Mathematics-I
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Lecture - 24
Derivatives

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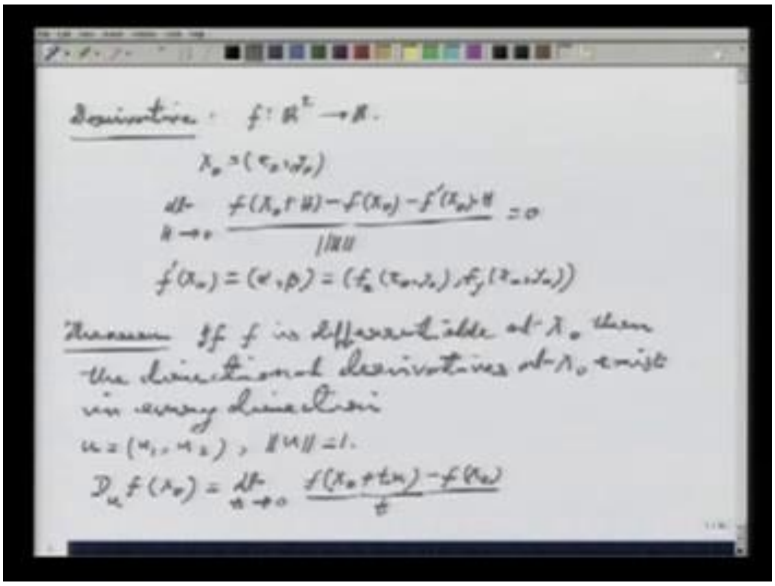

Today, we start with the connection between derivative and total derivative. So, we know the definition of derivative. So, let me recall it again. So, f is a function from R 2 to R. We are talking about derivative at a point x naught, so x naught looks like x naught y naught. So, we say f is differentiable at x naught. If this limit h going to 0 f of x naught plus h minus f of x naught minus f prime at x naught dot h divided by. That means, multiplied by 1 by norm h, if this is equal to 0. Here, f prime at x naught is a vector alpha beta.

And we have seen in our previous lecture, that what these alpha betas are. They are nothing but, the partial derivatives at the point x naught y naught. Now, what I want to prove is the following theorem, that if f is differentiable at x naught, then the directional derivatives at x naught exist in every direction. So, let me again recall, what is the definition of the directional derivative. So, I take a vector u which looks like u 1 u 2 such that, norm of u is equal to 1.

Then, the directional derivative of f at x naught in the direction of the vector u is called D u f at x naught, and is given by the following limit going to 0. Here t is a number f of x naught plus t u minus f of x naught divided by t. This is the definition of the directional derivative. So, all I have to show is assuming that f is differentiable at the point x naught. I am going to show that the directional derivative exist in all possible directions.

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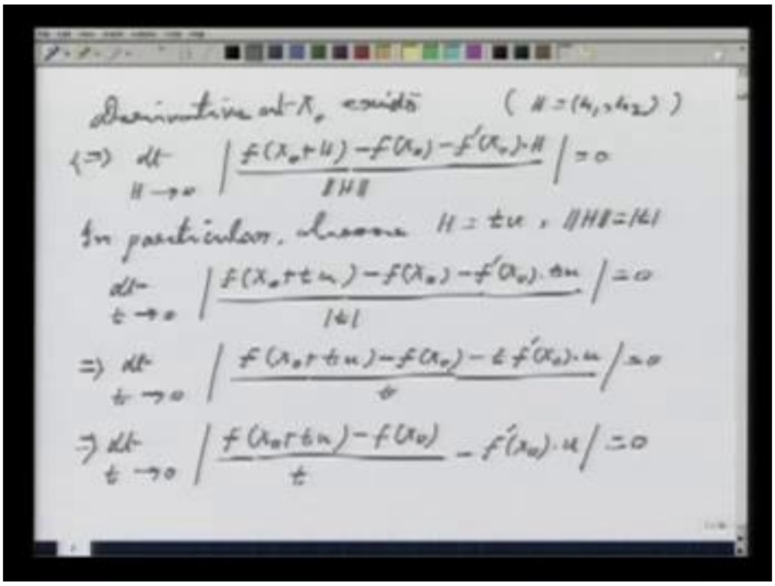

So, derivative at x naught exists implies and implied by that limit H going to 0. Modulus f of x naught plus h minus f x naught minus f prime at x naught which remember is a vector. So, it makes sense to take a dot product with H. Where H looks like, I will write in the bracket here H equal to H 1 H 2 divided by norm H. This is equal to 0 derivative exist means this.

So in particular, choose H to be t where u is a vector, t is a number. Then norm of H is nothing but mod t, because u is an unit vector. So, while taking norm H going to 0, we can as well take t going to 0. So, if instead of H, I put t u in the above limit. What I get is, that limit t going to 0 modulus f of x naught plus t u minus f of x naught minus f prime x naught dot the vector t u divided by mod t this is equal to 0.

Just in the previous limit, I have put h equal to t u. But, this then implies that limit t going to 0. Since, I have the modulus it is f of x naught plus t u minus f of x naught minus t times f prime x naught dot u. Because, t is a number divided by t this is equal to

0. This implies limit t going to 0 modulus f of x naught plus t u minus f of x naught divided by t minus f prime x naught dot u is equal to 0.

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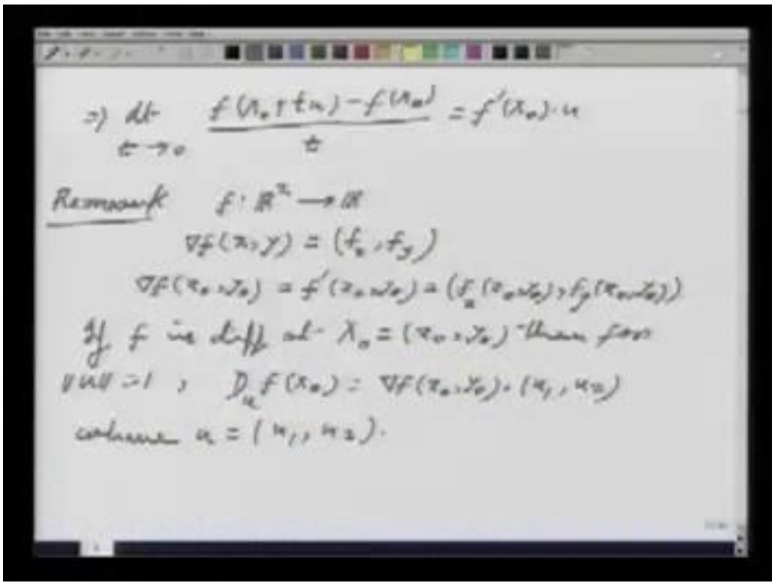

This then implies that limit t going to 0 f of x naught plus t u minus f of x naught. This whole thing divided by t is equal to f prime at x naught dot u. So, the limit exist and the limit is equal to f prime x naught dot u. So, it implies that if the function is differentiable at x naught. Then partial derivatives exist in all the directional derivatives exist in all possible directions.

