Mathematics-I
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Lecture - 23
Differentiation

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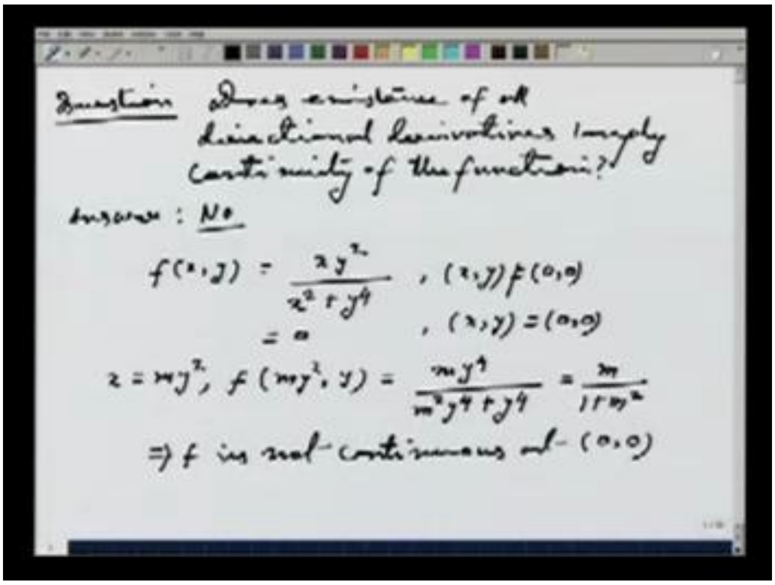

The question, we ask is does existence of all directional derivatives imply continuity of the function. So, this is the question, does existence of all directional derivatives imply continuity of the function. Well the answer to this question is no that can be shown by the following example. Let us look at this function f of x y equals to x y square divided by x square plus y to the power 4. If x y is not the origin it is 0 if x y is equal to 0.

Now, I am going to check that f is not continuous at the origin. So, what I do is, I approach 0, 0 through the path x equal to m y square. If I do that, then f of m y square y that turns out to be m y square times y square. That is m y to the power 4 divided by x square now means, m square y to the power 4 plus y to the power 4 which is m by 1 plus m square.

So, if I take limit y going to 0, it is any way m by 1 plus y square. It does not depend on y and the limit depends on m. So, if I go through different paths I get different limits. So, this implies f is not continuous at 0, 0. But then what about the directional derivatives, first naught that the usual partial derivatives exist.

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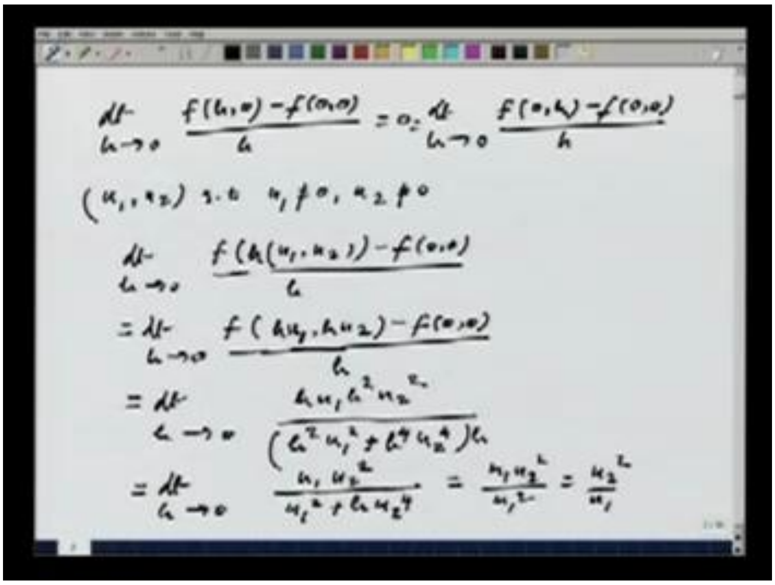

Because limit h going to 0 f of h 0 minus f of 0, 0 divided by h turns out to be equal to 0. Because, f of h 0 is anyway 0 follows from the definition of the function, so is f of 0, 0. So, this quotient is always 0, hence the limit is also 0. Similarly, limit h going to 0 f of 0 h minus f of 0, 0 divided by h that is also 0, because f of 0 h is 0 and so is f of 0.

Now, the question is about other directional derivatives. So, let me choose a vector u 1 u 2 such that, u 1 not equal to 0 and u 2 not equal to 0. Because, if one of them is 0 that case I have already taken care of in the usual partial derivatives. Now, we will look at the limit h going to 0 f of t u 1 u 2 minus f of 0, 0 divided by h. This is same as limit h going to 0, f of t u 1 t u 2 minus f of 0, 0 which I know is 0. This is instead of t it should be h, instead of t here I have h, because the increment is given in terms of h divided by h.

Now, I just write down the definition of the function this is limit h going to 0. Now, x y square, that means h u 1 into h square u 2 square divided by x square that means, h square u 1 square plus y to the power 4. That means, h to the power 4 u 2 to the power 4, but then the reason 1 by h. So, that sits here if I calculate, what I get is limit h going to 0, u 1 u 2 square, because in the numerator I have an h cube which cancels with the denominator.

I get u 1 square plus h u 2 to the power 4. That limit certainly is u 1 u 2 square divided by u 1 square, that is u 2 squares by u 1. So, the limit exist and it certainly makes sense as I have chosen my u 1 u 2 both to be not equal to 0. So, you see that directional derivatives exist in the direction of any vector. But, at the same time we have proved that the function is not continuous. So, existence of all directional derivatives can never be a replacement of existence of derivatives. So, in the next lecture we are going to see how to define a derivative function of two variables.

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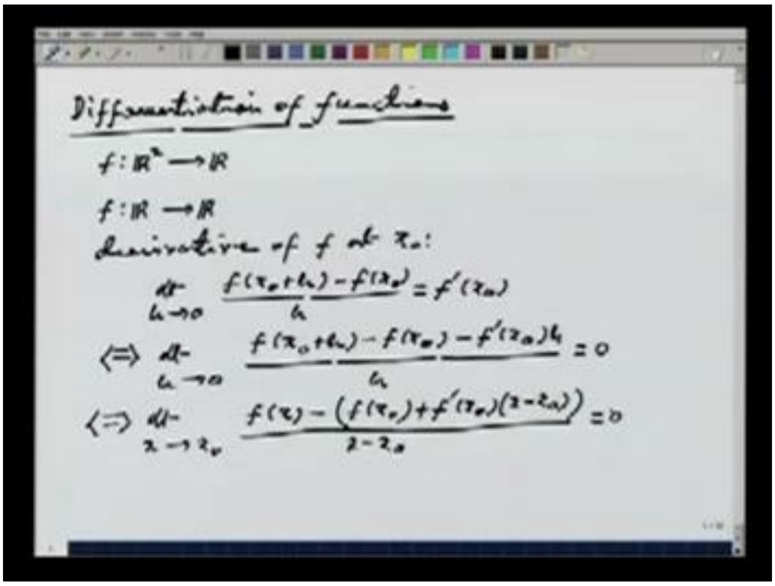

Now, we come to the question of differentiation of functions. So, for the sake of simplicity we will assume that the functions we are going to look at, they are function from R 2 to R. And the theory we are going to develop it essentially works, similarly if you go to R 3, R 4 or R n. The most fundamental case is to understand the case of R 2 first.

