives exist, but the function is not continuous.

So, this cannot be an alternative for derivative that is very clear. But, do you see later it is not entirely an useless exercise to look at partial derivatives. It helps us to reach up to the derivative. Now, there are different ways of generalizing I can look at something called directional derivative. So, what is the directional derivative. Here, the derivatives are actually looking at the difference quotients, where in the numerator you given increment to the function. The directional derivative means, in which direction you give the increment. So, we start with a unit vector let us say u 1, u 2 this is a unit vector in R 2. That is the norm of this is 1 square root of u 1 square plus u 2 square is equal to 1 and I call this u.

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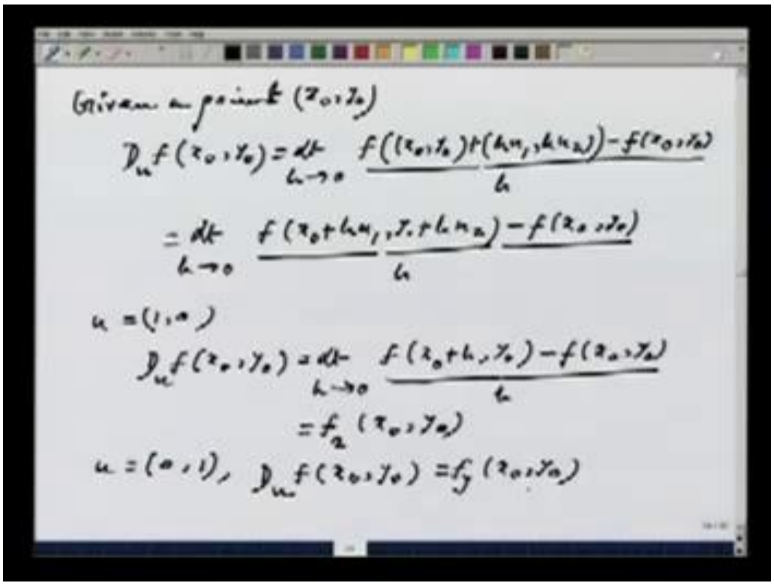

Then I define given a point x naught, y naught I define D u f. This stands for the partial the directional derivative in the direction of u at x naught, y naught to be limit h going to 0 f of x naught, y naught. Plus h u 1, h u 2 minus f of x naught, y naught divided by h using the rule of addition in R 2. I write it as limit h going to 0 f of x naught plus h u 1 y naught plus h u 2 minus f of x naught, y naught divided by h. Now, if this limit exists then I call that the directional derivative of f in the direction u exists.

Now, if you choose u to be equal to 1, 0. Then what is D u f at x naught, y naught by the definition it is limit h going to 0. Now, f of x naught plus h I get because, u 1 is 1, but u 2 is 0. So, I just get y naught here minus f of x naught, y naught divided by h and this is a known quantity, we have seen it this is nothing but f x at x naught, y naught. So, the partial derivative del f del x is nothing but the directional derivative in the direction of the vector 1, 0. Similarly, you can check that if u it is equal to 0, 1. Then D u f at x naught y naught is nothing but f y at x naught, y naught.

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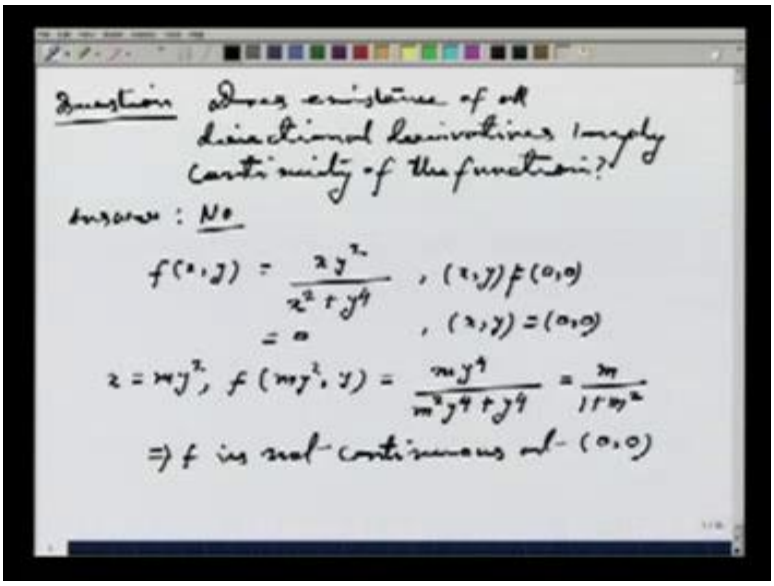

Now, the question, we ask is does existence of all directional derivatives imply continuity of the function. So, this is the question does existence of all directional derivatives imply continuity of the function. Well the answer to this question is no, that can be shown by the following example. Let us look at this function f of x y equals to x y square divided by x square plus y to the power 4, if x, y is not the origin it is 0, if x, y is equal to 0, 0. Now, I am going to check that f is not continuous at the origin.