Not only that, it also tells me how to compute the directional derivatives. So, for that I will just put it as a remark. And I will start with a notation. Let us say, f is from R 2 to R. Then I define this, it is notation it is called gradient of f at the point x y. It is just f x at the point x y comma f y, so it is a vector. And we know this gradient if f is differentiable, this gradient actually gives me the derivative.

So, grad f at x naught y naught is nothing but, f prime at the point x naught y naught, which is given by the definition of grad as f x at x naught y naught comma f y at x naught y naught. Now in this light, what I have proved is that if f is differentiable at x naught which is little x naught little y naught. Then for a unit vector u I have D u f at x naught which is nothing but, grad f at x naught y naught dot product with the vector u; which is given by u 1 u 2, where u equals to u 1 u 2.

So that means, to calculate partial the directional derivatives of a function. All we have to do is we have to calculate the partial derivatives first. Evaluate the partial derivatives at the point, where we are going to talk about directional derivatives. Let us say that point is x naught y naught.

So, what I do is, I calculate the partial derivatives at x naught y naught. Then I get a vector if I look at f x at x naught y naught comma f y at x naught y naught. Then, I just take the dot product of this vector with the vector u 1, u 2 in whose direction I am calculating the directional derivative. This quantity gives me the directional derivative of the function at the point x naught y naught. So, this comes very simply, but some care has to be taken here. That the just the mere existence of the gradient vector. That is grad f does not give me the existence of the total derivatives, the existence of the directional derivatives.

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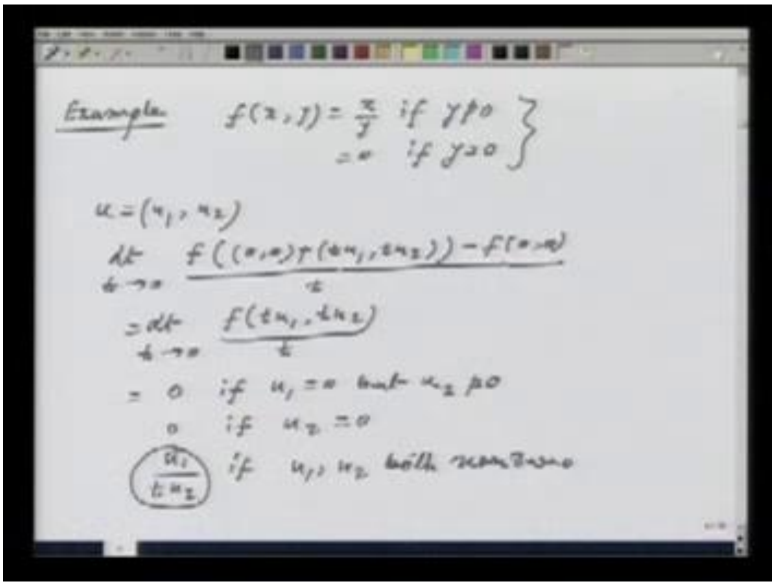

For that let me just, look at this simple example. See you might think it this way that suppose I know that I can calculate the gradient of the function, that means f has partial derivatives. Once I have this, then if I believe the previous theorem. Then the directional derivative of the function in any direction should exist as grad f dot u. But then, we are forgetting the first line of the theorem.

That if f is differentiable at x naught, then it is true, if f is not differentiable at x naught grad f may exist. But, the directional derivatives in many other directions may 0 exist.

So, that is what we are going to see in this example. So, let us look at this function f of x y to be equal to x by y. If y is not equal to 0, it is 0 if y is equal to 0. Let us look at this function. And now, I want to talk about it is partial derivatives, directional derivatives and all such things.

So, I take a vector u which I call u 1 u 2. Then I want to calculate limit t going to 0. F of I am going to calculate the directional derivatives at let us say 0, 0. So, it is 0 0 plus t u 1 t u 2 minus f of 0 0 divided by t. What I get is, limit t going to 0 f of t u 1 t u 2 divided by t. Now, I say this is 0, if u 1 equal to 0, but u 2 not equal to 0. And this is also 0. By definition, if u 2 equal to 0 and this is equal to u 1 by t u 2.

If u 1 u 2 both non-zero that means, if I look at the third case it is very clear. That as t goes to 0, this limit does not exist. But, in the other cases the limits do exist now. What are the other cases, I take u 1 to be equal to 0. That means, what I am getting is a vector in the direction of the y axis. If I take u 2 to be equal to 0, I get a vector in the direction of the x axis. Those two directional derivatives exist, but if u 1 and u 2 both are non zero. Then I get vectors in different directions in those directions the limits does not exist.

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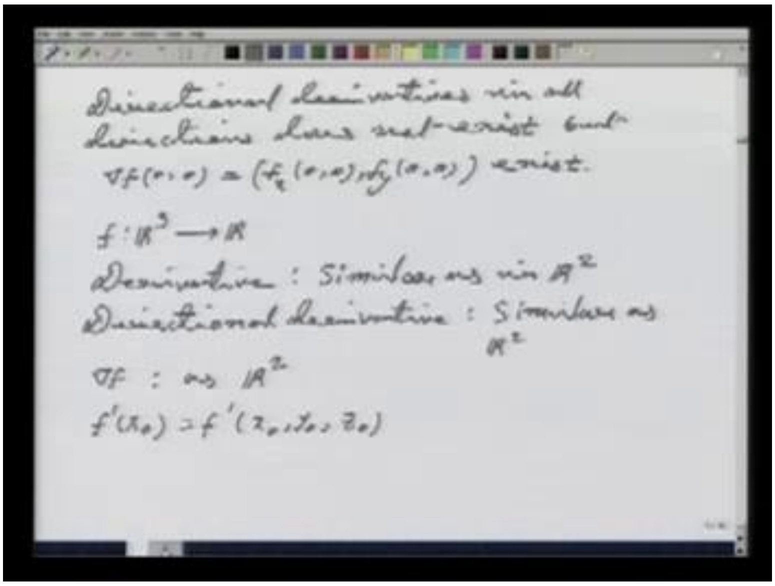

So, the conclusion is that directional derivatives, in all directions does not exist. But, grad f at 0, 0 which is given by f x at 0, 0 f y at 0, 0 exist. So, why it does not contradict the previous theorem simply, because the function is not differentiable. So, this is something to remember that if the function is differentiable at X naught. Then to calculate the directional derivatives which now I know exist, if f is differentiable.

To calculate the directional derivatives in any direction all you calculate is a gradient that is grad f at the point. And take the dot product with the vector u 1 u 2. That will give me the directional derivative of the function in any direction. Now so far, whatever we have developed goes through for arbitrary R n. So, in particular we will concentrate on functions from R 3 to R. So, definition of derivative this is similar as in R 2, so is directional derivative. This is also similar as R 2 even gradient as in R 2. So, formally what do I mean by f prime x naught here. So, this should be f prime at x naught y 0 z 0, because I am on R 3.