So, question again is how to define derivative of a function. So for that, let us again look back first at the definition of derivative of a function at a point x naught, if the function is from R to R. So, let us look at a function f from R to R. Then derivative of f at x naught, how do I define this? Well, it says that limit h going to 0 f of x naught plus h minus f x naught divided by h. If this limit exists, then it says a that the function is differentiable at x naught. And that we call f prime x naught. So, the derivative of a function at a point x naught is just a number.

So, an analogous way of expressing this is. That limit h going to h naught, h going to 0, f of x naught plus h minus f of x naught minus f prime at x naught times h divided by h equal to 0. So, we say that a one variable function f from R to R is differentiable at x naught. If there exist a number called f prime x naught, such that f of x naught plus h minus f x naught minus f prime x naught times h divided by h is equal to 0. An equivalent expression again is limit, x going to x naught, f x minus f x naught plus f prime x naught into x minus x naught. This whole thing divided by x minus x naught is equal to 0. This is another way of saying the fact that f is differentiable at x naught.

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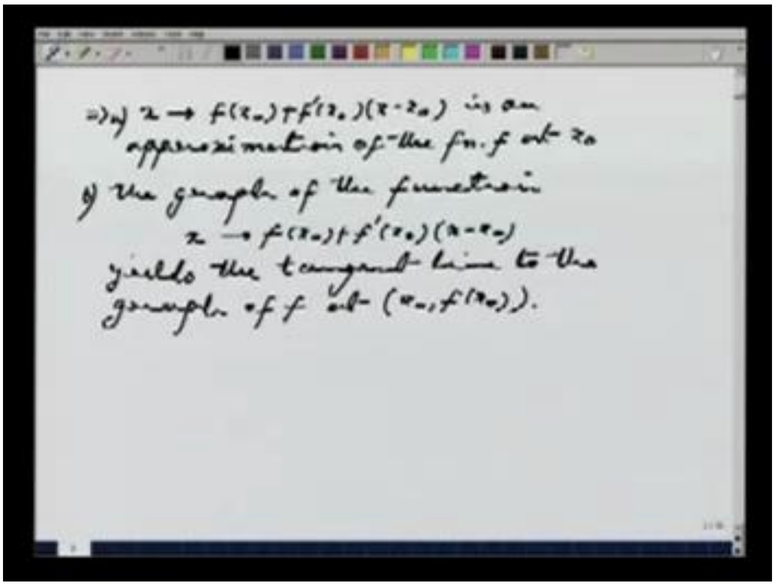

Now, we are going to generalize using this view point on R 2. So, what does this say, this implies that the function x going to f x naught plus f prime x naught into x minus x naught is an approximation of the function f at x naught, and also the graph, so this is 1. Second one is the graph of the function x going to f x naught plus f prime x naught into x minus x naught, yields the tangent line to the graph of f at x naught f x naught. So, here existence of derivatives gives you the best linear approximation of f at x naught. And that line which you get is essentially the tangent line. Now, this is the view point we are going to take, when we go to functions of several variables.

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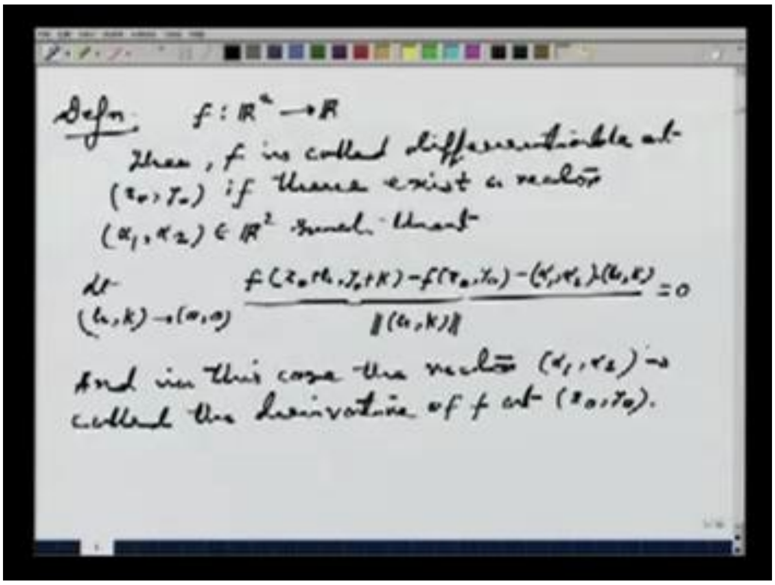

So, what is the definition now. So, we come to the definition of derivative, suppose f is a function from R 2 to R. Then f is called differentiable at a point x naught y naught. If there exist a vector alpha 1 alpha 2 in R 2, such that limit h k going to 0, 0 f of x naught plus h y naught plus k minus f of x naught y naught minus the dot product of the vectors alpha 1 alpha 2. By this dot I mean the dot product h k divided by the norm of h k, this is equal to 0.

And in this case, that is if the limit exists. The vector alpha 1 alpha 2 is called the derivative of f at the point x naught y naught. Now, the question is, there are so many questions to answer now. That given the function f how do I calculate alpha 1 alpha 2, that is the most fundamental question. Second one is this definition of derivatives in the sense, does this imply that the function f is continuous. So, we will go to answer those questions. First let me tackle the question. That if I already know that the function f is differentiable, how to calculate this vectors alpha 1 and alpha 2, which is going to be the derivative of f.

