Mathematics - I
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Lecture - 26
Maxima-Minima

In today's lecture, we are going to continue with Maxima, Minima again. The aim is, to get an analog of the second derivative test.

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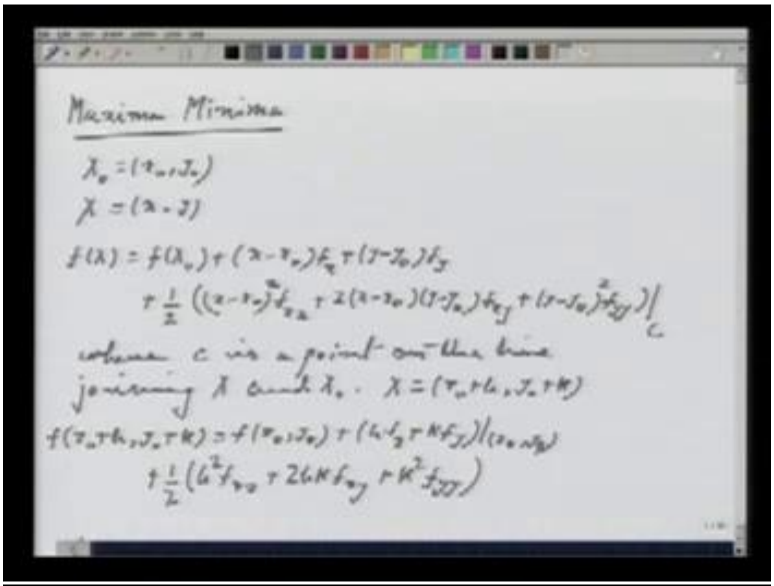

So, we start with the extended mean value theorem. What, we have proved in the last lecture. It says that, if x naught is the point x naught y naught, f is a differentiable function. And let us say X is given by x, y. Then I have that f of x equal to f of x naught plus x minus x naught times f x plus y minus y naught f y plus, the extra terms are half x minus x naught square f x x plus 2 times x minus x naught into y minus y naught f x y plus y minus y naught square f y y. This is at a point c, where c is a point on the line, joining x and x naught.

So, this was the extended mean value theorem. The way we have written it now, which is useful for us, that let us say x is X naught plus h y 0 plus k, then f of x naught plus h y naught plus k. That is equal to f of x naught y naught plus h f x plus k f y. At the point, x naught y naught plus half h square f x x plus twice h k f x y plus k square f y y. This we have proved in the last lecture.

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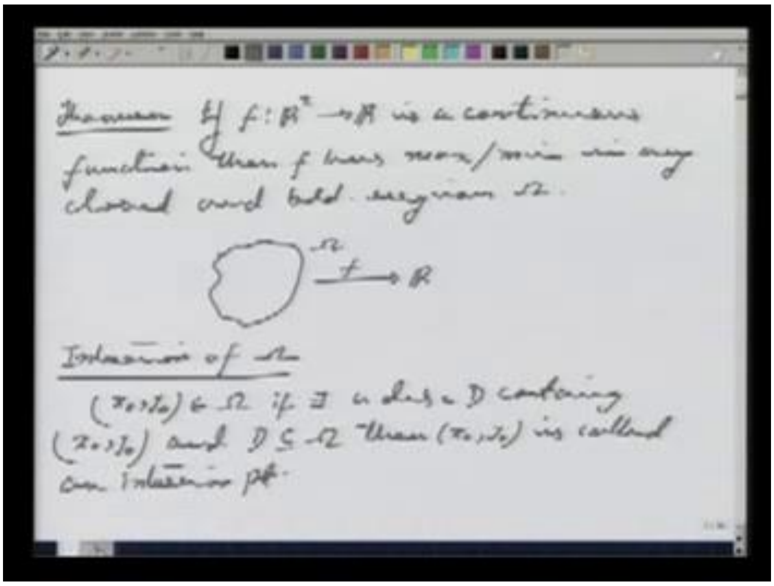

Now, about existence of maxima minima, we are going to use a theorem, without any proof. But, this is analogous to the one variable case. That, if you have a continuous function on a closed interval, closed and bounded interval. Then, the function achieves it is supremum and infimum on that interval. That is, the maximum and minimum of the function exist in that interval.