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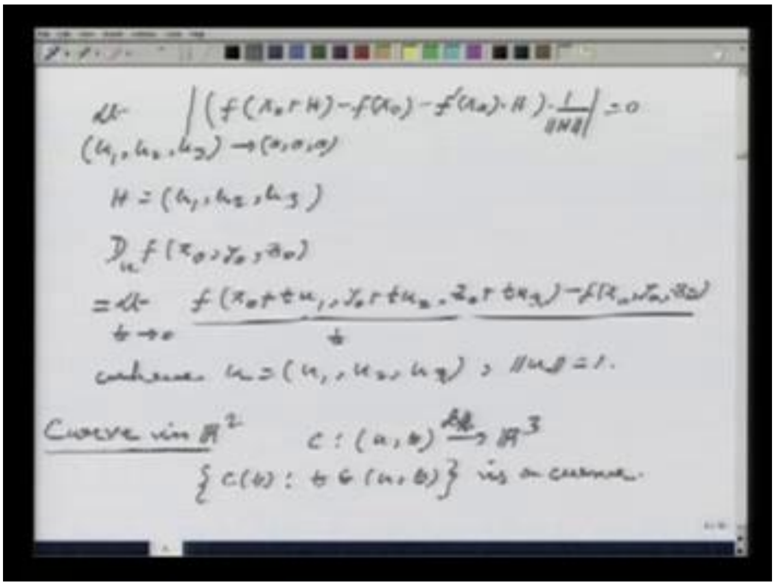

So, how is that defined, this is limit h 1 h 2 h 3 going to 0 0 0. This is limit h going to 0 of the quantities f of x naught plus H minus f of x naught. Exactly as in the case of R 2 minus f prime X naught dot H times 1 by norm H if this is equal to if this is equal to 0. Here, H is h 1 h 2 h 3 and directional derivative is exactly as in the previous case. I can define D u f at x naught y naught z naught.

This is limit t going to 0 f of x naught plus t u 1 y naught plus t u 2 z naught plus t u 3 minus f of x naught y naught z naught whole divided by t, where u is the vector u 1 u 2 u 3 with norm 1. So, everything is analogous here, whatever we have proved all the results work including the chain rule which I am going to use now. Now in all this, I could suddenly see that gradient of f is playing some role. So, you want to understand the geometric interpretation of gradients. So, that is what I am going to start with now.

So, we start with definition of curves. So, what is a curve in R 3, it works similarly in R 2 also. It is just a differentiable function c from an interval a b to R 3. Then, if I look at c t, t belongs to a b this is a curve, so it is just image of differentiable functions. So, c is differentiable.

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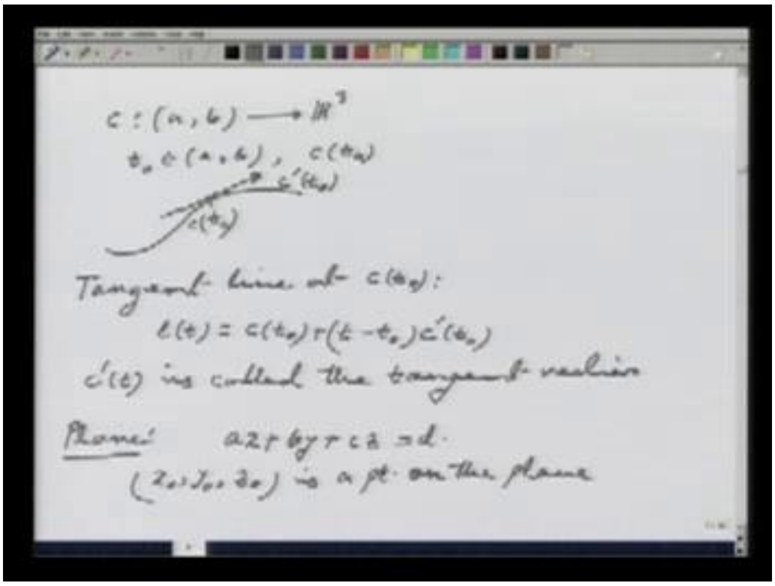

Now, I want to talk about the tangents to curves. How does 1 define tangent to a curve, So, let us say I have a point, so c let us say is a curve a b to R 3. Let us say t naught belongs to a b and I look at the point c t naught. So, the image of the curve let us say looks like this and this is c of t naught. I want to talk about the tangent line at the point c t naught. So, analogous to the one dimensional case, I can tell you the parametric equation of the line in the space, it is given by if I call it l t. This is the line which passes through c t naught. That is the point on the curve with the direction of the vector c prime t naught. So, it is t minus t naught times c prime t naught.

So, what I get, is a line which passes through the point c t naught. Because, if I put t equal to t naught. Then l t equals to c t naught and the direction of the line is in the direction of the vector c prime t naught. That why c prime t naught is called the tangent vector. This is at so c prime t is called the tangent vector. So, this would give you the direction of c prime t naught with t of course, the tangent changes. So, the tangent vector is also a function of t which I have written as c prime of t. Now, I need some more notations about planes, so how do we view planes. So, what is a plane, we know the usual equation of planes it is of the form a x plus b y plus c z equal to t. Let us say x naught y naught z naught is a point on the plane.

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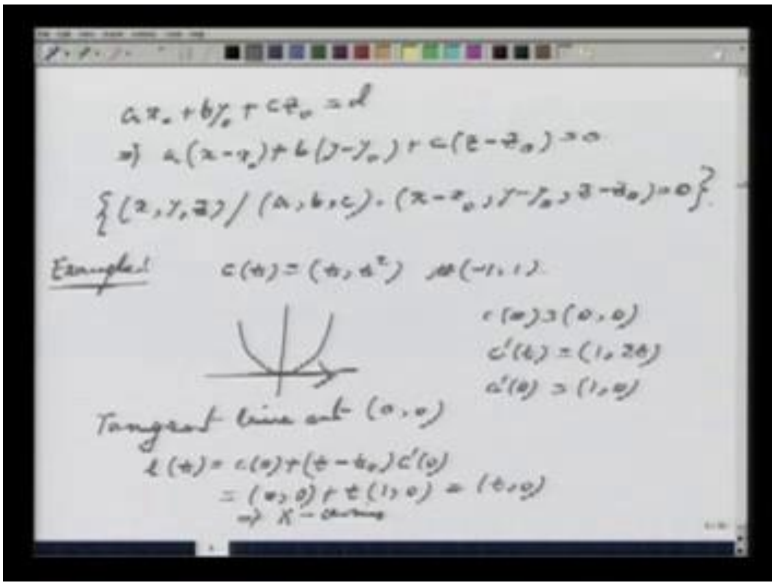

Then, a x naught plus b y naught plus c z naught. That is also equal to d, that implies a of a times x minus x naught plus b times y minus y naught plus c times z minus z naught is equal to 0. So then, I can write this as the collection of all points x y z. Such that, the dot product of the vectors a b c dot x minus x naught y minus y naught z minus z naught is equal to 0.