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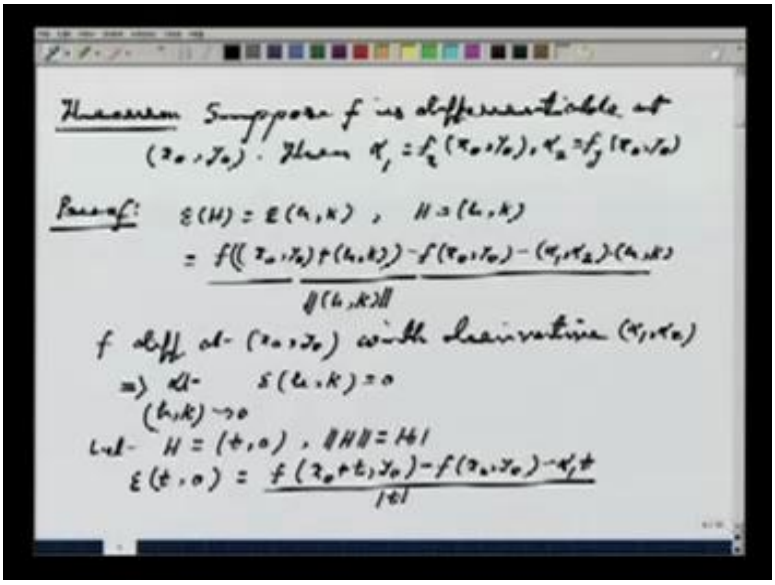

So, let us go to this theorem now. So, suppose f is differentiable at x naught y naught. Then assume the previous definition, we know there exist vectors alpha 1 and alpha 2. Such that, this limit is 0 I say, then alpha 1 is f y at x naught y naught and alpha 2 is f y at x naught y naught. That is if the function is differentiable. Then, the derivative is given by being a vector, now it should get triple of 2 numbers.

Then the first number is the partial derivative of f with respect to x at the point x naught y naught. And the partial derivative of f with respect to y at the point x naught y naught. So, let us come to the proof of this. Let us say I define epsilon of H which is epsilon h k. So, H always stands for the vector h k. Then what is this epsilon, this is f of x naught y naught plus h k minus f of x naught y naught minus alpha 1 alpha 2 dot h k divided by norm h k.

So, it is essentially the difference quotient which I am giving a name I call it epsilon H. Then, I know f differentiable at x naught y naught with derivative alpha 1 alpha 2 implies. That limit h k going to 0 epsilon h k equal to 0, this is the definition of derivative. Now, I am going to have a particular choice. So, let instead of h k I take my H to be equal to t 0. So, I am making k to be equal to 0 and h is equals to t. Then, what is epsilon t is 0. I just write down the quantity it turns out to be f of x naught plus t y naught minus f of x naught y naught minus. Now, alpha 1 alpha 2 dot t 0. But, since alpha 2 times 0 is 0, what I get is, alpha 1 t.

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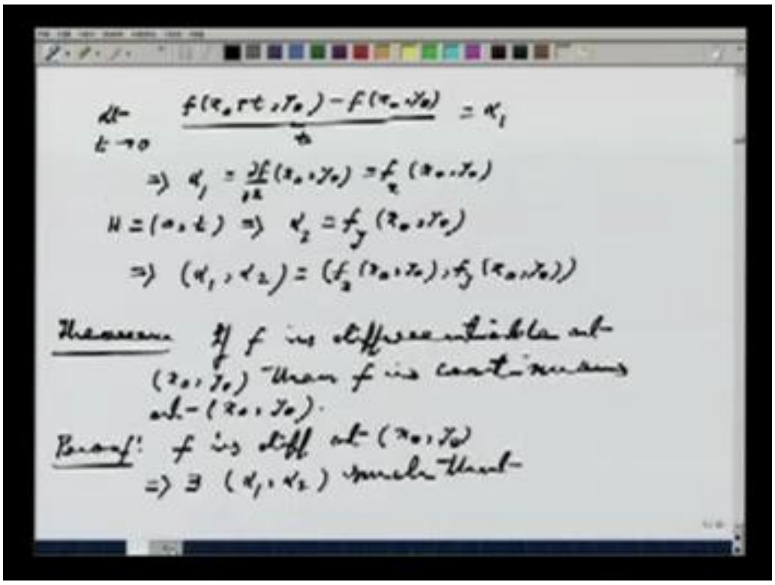

Now, I have to look at norm of t 0. Now, norm of h in this case is just mod t, so I get mod t here. This then implies that limit t going to 0 f of x naught plus t y naught minus f of x naught y naught divided by t equals to alpha 1. This precisely implies by my definition that alpha 1 is equal to del f del x at x naught y naught or f x at x naught y naught.

The exactly similar kind of argument by choosing H equal to 0 t, implies that alpha 2 is f y at x naught y naught. This implies finally, that the vector alpha 1 alpha 2 is actually given by f y at x naught y naught comma f y at x naught y naught. So, if I know that a function is differentiable. Then calculating its derivative is not very non trivial. All I have to do is, I have to calculate two partial derivatives of f with respect to x an y. And evaluate them at the point x naught y naught, that gives me a triple of numbers, that is the derivative of the function at x naught y naught.

But, some care which to be taken here, because if you follow the proof, then you must have noticed by now. That it is very important to use the fact that f is differentiable. If I assume that f is differentiable, then the derivative of the function is given by the partial derivatives. It is no way true, that if the partial derivatives just exist at a point, then the function is differentiable there. We will come to that just at this point, I will note that I assume that f is differentiable, then the derivative is given by the partial derivatives.

Now let us go to the another important question, that also I will put as a theorem. If f is differentiable at x naught y naught, then f is continuous at x naught y naught. Now, the proof of that is not very difficult, so we come to the proof. Now, f is differentiable at x 0 y 0 implies, there exist a vector alpha 1 and alpha 2.

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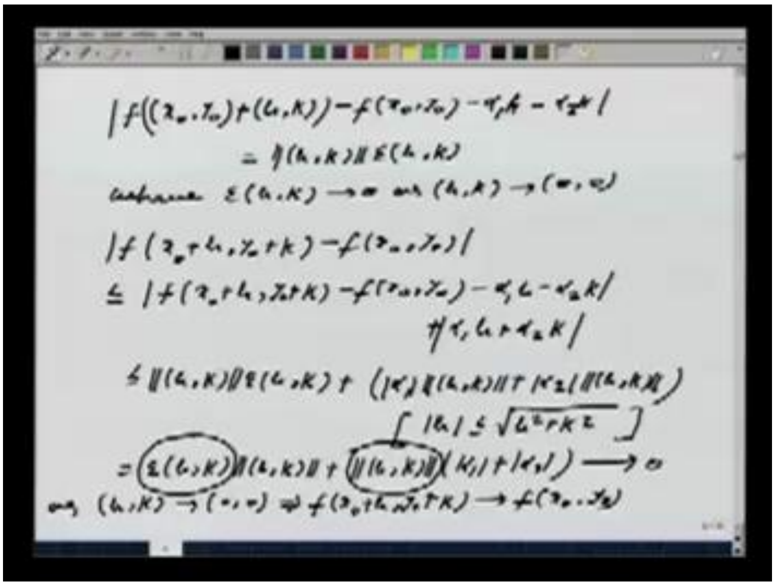

Such that, modulus of f of x naught y naught plus h k minus f of x naught y naught minus alpha 1 h 1 plus well alpha not h plus alpha 2 k. This modulus is equal to norm of h k times epsilon h k where epsilon h k goes to 0, as h k goes to 0, 0. This is just the definition derivative of a function. Now, I just look at f of x naught plus h y naught plus k minus f of x naught y naught.