So, there exists an analogous theorem here. That suppose, if f from R 2 to R is a continuous function. Then, f has maximum and minimum in any closed and bounded region omega. So, you take any closed and bounded region omega, bounded means, it can be put inside a disc. And suppose, f is a continuous function from here to R, then maximum and minimum of f has to be there in omega.

Now, our aim is to get hold of maximum and minimum, if they occur in the interior of omega. So, for that I need the definition of interior of omega. What do you mean by interior of omega? It means suppose x naught y naught is a pointing omega. If there exist a disc D containing x naught, y naught and D is a subset of omega. Then, x naught, y naught is called an interior point.

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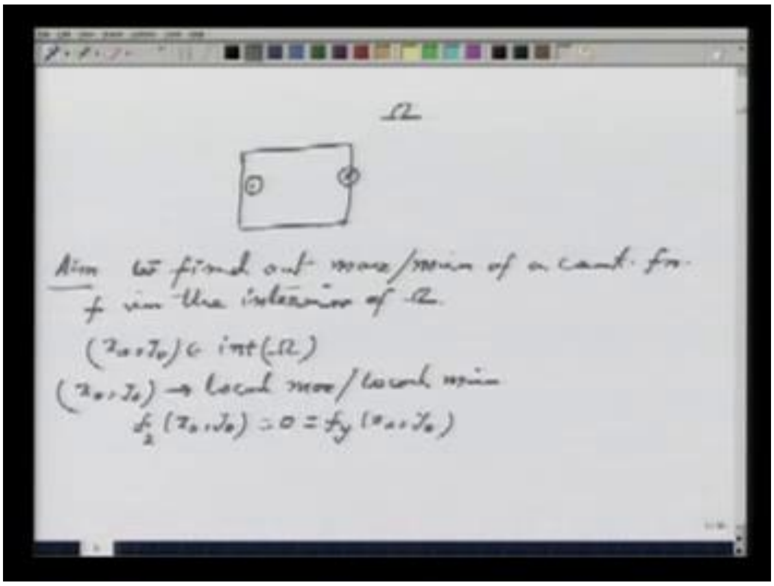

So, for example, suppose this is my omega the boundary points included. Then, if I take any point here inside, I can put a small disc around that point which is inside the square. But, if I take a point here on the boundary, then whatever disc I choose it goes out of omega. So, the points on the boundary of this square are not interior points, but the points on the square, which are not boundary points they are interior points.

So, our view point is that, when any point in the interior is a maximum or minimum of a function that I want to find out. So, our aim is to find out local maximum or local minimum of a continuous function f in the interior of omega. So, we start with the calculation. So, let us say x naught, y naught belongs to interior of omega. Then, if x naught y, naught is an extreme value. That is either it is a local maximum or a local minimum, then the partial derivatives has to vanish. This would imply that f x at x naught, y naught, this is 0.

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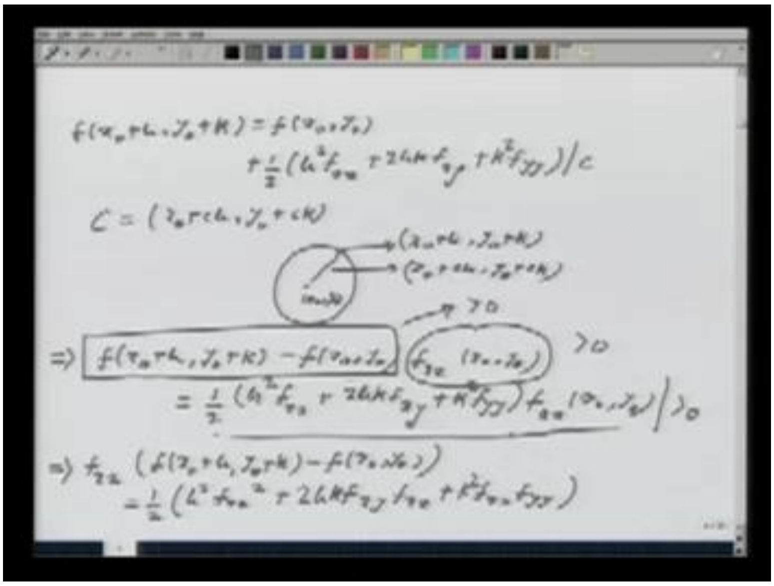