So any plane, actually looks like this kind of a set x y z such that a b c dot x minus x naught y minus y naught z minus z naught is 0. Where x naught y naught z naught is a point on the plane and a b c are some given numbers. So, any expression of this form always gives me a plane, now this formulation of plane I am going to use.

Now, let us come back to tangent line again, let us see an example. So, let us look at the curve c t equal to t t square, where let us say t belongs to minus 1 1. Then what is the, what does the curve look like. This is clearly a parabola y equal to x square form, so it looks like this. And I want to calculate the tangent line, let us set the origin. Let us say 0 0, so what would be the equation.

Equation of the line should be l t equal to c of 0. Because, I see that c of 0 is 0 0 plus t minus t naught into c prime t naught. That means c prime at 0. So I calculate, what is c prime at t it turns out to be 1 2 t. So, c prime at 0 is 1 0. So, what I get is 0 0, plus here t naught also is 0, so it is t times 1 0. So, it is just t 0, where t varies over the set of real numbers. So, all I get is the x axis which is very obvious from the picture, that this line at 0 0 is the tangent to the curve.

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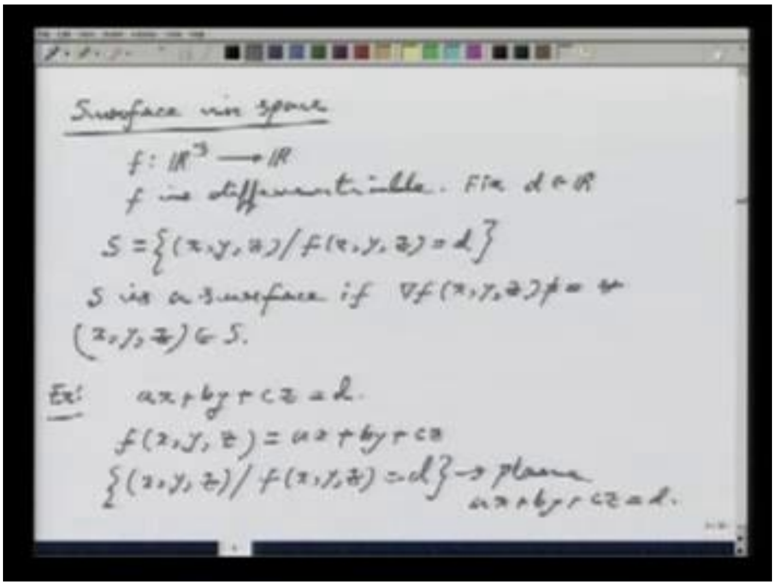

Fine, now I define something called surface. So, what is a surface, so I would say surface in space. Let us say f is a function from R 3 to R, f is differentiable. I look at the set S fix a constant c fix some let us say d in R. I look at all x y z which are mapped by f to d. I call S is a surface, if grad f at x y z is not equal to 0 for all x y z in S.

So, what is the surface then, I have a differentiable function f I look at I fix a constant d. I look at all x y z such that, f of x y z equals to d, assuming the set is non empty of course. Then, if the gradient of the function at all those points x y z is non zero, then I call S a surface.

So, some obvious examples of surface I can try to see let us look at the plane a x plus b y plus c z equal to d is this a surface. So, what I do is, I concentrate on the function f of x y z equal to a x plus b y plus c z. And then I look at all x y z such that, f of x y z equal to d.

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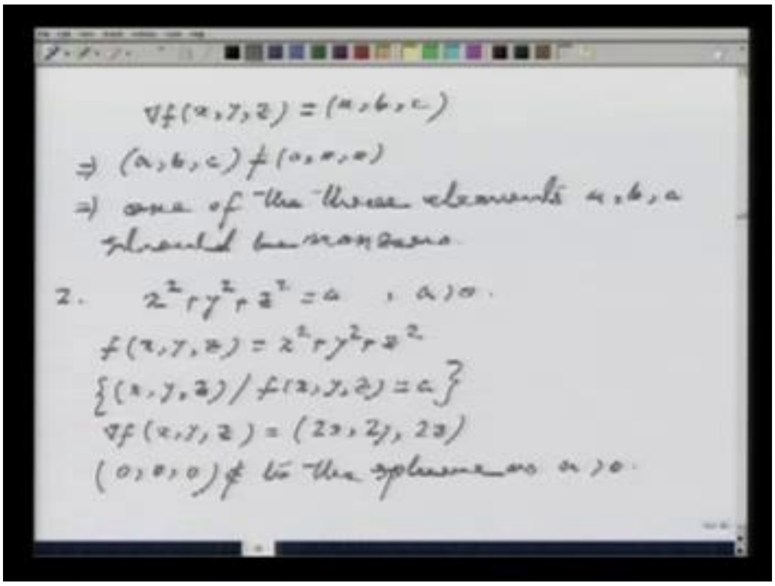

Now, we can see from the definition of f. That this is the same plane, given by a x plus b y plus c z equal to d. Question is when is this a surface. So, for that I need to look at gradient of f at any x y z. Well gradient of f at x y z turns out to be a b c. So, my definition demands that this gradient should be non zero. That means, I want a b c to be not equal to 0 0 0.

This would imply that one of the three quantities a b c should be non zero. If all of them are 0, then I do not have the plane you see. That means, any plane is a surface. So, let us look at another example, let us look at sphere of radius. Let us say some a where a bigger than 0. For this, I look at the function f of x y z to be equal to x square plus y square plus z square. Then the sphere is just the collection of x y z. Such that, f of x y z is equal to a, f is certainly differentiable function.

All I need to check is that the gradient of f at every point on the surface is non zero. So, what is gradient of f at the point x y z. What I get is, twice x twice y twice z. Now, this should be not equal to 0 0 0, that means one of those x y z should be not equal to 0 0 0. Now, what happens if all these quantities are 0, x y and z both are 0. Then I am getting the point 0 0 0 which fortunately does not belong to the sphere as a is strictly bigger than 0.

So, at every point on the sphere x square plus y square plus z square equal to a is strictly bigger than 0. The gradient is non zero, because all these quantities cannot be 0 then. So, any sphere of any positive radius is always surface for us according to our definition.