I just write this now as I can say that this is less or equals to f of x naught plus h y naught plus k minus f of x naught y naught minus. Well, I will have minus here also minus alpha 1 minus alpha 2 k plus alpha 1 h plus alpha 2 k. Now, this then is equal to norm h k times epsilon h k. And now let me just write down this extra portion as I will say less or equal to alpha 1 norm h k plus alpha 2 norm h k.

And just using the fact that mod h is lesser equal to square root of h square plus y square. So, I finally get that, this is equal to epsilon h k times norm h k plus norm h k times a constant which is mod alpha 1 plus mod alpha 2. Now, as h k goes to 0, this implies then. That epsilon h k anyway goes to 0 norm h k goes to 0 the rest is a constant. So, this implies that f of x 0 plus h y 0 plus k goes to f of x naught y naught.

So, the argument is this right hand side goes to 0 as h k goes to 0, 0. Because, epsilon h k goes to 0 and norm h also goes to 0. That means the left hand side by Sandwich theorem also goes to 0. That means, f of x 0 plus h y 0 plus k goes to f of x 0 y. But that is precisely continuity of f at x naught y naught. So, this implies that the function f is continuous.

So, then these definition of derivatives looks like will be a truthful generalization of the one variable case of differentiability of a function, because it is giving me continuity. The geometric interpretation of it we will come to it later. We will now try to enquire about the connection of differentiability with partial derivative.

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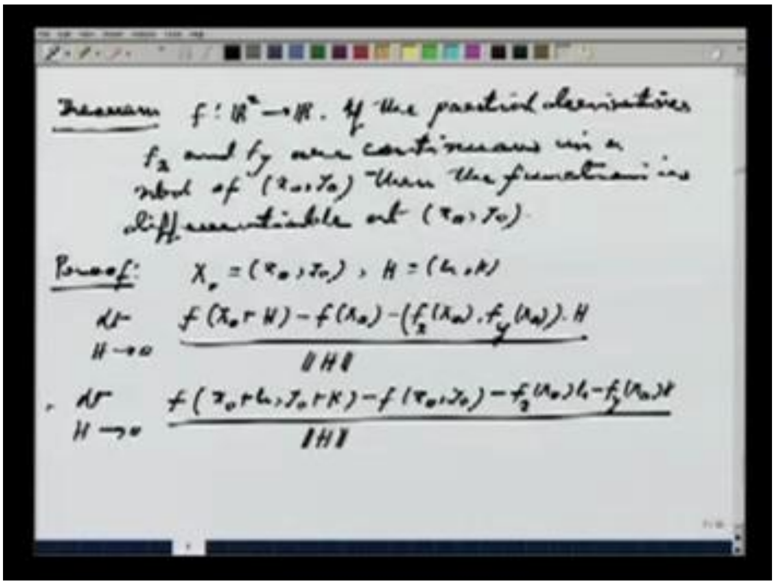

Now, let us come to sufficient criteria for existence of derivatives. So, I will prove it as a theorem, it is about that how to connect differentiability of a function. So, the partial derivatives of the function. So, suppose f is a function from R 2 to R. If the partial derivatives f x and f y are continuous in a neighborhood which I will write in short as nbd in a neighborhood of x naught y naught. Then the function is differentiable at x naught y naught.

Notice that as I said that the partial derivatives f x and f y are continuous in a neighborhood of x naught y naught. That certainly means that partial derivatives exist otherwise there is no meaning of saying that they are continuous. But the point to note, is that the mere existence of partial derivatives. Does not tell me that the function is differentiable at x naught y naught. I am imposing an extra condition of continuity on the partial derivatives, that gives me differentiability of the function.

Now, we will prove this result. And after that we come to the question that is it true that if just the partial derivatives exist. Then the function is continuous. Well that can be answered very simply we will come to that first let us try to prove this result. So, how do you proceed, let us say X naught stands for x naught y naught and H stands for h k. In that light, let us look at limit H going to 0 f of x naught plus H minus f of x naught minus f x at x naught times.

If I want to show that the function f is differentiable. I should check somehow that this limit is 0. Now let me write down all these quantities f of x naught plus h y naught plus k minus. Minus f x at x naught times h minus f y at x naught times k divided by norm h.

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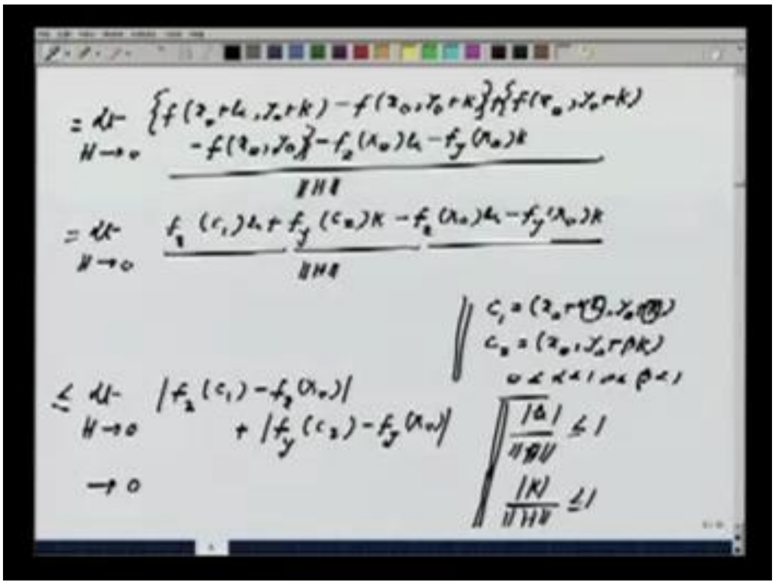

Now, this quantity I write as limit h going to 0 f of x naught plus h y naught plus k minus f of x naught y naught plus k; these are the terms I am introducing plus. So, these two terms was not there. Earlier I have introduced them minus f of x naught y naught, then the rest f the terms as a result f x at x naught. Now, look at this first term, if I fix y naught plus k, then it becomes a function of one variable.

If I fix the second coordinate, then I get a function of one variable. And there I can apply the mean value theorem. Similarly, I am going to look at this and this. Here what

I do is, I fix the first coordinate, then the second variable I can apply the one variable mean value theorem. So, what do I get, if I do that, I get limit H going to 0. Let me write it in this form I get f x at some point c 1 times h plus f y at some point c 2 h k.