Now, I apply the extended mean value theorem. That would give me, that f of x naught plus h y naught plus k. That is, equal to f of x naught, y naught plus half h square f x x plus twice h k f x y plus k square f y y at a point c, where c is the point. So, if I take this point to be x naught, y naught, suppose this point is x naught y naught. Then, if I look at any points here, they are of the form x naught plus c h y naught plus c k, if this point is x naught plus h y naught plus k.

So, by varying h k and c, I can actually get hold of points in a disc around x naught, y naught, that is the whole point. Notice, in the above expression the f x and f y do not appear, because f x and f y are actually assumed to be 0. Because, x naught, y naught is supposed to be a local extremum, that is either a local maximum or a local minimum. Then, what you do is, that this implies f of x naught plus h, y naught plus k minus f of x naught, y naught. I just write this as half, that means, I am just taking one thing from the other side of the equality, this is h square f x x plus twice h k f x y plus k square f y y. Now, what I do is, I multiply both side of this equality with f x x.

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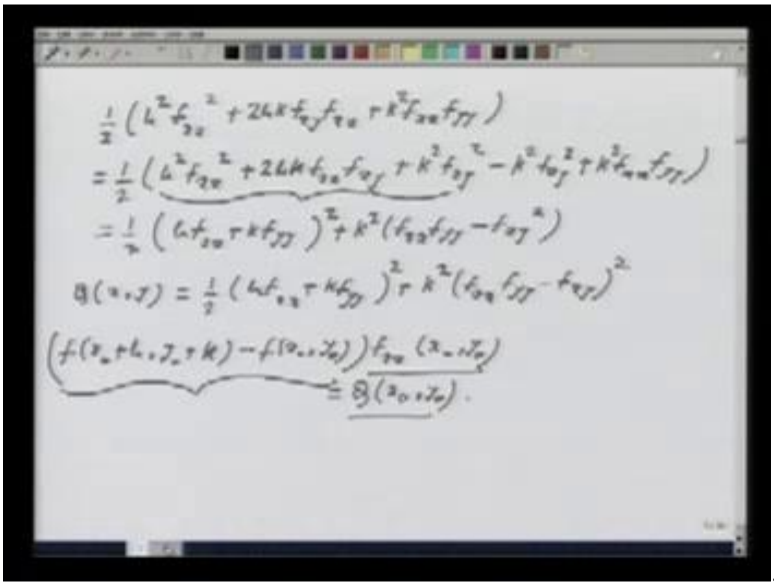

So, now, I am going to concentrate on the right hand side. So, what it is? It is half h square plus k square f x x into f y y. This can be written again as half h square f x x square plus twice h k f x x into f x y plus k square f x y square minus k square f x y square, then plus the remaining term k square f x x times f y y. Now, the first terms together, they form a whole square.

So, I get h f x plus k f y y whole square minus, I would look at write at it at plus k square f x x f y y minus f x y square. So, this is the difference. So, let me denote this as q x y. So, in a nutshell, what I have is f of x naught plus h, y naught plus k minus f of x naught, y naught times f x x at x naught y naught equal to Q. Now, Q being a continuous function, if it is bigger than 0 at some point x naught, y naught or it is less than 0 at some point.

That means, in a disc around that point the function is bigger than 0 or less than 0, just like one variable continuous function. What does it say in one variable, if I have a continuous function f, which is bigger than 0 at a point. Then, there exist an interval around at that point where the function is positive. Similarly, the same logic applies in two dimensions. That is, if I have a function f continuous and positive at a point x naught, y naught. Then, there exist a disc around that point, where the function is positive, the same thing works for negative.