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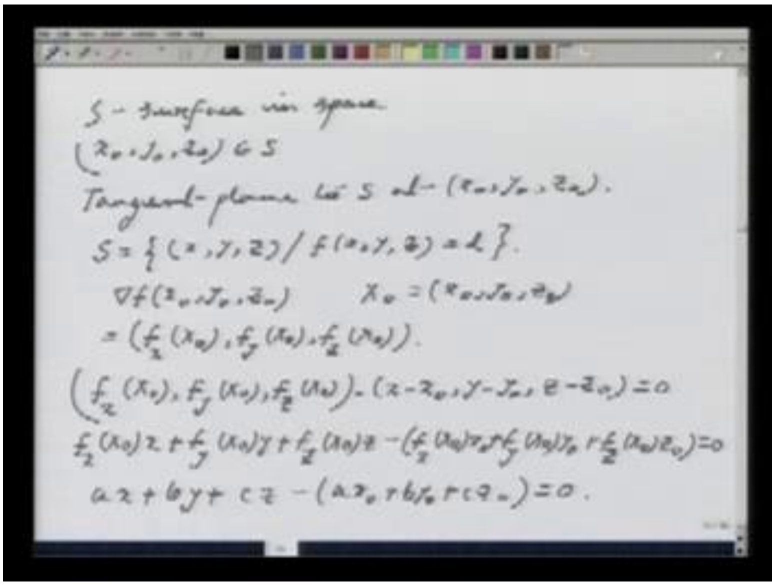

Now the next, let S be a surface and x naught y naught z naught. So, it is a surface in space. Let us say this is a point on S, I want to talk about something called tangent plane to S at the point x naught y naught z naught. So, first of all it should be a plane, I will first tell you, what is the definition of the tangent plane? Since S is a surface, S must be given by a function. So, it must be collection of all points x y z. Such that, f at x y z is some constant d. S must be of this form either look at grad f at x naught y naught z naught.

So, I know its expression, what it is it is f x at the point, let us say X naught is equal to x naught y naught z naught. Then, grad f at x naught y naught z naught is f x at x naught comma f y at x naught f z at x naught. Then the equation of the tangent plane is nothing but this that f x at x naught comma f y at x naught comma f z at x naught. This vector dot x minus x naught y minus y naught z minus z naught equal to 0.

So, if I write it down the equation of the plane turns out to be f x at x naught times x. Plus f y at x naught times y plus f z at x naught times z minus f x x naught times x naught plus f y x naught at y naught plus f z x naught times z not equal to 0. Notice here, that X naught that is the points x naught y naught z naught is given to me. So, f x at x naught that is a number f y at x naught is a number f z at x naught u is a number.

So, it is of the form a x plus by plus c z minus a x naught plus b y naught plus c z naught equal to 0 which is certainly equation of a plane. Because, x naught y naught z naught is a constant given coordinates of a point. So, the question in the bracket is actually a number. So, it certainly represents a plane and given a surface. And given the points I can certainly find out this plane, that we will do in the sequel. But, the question remains is, why exactly this plane is called the tangent plane to the surface. Is the plane in some sense tangential to the surface, we will like to know this.

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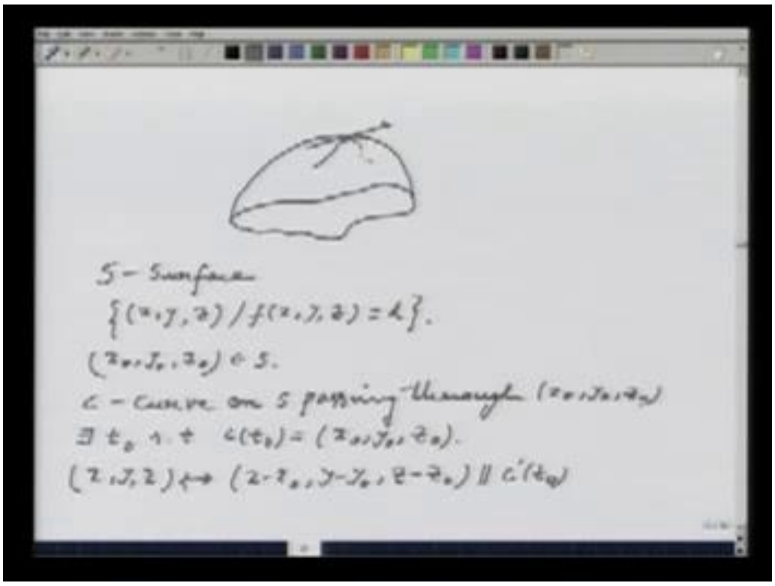

So, let us intuitively try to fix it out. So, suppose this top portion is a surface. And at this point I want to write the equation of the tangent plane. Now, I know the equation, I want to verify whether the plane is tangential to this surface. So, what I do is, I choose a curve you know this is a curve. So, this dotted portion is on the other side of the surface and I will look at then the tangent lines. So, if I take a different curve, I will get a different tangent. I will just like to check whether these tangents lie on the plane which I call tangent plane.

So, let us say S is the surface. Then I know that S is given by all x y z such that f of x y z is equal to d. Let us, fix a point x naught y naught z naught belonging to S, I take a curve c, so c is a curve on S passing through the point x naught y naught z naught. It means then there exists t naught such that c of t naught is equal to x naught y naught z naught.

Then, I can try to write down the tangent line at the point c t not, so whatever we have discussed. So, far the tangent line contains the points x y z which satisfies the following that the vector x minus x naught y minus y naught z minus z naught, this is parallel to the vector c prime t naught.

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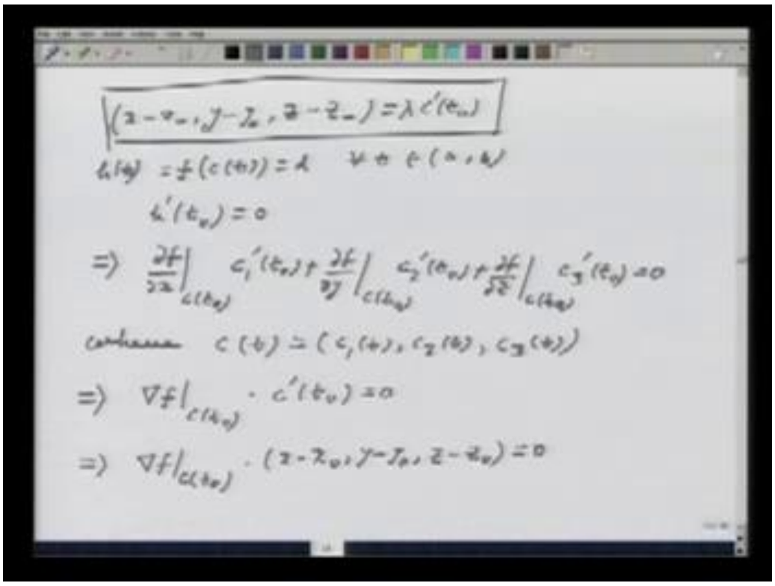

This is, what is the equation of the tangent line means, x minus x naught y minus y naught z minus z naught is a scalar multiple of c prime t naught, so it is lambda times c prime t naught. On the other hand, since the curve c lies on the surface, I also have that f of c t equal to d. For all t in that interval, let us say t belongs to a b. This one variable function I call h t, since the function is constant for all t in a b, I know that h prime at t naught certainly is equal to 0. Here, h prime t naught stands for the usual one variable derivative of h at t naught.