I will explain what are these c 1 c 2 minus f x x naught h minus f y x naught k divided by norm h. Where c 1 it is a point of the form x naught plus alpha h y naught plus k. And c 2 is a point of the form x naught y naught plus beta k where alpha less 1 beta less 1. This is by the standard one variable mean value theorem. Now, the point to note is, that I can write that this is less than or equal to limit H going to 0 modulus of f x c 1 minus f x at x naught plus modulus of f y c 2 minus f y x naught.

This simply follows by the fact that modulus of h by norm h is strictly less than 1. And modulus of k by norm h is less than or equals to 1, if I use this then I come to this step. And now, I am going to use that fact that the partial derivatives f x and f y are continuous. Notice that if h k goes to 0, 0 if little h and little k if they go to 0. Then the point c 1 and c 2 they go to x 0, y 0. That is very simple from their expressions.

So, once again if little h comma little k goes to 0, 0, that implies the point c 1 c 2. They converge to x naught y naught. But since, f x and f y are continuous functions, then this goes to 0. So, in the last step I am actually using the fact that partial derivative of f at x, in the x direction and in the y direction are actually continuous functions. So, finally, this limit goes to 0. ((Refer Time: 36:02)) And hence, the difference quotient which I have started with that is this difference quotient now goes to 0.

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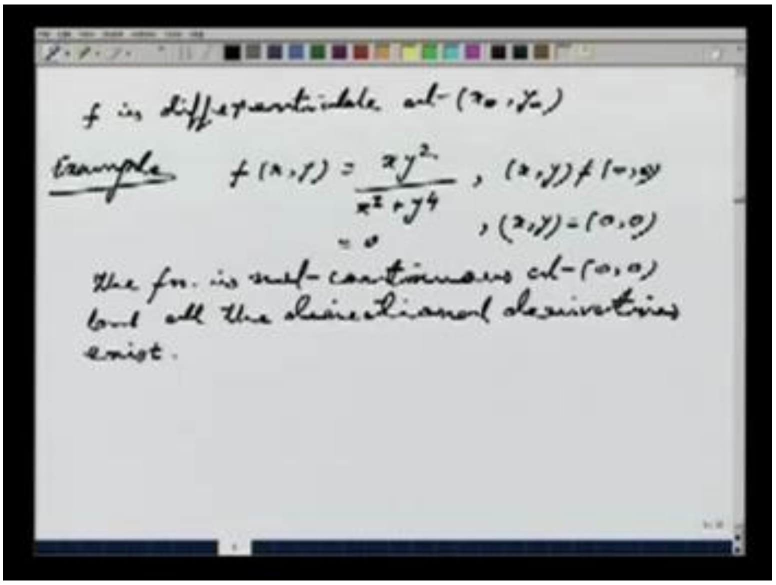

This implies that f is differentiable at x naught y naught. Of course, then it has to follow that the derivative of f is given by the partial derivatives. That is why in the first place I have chosen my alpha 1 alpha 2 the vectors to be f x an f y, and started calculating the difference quotient. So, we assume now that if the partial derivatives are continuous in a neighborhood of a point.

See the neighborhood is important. Because c 1 and c 2 lie in the neighborhood of x naught y naught they are not really equal to x naught y naught. See, if those points converge to x naught y naught, then the partial derivatives should also converge to the partial derivatives at x 0 y 0. That is the continuity which you are going to use. So, the continuity of the partial derivatives in a neighborhood of a point x naught y naught. Implies that the function is differentiable, and then the derivatives are given by partial derivatives.

So, now the question remains that if it is the mere existence of partial derivatives, does that imply f is continuous. Well we have already seen some such examples, I will again repeat that example here. I look at the function f of x y is equal to x y square divided by x square plus y to the power 4. Where x y is not equal to 0 0 and it is 0. If x y is equal to 0, 0, but this function we have seen that, the function is not continuous at 0, 0. But, all the directional derivatives exist.

So, in particular the partial derivatives exist. That is in the direction of x and y. But then, the function cannot be differentiable, simply because the function is not continuous. If f is differentiable f had to be continuous. But, this function is not differentiable, because it is not continuous at 0, 0.

But, see all the directional derivatives exist; that means, in particular partial derivatives also exist. That means, just existence of mere partial derivatives can never give you differentiability, simply because mere existence of partial derivatives will never give you continuity of the function. And if the function f has to be differentiable it has to be continuous.

So, you need continuity of the partial derivatives, that continuity cannot be resolved cannot be dropped. So, if the partial derivatives are continuous. Then the function is certainly differentiable and the derivative is given by partial derivatives. On the other hand if I already know that the function is differentiable. Then I know its derivatives are given by partial derivatives. So, that is all I wanted to say about the connection with partial derivatives.

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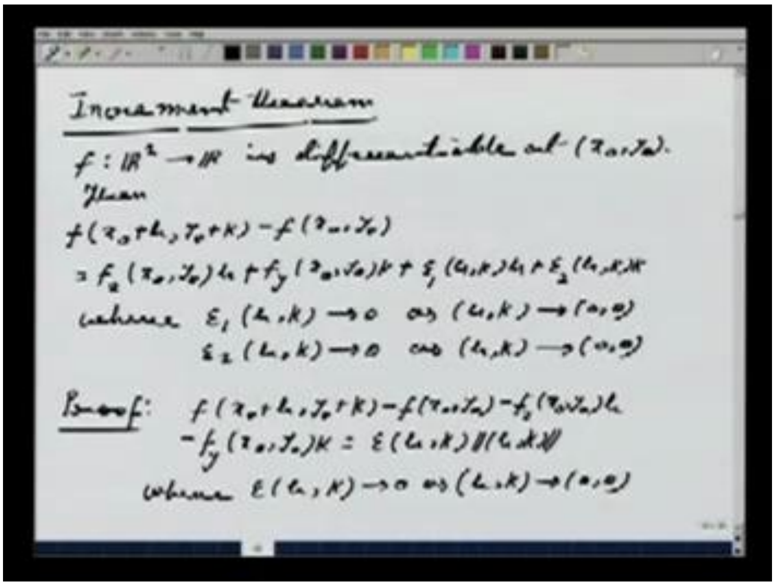

Now, let us proceed with the lecture our next topic is bit technical in nature. It is called the increment theorem. I am going to prove it, because I am going to use it very soon. So, let us start with the statement of increment theorem. So, what does it say, so suppose f from R to R. So, the function is from R 2 to R is differentiable at x naught y naught. Then, we can write that f of x naught plus h y naught plus k minus f of x naught y naught is equal to f x at x naught y naught times h plus f y at x naught y naught times k nothing new so far, but now plus epsilon 1 h k. So, it is a epsilon 1 depends on h k that is what I mean here, times h plus epsilon 2 h k. That means, epsilon 2 also depends on h k times k, where epsilon 1 h k goes to 0 as h k goes to 0, 0. Similarly, epsilon 2 h k also goes to 0 as h k goes to 0, 0. So, the only difference here with the definition of derivative is there I just had epsilon h k. Here I am breaking it into two parts epsilon 1 h k epsilon 2 h k this form is going to be handy to prove certain result, that is why I am going to prove it.