And since, I am assuming that all the partial derivatives, which exist here, are continuous. I will get my function Q also to be continuous. And hence, if it is positive at some point, it is positive in a neighborhood of that point. Now, the whole issue is, that given a point x naught, y naught, whether the function is a local maximum or a local minimum.

Now, notice suppose x naught, y naught is a local maximum. That means, f x naught plus h y naught plus k must be lesser equals to f of x naught, y naught. Because, f of x naught, y naught is local maximum. Now, that will be reflected on the sign of the function Q and the sign on the function f of x x. Because, f of x, x is multiplied here and this function I have here. So, under what conditions using signs of f x s and Q, I can impose certain sign condition on this quotient. That is the whole issue now. So, let me attend that problem now.

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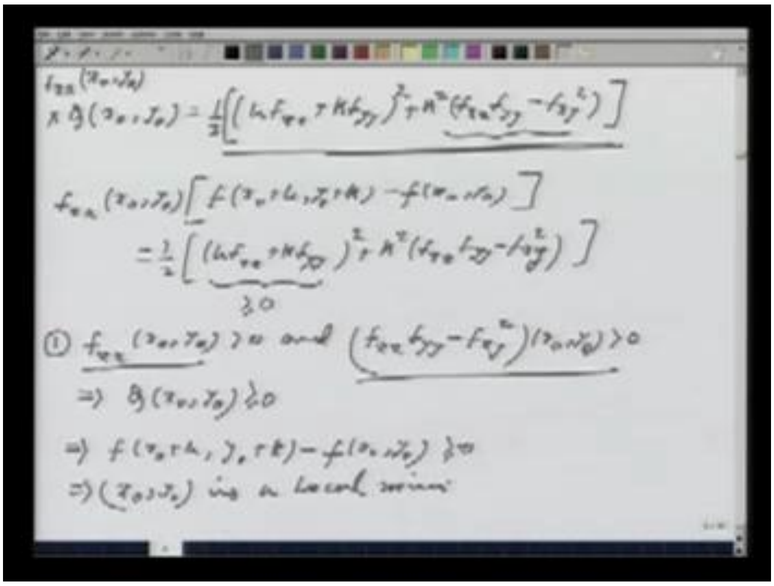

So, I consider q x naught, y naught. This is half h f x x plus k f y y square plus k square into f x x. So, and I also have that 2 times, then f x x, well this expression, which I have written here is it is multiplied with f x x at x naught, y naught. So, f x x at x naught, y naught times q x naught, y naught is actually this quantity on the right hand side. Now, because of this, now I have that f x x at x naught, y naught times.

Now, notice that, this quantity here is always bigger than or equal to 0. Because, it is a perfect square, so there are several cases, now let me look at that. Suppose, f x x at x naught y naught is bigger than 0 and f x x f y y minus f x y square, sorry, I do not have square here. What I have is and suppose this quantity is at x naught y naught, strictly bigger than 0.

Then of course, in a neighbor hood of x naught, y naught both the functions are bigger than 0. So, this would then imply that q of x naught y, naught is bigger than or equal to 0. Because, q of x naught y naught, if I look at, that gives me just this whole square quantity plus k square times the object. So, this whole quantity then is bigger than or equal to 0. That would then imply, that f of x naught plus h y naught plus k minus f of x naught, y naught, this is bigger than or equal to 0.

Because, let me look at this expression, which I had here. That this quantity was this, that we have shown. Then, this was multiplied with f x x times x naught, y naught, this was then multiplied with f x x at x naught y naught. Now, this multiplication on the right hand side produced a whole square plus some term. The term, which I got exactly, I have written it down; it is this term, after multiplication.

So, when this quantity, if this quantity is bigger than 0, then the whole thing is strictly bigger than 0, fine. Now, we go back again; that means, this whole right hand side here, what I got, this is strictly bigger than 0. Now, I am also assuming that this is strictly bigger than 0. That would certainly imply, that the quantity I have here, this portion must be strictly bigger than 0.