Now, I want to write down what is h prime at t naught for that I need to apply chain rule. So, what does this say by chain rule, then what I get is, that I get del f del x at c t naught times c 1 prime t naught plus del f del y at c t naught times c 2 prime at t naught plus del f del z at c t naught times c 3 prime t naught equal to 0, where what are these c 1 c 2 c 3, well c t is given by c 1 t c 2 t, because it is a curve in R 3 c 3 t.

But then this precisely implies that grad f at c t naught dot product with the vector c prime at t naught is equal to 0. But then this implies since the vector x minus x naught y minus y naught z minus z naught is parallel to c prime t naught. I get that grad f at c t naught dot x minus x naught y minus y naught z minus z naught s equal to 0.

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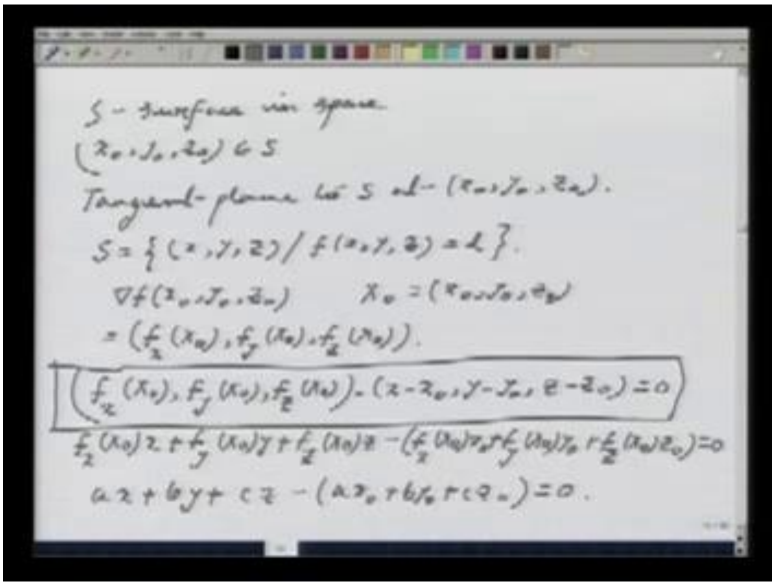

Now, if I expand this what I get is exactly the equation which I had written here where grad f at c 2 is given by the first quantity in the parenthesis.

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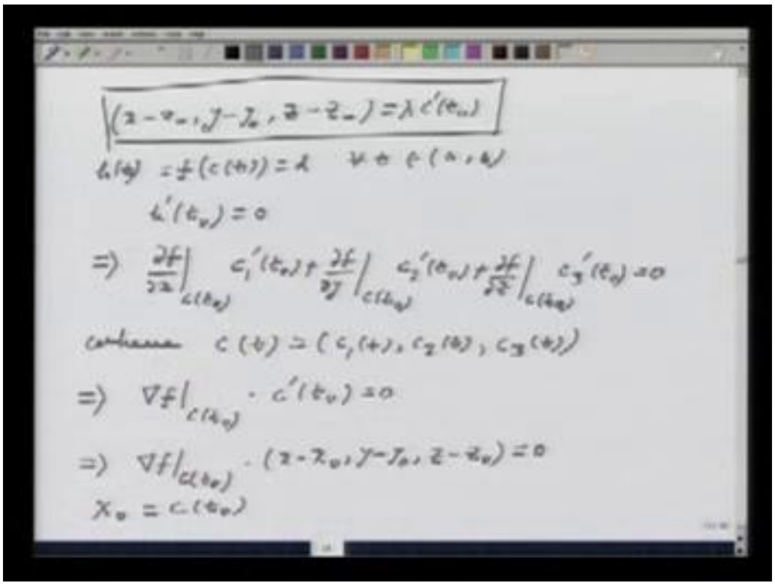

That is I just have called x naught is c t naught if I just use this, then what I get is the equation of the tangent plane; that means the tangents to the vectors c t actually lie on the plane, that is why this plane is called the tangent plane, it consists the tangents to the curves passing through the points t naught.

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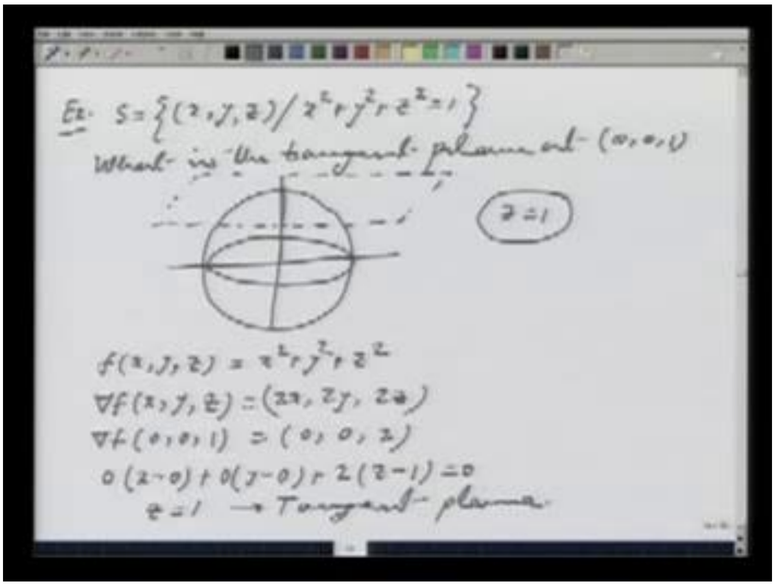

So, let us see some examples now, whether it matches with our intuitions. So, first example the obvious one I take the sphere that is my surface S is all x y z, such that x square plus y square plus z square equal to 1. Question is what is the tangent plane, at the points let us say 0 0 1.

So, let us look at the picture, so my sphere is this let us say this is the point the intersection is the z axis is the point 0 0 1 and I want to know what is the tangent plane, my intuition suggests that the plane should be looking something like. It should be parallel to the x y plane just at the height z equal to 1, so the tangent plane should be the plane z equal to 1.

Just think of a ball and on the top of the ball, you want to have a plane, which is tangential to the ball just feel think like putting a post card on the ball. So, from the definition on the tangent plane I am planning to get back the post card that is z equal to 1. So, let us go by the formulas, so what is my f here, f of x y z is x square plus y square plus z square, then what is grad f at x y z, this is 2 x 2 y 2 z. So, what is grad f at 0 0 1, this is 0 zero 2.

Then, what is the equation of the tangent plane, that is 0 times that is f x at the point times x minus x 1; that means, x minus 0 plus 0 times y minus 0 plus 2 times z minus 1 equal to 0; that means, z is equal to 1 this is the tangent plane, so it matches with our intuition.