The proof is not very difficult, so we start with the proof the following way. So, as f is differentiable, I know that f of x naught plus h y naught plus k minus f of x naught y naught minus f x at x naught y naught times h minus f y at x naught y naught times k that is equal to some epsilon h k times norm h k, where epsilon h k goes to 0 as h k goes to 0, 0.

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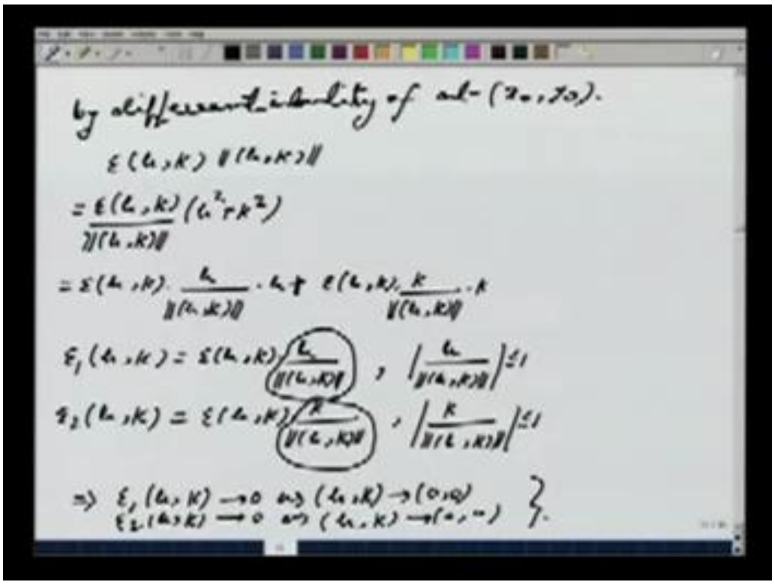

By differentiability of f of f at x naught y naught, now, I am going to play around with this epsilon h k times norm h k, what I do is? I write this as epsilon h k divided by norm h k times h square plus k square, that I can do because the expression of norm h k is square root of h square plus k square. So, what I am doing is, I am multiplying numerator and the denominator by square root of h square plus k square.

So, this now I write as, epsilon h k times h by norm h k times h plus epsilon h k times k by norm h k times k. So, now I define epsilon 1 h k to be equal to epsilon h k times h by norm h k, similarly epsilon 2 h k means epsilon h k times k by norm h k. Then I got the required form only thing I have to prove that as h k goes to 0, 0 epsilon 1 h k and epsilon 2 h k they go to 0.

Now, we just simply notice that modulus of h by norm h k is lesser equal to 1. Similarly, modulus of k by norm h k, that is also lesser equal to 1 that means, this is bounded quantity. So, is this and as h k goes to 0, 0, I know that epsilon h k goes to 0. So, anything bounded times a quantity which goes to 0 then the product also goes to 0. So, this implies that epsilon 1 h k goes to 0, 0 goes to 0 as h k goes to 0, 0. Similarly, epsilon 2 h k also goes to 0 as h k goes to 0, 0.

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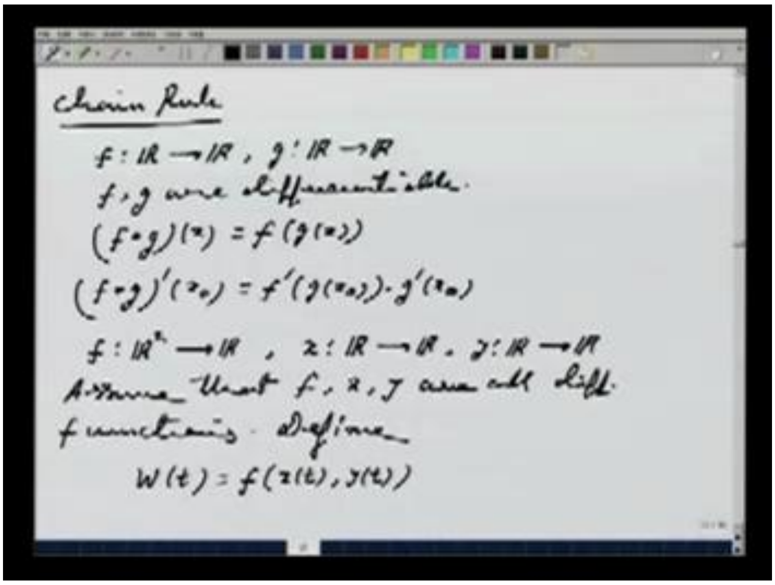

That proves the increment theorem which now I am going to use in something called Chain rule. So, what is Chain rule, so suppose I have a function f from R to R and g is another function from R to R, assume that f g are differentiable. Then, I can look at the function f compose g at x it is f of g x. Now, I want to know what is f compose g prime the derivative at x naught. This we know by the one variable chain rule is f prime at g x naught times g prime at x naught. This is quite frequently used now I want several variable analog of this that is in two dimensions.

So, here the setup is exactly like this that I have a function f from R 2 to R. And I have two functions one is x from R to R. Other is y from R to R, assume that f x y are all differentiable functions. Once I know this I can create another function. So, I define W of t equal to f of x t comma y t and I want to know about the differentiability of W at a point t naught.

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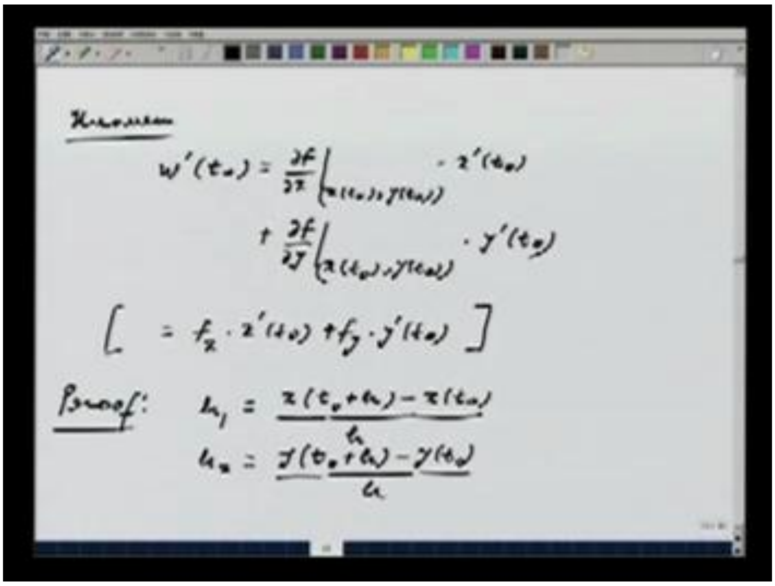

This is precisely the Chain rule which we are going to prove. So, the theorem I am going to prove is, it is that w prime at a point t naught is del f del x at x t naught y t naught times x prime t naught plus del f del y at x t naught y t naught. Let me put brackets here times y prime t naught. A standard way of expressing is that this is f x times x prime t naught plus f y times y prime t naught. So, this f x f y here stand for that the partial derivatives at x t naught y t naught.