That means what? That means, f of x naught plus h y naught plus k is always bigger than or equal to f of x naught, y naught, in a neighborhood of x naught, y naught; that means in a disc around x naught, y naught. That means, f of x naught y, naught must be a local minimum. So, the conclusion then is that x naught y naught is a local minimum.

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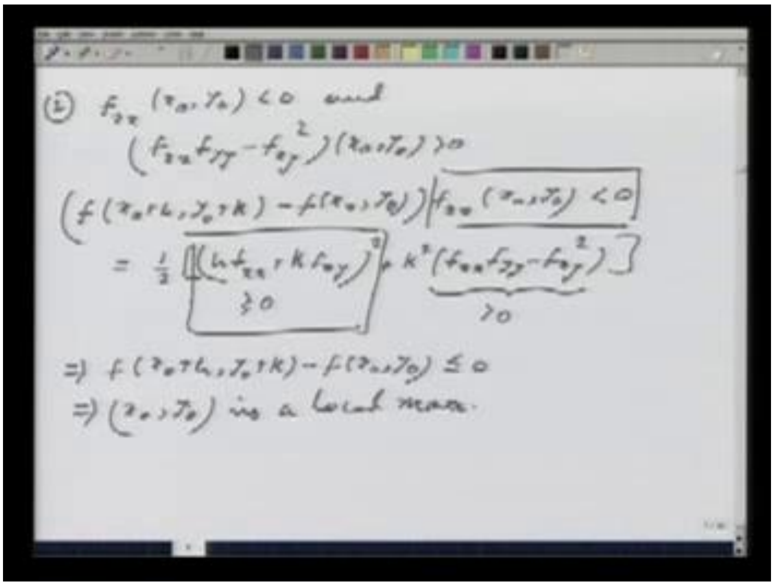

Then, the next case, assume now, that f of x x at x naught y naught is less than 0. And f x x f y y minus f x y square at x naught, y naught is strictly bigger than 0. So, I again write down, that f of x naught plus h, y naught plus k minus f of x naught, y naught, this times f x x at x naught, y naught. I know what this quantity is? This is half h f x x plus k f x y square plus k square times f x x f y y minus f x y square.

Now, I am assuming that, this is positive, but I am assuming that, this is negative, this quantity. Putting together, since this is always bigger than or equal to 0, this quantity. Putting together these three facts, what I get is? That this implies f of x naught plus h y naught plus k minus f of x naught, y naught, this is less than or equal to 0. This implies then that x naught, y naught is local maximum.

Because, x naught plus h, y naught plus k represents points in a disc around x naught, y naught. And from my conditions, I can see, that in all points in a disc, the value of the function is less than or equal to the value of the function at x naught, y naught. That means, the value of the function at x naught, y naught is larger than any other value in the disc. That means, it is a local maximum.

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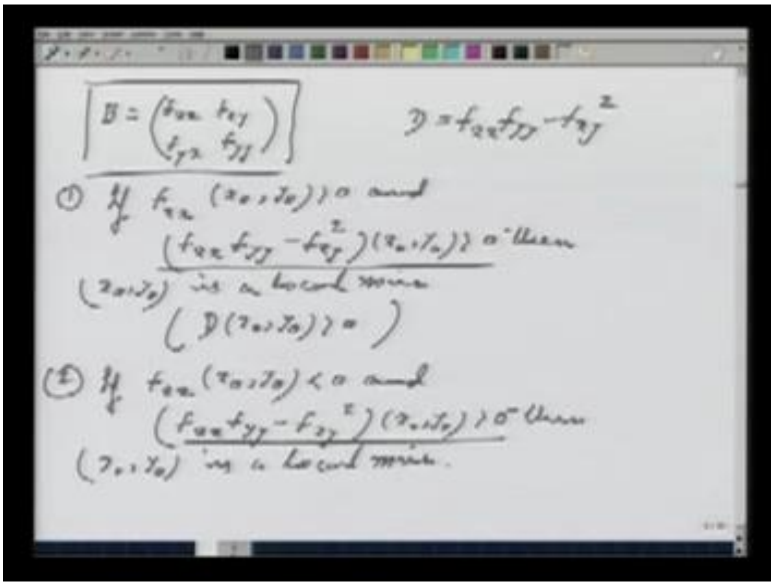