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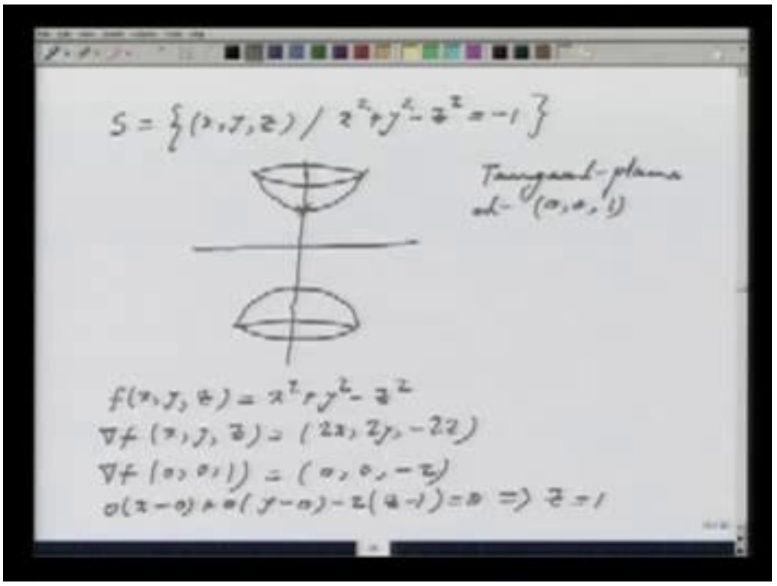

Now, let us look at another example, let us look at this surface, so S equal to all x y z such that x square plus y square minus z square equal to minus 1. So, what is the surface then, I can try to draw this, this is actually two sheeted hyperboloid. So, look at the height one and minus 1 and there, so this is my surface. Let me see, whether I can find the tangent plane at the point again. I want to find tangent plane at let us say 0 0 1; that means, at this point again you feel like the tangent plane should be z equals to 1.

So let us see, what is my f x y z now, this is x square plus y square minus z square, so grad f at x y z, then is 2 x 2 y minus 2 z. So, grad f at 0 0 1 that is 0 0 minus 2, and then the equation of the tangent plane at 0 0 1 is 0 into x minus 0 plus 0 into y minus 0 minus 2 into z minus 1 equal to 0. This implies again the plane is z equal to 1, what happens to the tangent plane, at 0 0 minus 1.

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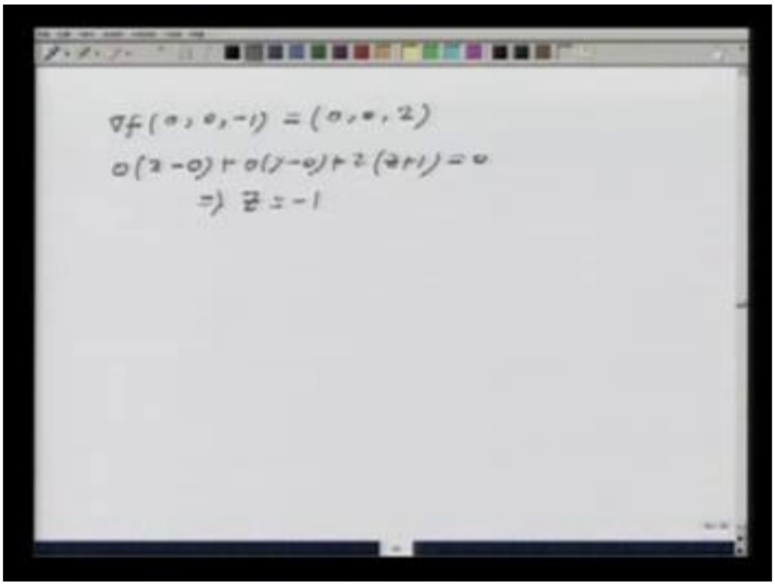

So, I have to calculate then grad f at 0 0 minus 1. That means, I get 0 0 2, then the equation of the tangent plane is 0 x minus 0 plus 0 y minus 0 plus 2. Now, it is z plus 1 because my point is 0 0 minus 1 equal to 0.

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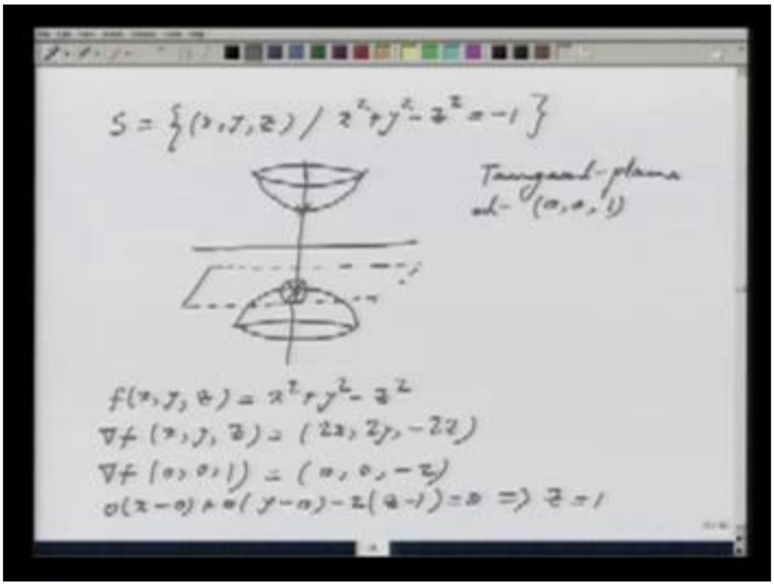

This implies the tangent plane is z equal to minus 1 look back at the picture. The tangent plane at this point is certainly. It looks like the tangent plane should be parallel to the x y x y plane at the height z equal to minus 1.

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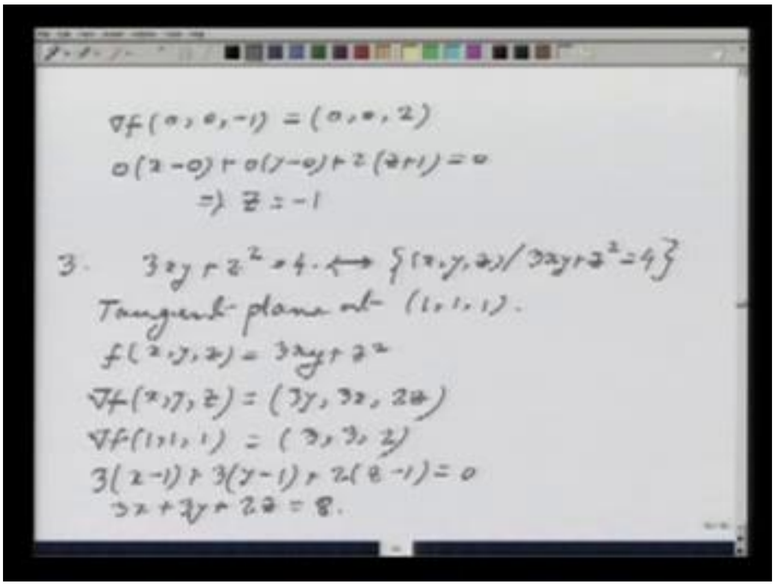

This is precisely, what is coming through the calculation also. Let us look at one last example I look at this surface 3 x y plus z square equal to 4. I want to calculate the tangent plane at the point 1 1 1. What is the idea, I look at the function first, what is f x y z, this is 3 x y plus z square. So, my surface is actually all x y z such that 3 x y plus z square equal to 4. So, if I calculate it grad f at x y z that is 3 y 3 x 2 z, so at the point 1 11, what I get is 3 3 2. So, the equation of the tangent plane is then 3 into x minus 1 plus 3 into y minus 1 plus 2 into z minus 1 equal to 0, that is 3 x plus 3 y plus twice z, that is equals to 8, so far so good for tangent planes.