So this is, what we are going to prove now, fine. So, how do you start here, first I define certain things I define h 1 to be equal to x of t naught plus h minus x t naught divided by h and h 2. That is equal to y of t naught plus h minus y of t naught divided by h. Notice here little x and little y stands for functions, in fact differentiable functions.

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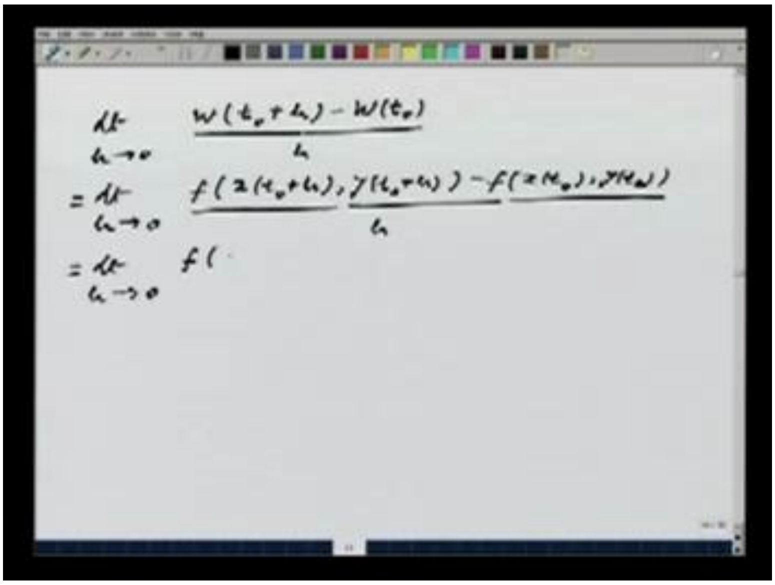

Now, using this let me start with limit h going to 0. W of t naught plus h minus W t naught divided by h. Because W is a function form R to R. So, I use the standard function of differentiation. This now is same as just by writing down definition writing down the definition of W. That this f of x t naught plus h y t naught plus h minus f x t naught y t naught divided by h.

So, this then is same as limit h going to 0, f of I look at now my definition of h 1. That is x t 0 plus h minus x t 0 by h and h 2 equals to y t naught plus h I define h 1 equal to x t naught plus h minus x t naught. And I define h 2 equal to y t 0 plus h minus y t.

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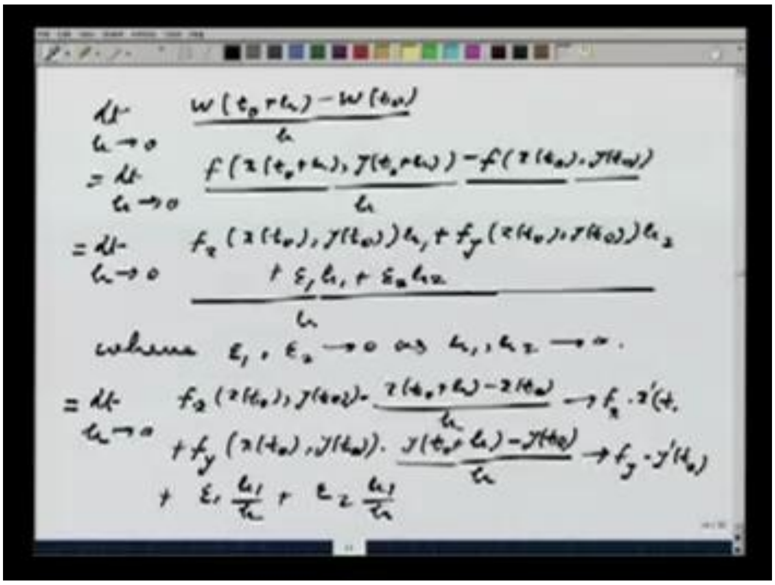

Now, I start with the usual definition of one variable derivative. That is limit h going to 0 W t naught plus h minus W t naught divided by h that is limit h going to 0. By writing down the definition of the function W it is f of x t naught plus h y t naught plus h minus f x t naught y t naught divided by h.

Now, I am going to use the increment theorem, by increment theorem. This is same as limit h going to 0 f x at x t naught y t naught times h 1 plus f y at x t naught y t naught times h 2 plus epsilon 1 h 1 plus epsilon 2 h 2 divided by h. Where I know that epsilon 1, epsilon 2, they both go to 0 as h 1 h 2 go to 0. This is from the increment theorem.

Now, I write it in a different form that this is same as limit h going to 0 f x at x t naught y t naught times the definition of h 1. That gives me x t naught plus h minus x t naught divided by h plus f y at x t naught y t naught times y t naught plus h divided by h. This is by using the definition of h 1 and h 2. Then plus the extra terms plus epsilon 1 h 1 by h plus epsilon 2 h 1 by h.

Now, let us look at the limits of each terms the first term this certainly goes to f x at the point times x prime at t naught. The second term goes to f y at the point x t naught y t naught times y prime at t naught. Here also I have at t 0 x prime at t 0. So, now I am bothered about the last terms, all I know is as h goes to as h 1 h 2 goes to 0, 0 epsilon 1 and epsilon 2 goes to 0.