So, what I do is I approach 0, 0 through the path x equal to m y square. If I do that then f of m y square y that turns out to be m y square times y square. That is m y to the power 4 divided by x square. Now, means m square y to the power 4 plus y to the power 4 which is m by 1 plus m square. So, if I take limit y going to 0 it is any way m by 1 plus y square, it does not depend on y and the limit depends on m. So, if I go through different paths I get different limit, so this implies f is not continuous at 0, 0.

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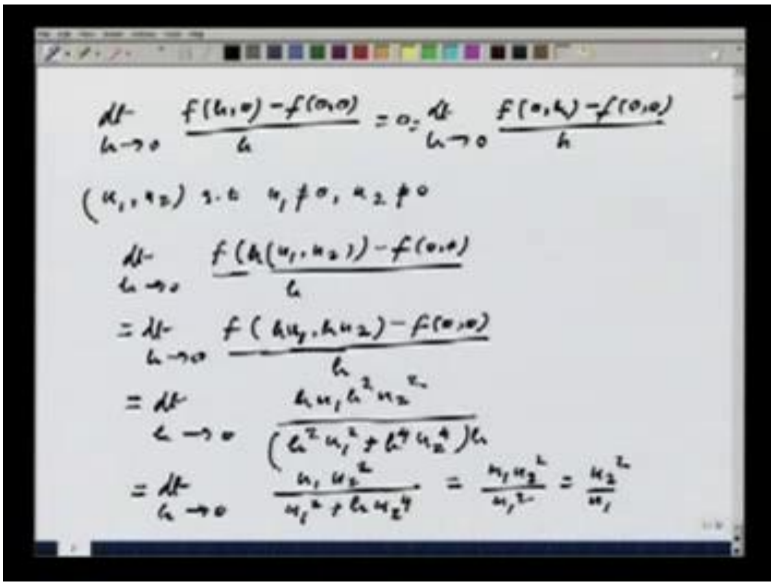

But then what about the directional derivatives first note that the usual partial derivatives exist. Because, limit h going to 0 f of h 0 minus f of 0, 0 divided by h turns out to be equal to 0. Because, f of h 0 is anyway 0 follows from the definition of the function, so, is f of 0, 0, so this quotient is always 0, hence the limit is also 0. Similarly, limit h going to 0 f of 0 h minus f of 0, 0 divided by h that is also 0, because f of 0 h is 0 and so is f of 0.

Now, the question is about other directional derivatives. So, let me choose a vector u 1, u 2. Such that, u 1 not equal to 0 and u 2 not equal to 0 because, if one of them is 0 that case I have already taken care of in the usual partial derivatives. Now, we will look at the limit, limit h going to 0 f of t u 1, u 2 minus f of 0, 0 divided by h. This is same as limit h going to 0 f of t u 1, t u 2 minus f of 0, 0 which I know is 0 this is instead of t it should be h, instead of t here I have h, because the increment is given in terms of h divided by h.

Now, I just write down the definition of the function. This is limit h going to 0, now x y square. That means, h u 1 into h square u 2 square divided by x square. That means, h square u 1 square plus y to the power 4, that means h to the power 4 u 2 to the power 4, but then the reason 1 by h. So, that it is here, if I calculate what I get is limit h going to 0 u 1, u 2 square. Because, in the numerator I have an h cube which cancels with the denominator I get u 1 square plus h u 2 to the power 4. That limit certainly is u 1, u 2 square divided by u 1 square that is u 2 square by u 1.

So, the limit exist and it certainly makes sense as I have chosen my u 1, u 2 both to be not equal to 0. So, you see that directional derivatives exist in the direction of any vector. But at the same time we have proved that the function is not continuous, so existence of all directional derivatives can never be a replacement of existence of derivatives. So, in the next lecture, we are going to see how to define a derivative function of two variables.