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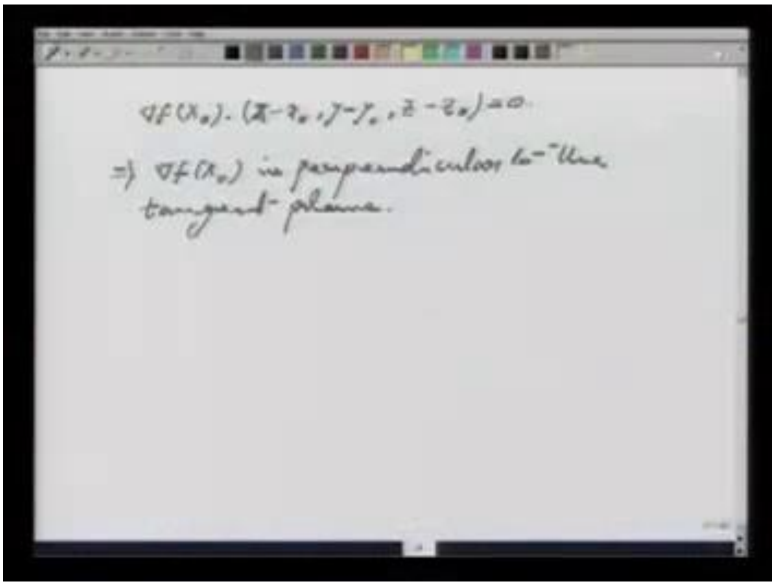

But what is the geometric interpretation of grad f, then well I again look back at the equation of the tangent plane. So, it is grad f at x naught times x minus x naught y minus y naught z minus z naught this is equal to 0. Now, x naught y naught z naught is a point on the tangent plane and so is x y z, but grad of f is equal to 0. That means, grad f at x naught is a vector which is orthogonal to the tangent plane. This implies grad f at x naught I would say is perpendicular to the tangent plane, because if we know that if dot product of two vectors is 0. Then the vectors are perpendicular to each other I am using the same technique here. That if I take any vector in the tangent plane, then the vector grad f is perpendicular to that. That means, the vector grad f x naught is actually perpendicular to the tangent plane.

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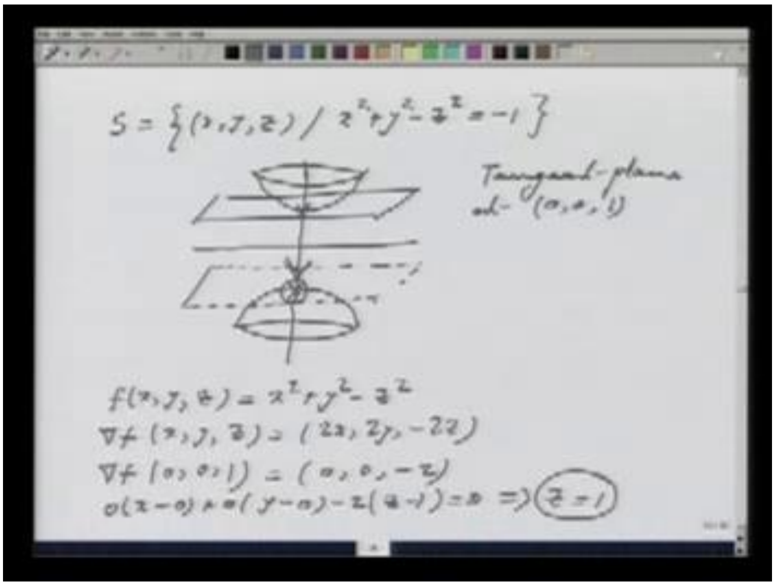

Let us look back at the previous example again which I have computed here, what was that tangent plane in the first case; I got z equal to 1.

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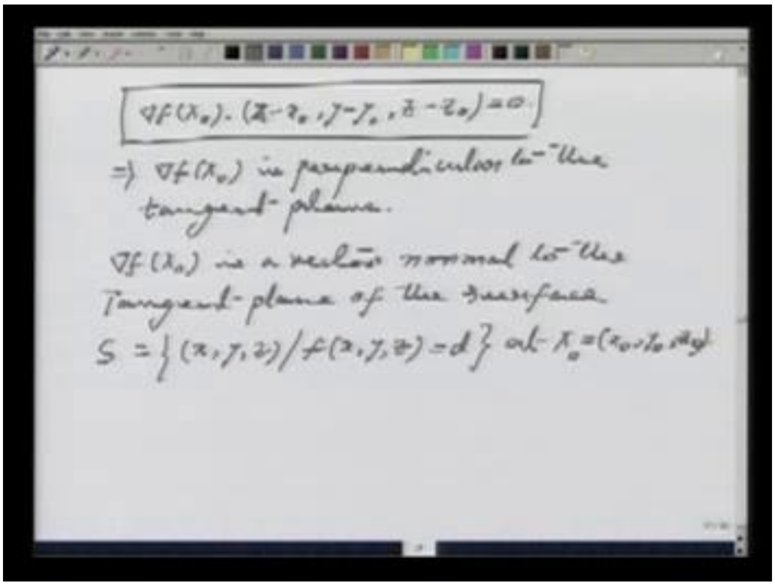

Then what was grad f at 0 0 1, that was 0 0 minus 2, so the direction was this it was going in this direction. And, the tangent plane was this see so grad f is actually perpendicular to the tangent plane which is apparent form this equation also. That the dot product of the vector grad f with the vector x minus X naught y minus y naught z minus z naught is 0. So, it is perpendicular to the tangent plane that is, why grad f x naught at the point, x naught is called the normal to the tangent plane at x naught.

So, what we got the geometric interpretation of grad f grad f at x naught is a vector which is normal to the tangent plane of the surface S surface is all x y z. Such that f of x y z equal to d at the point x naught given by x naught y naught z naught, so this is the geometric interpretation of the gradient, that it just produces the normal to the tangent plane of the surface at the point x naught y naught z naught. In the next lecture, we will move towards the mean value theorem for functions of several variables. And look at the consequences of mean value theorem in these cases.