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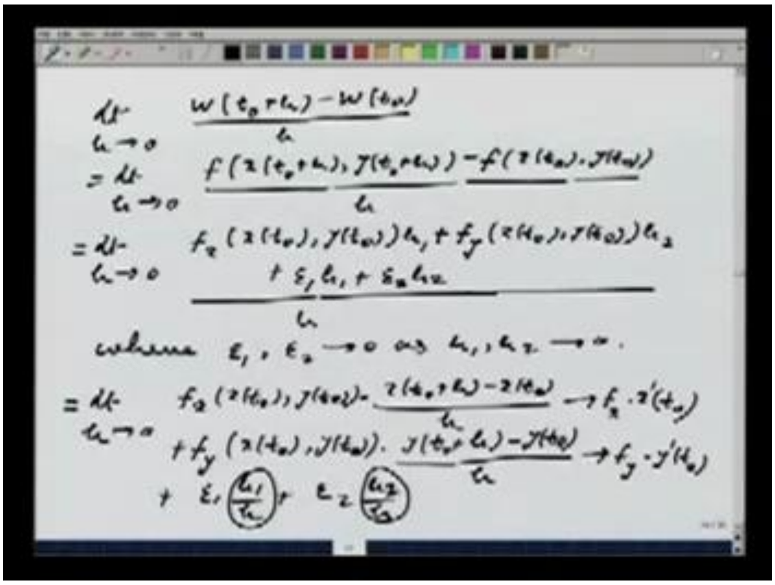

Now, as h goes to 0, then what is h 1. That is x t naught plus h minus x t naught, since x is differentiable it is certainly continuous. So, this goes to 0 as h goes to 0 and h 2 which is y t naught plus h minus y t naught that goes to 0 as h goes to 0. So, this implies epsilon 1 goes to 0 epsilon 2 goes to 0 as h goes to 0. Because as h goes to 0 h 1 goes to 0 h 2 goes to 0 and hence epsilon 1 goes to 0 epsilon 2 goes to 0. But, there are some more things to take care of what about limit h going to 0. H 1 by h that quantity also appears here you can see, and here I will get h 2 by h.

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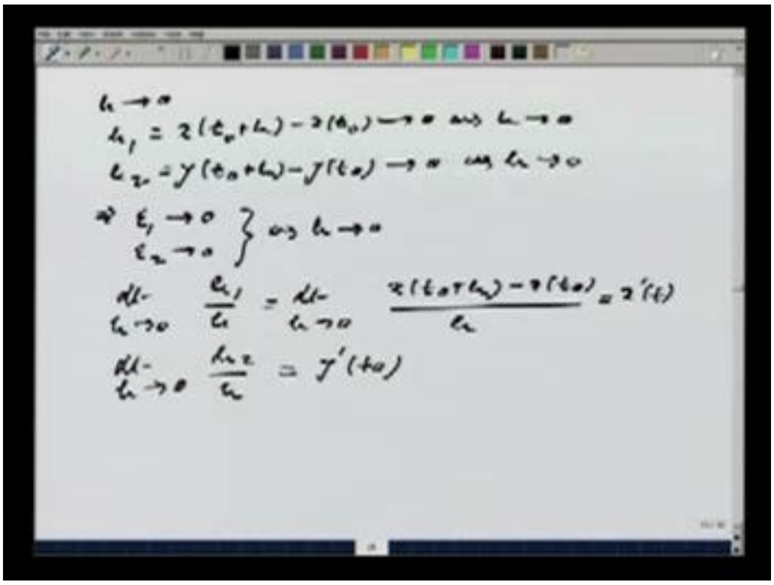

So, what do they go, again I write down the definition it is limit h going to 0. What is h 1? It is x t 0 plus h minus x t 0 divided by h. So, this limit I know is x prime at t 0.

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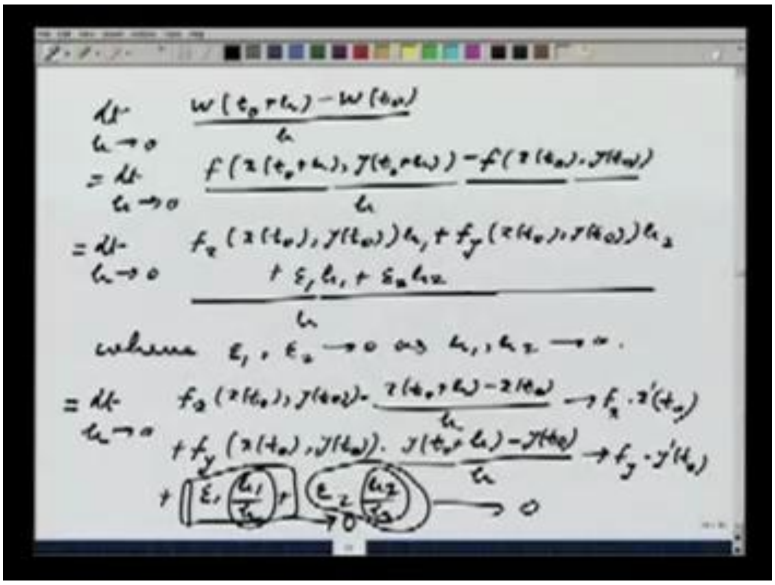

So finally, similarly h 2 by h limit h going to 0 h 2 by h, then is y prime at t 0. So finally, what do I get if I look at the final difference quotient. So, the first term goes to f x at x prime t naught, second term goes to f y at y prime t naught. The third terms that is the term here this goes to 0. Because, epsilon 1 goes to 0 and the remaining term h 1 by h that goes to x prime at t naught. That I have checked now I look at this term this also goes to 0, because epsilon 2 goes to 0 and h 2 by h that goes to y prime at t 0.

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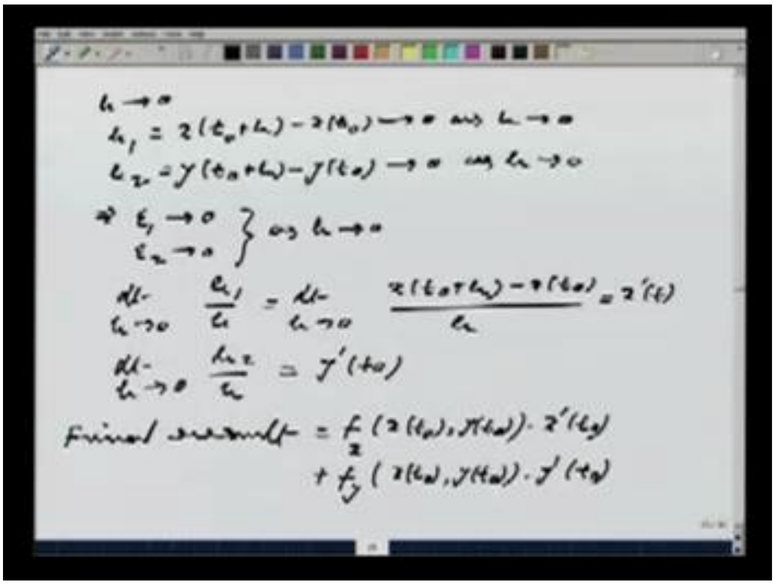

So, as a whole the whole limit turns out to be equal to So, final result of the limit that turns out to be f x at x t naught y t naught times x prime at t naught plus f y at x t naught y t naught times y prime at t naught. But, this is precisely, what has been claimed in the theorem. That the derivative of W is given by this some of the products, so this proves Chain rule for us. In the next lecture, we will go to the geometric interpretation of the derivatives its relations with tangents and things like that.