So, let us summarize what we have got so far. So, for that, I need a notation. I define; let us say b equal to this matrix of double partial derivatives. Now, then my condition one was, let us have a look at it again, the condition one was f x x is bigger than 0 and this quantity is bigger than 0. So, if f x x at x naught, y naught is bigger than 0 and f x x f y y minus f x y square at x naught, y naught is bigger than 0. Then, x naught y naught is a local minimum.

So, just look at this formal matrix, which I have written. If I formally calculates it determinant, which I would call D, that is precisely f x x f y y minus f x y square. So, all we are saying is, that D at x naught, y naught is bigger than 0. So, it can also be written as D at x naught, y naught is bigger than 0. Then, I got the second condition, that if f x x at x naught, y naught is less than 0 and f x x f y y minus f x y square at x naught, y naught is bigger than 0. Then, x naught, y naught is a local minimum.

Now, problem starts, when D is equal to 0 or D is less than 0. Now, in those cases, we are not going to prove anything. I will just let you know, what exactly happens in those cases, there is not much to say about those cases. And then we will go to some examples. To see how do one apply this results to characterize maximums and minimums.

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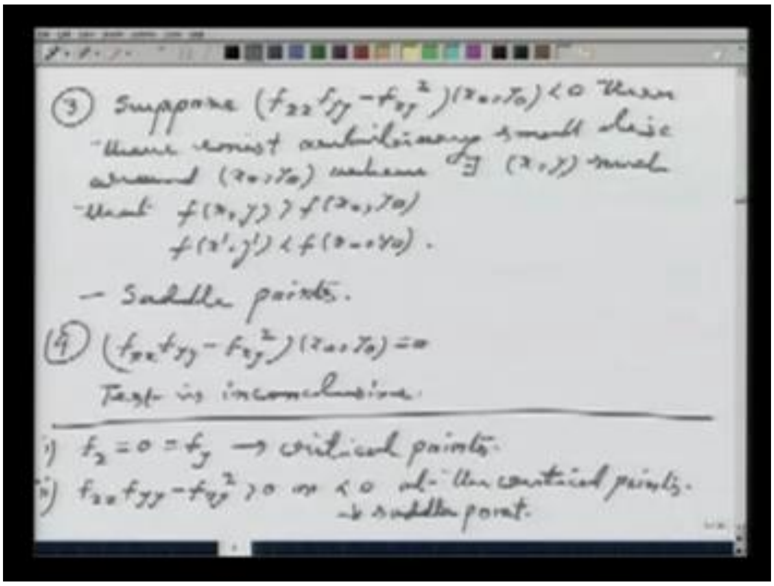

Now, the third case, I am not going to prove anything here. That suppose, f x x f y y minus f x y square at a point x naught, y naught is less than 0. Notice, in the previous case, everywhere, I have looked at this quantities is bigger than 0, this quantities bigger than 0, I never look at the case less than 0, which I am going to do now. Suppose, it is less than 0, then it turns out that x naught, y naught, this point is neither a maximum nor a minimum.

What happens is, then there exist arbitrary small disc, around x naught, y naught. Where, there exist x, y, such that f x, y is bigger than f of x naught, y naught. And f of x prime y prime that is some other point will remain in that neighborhood, where this happens. That means, when D is less than 0, one can actually show, that there exist arbitrarily small disc around those points. Such that in those discs, there are points x y and x prime y prime. Such that, f at x y is bigger than f of x naught, y naught and f of x prime y prime is less than f of x naught, y naught.

That means, this point x naught, y naught is neither a local maximum nor a local minimum. We will see examples of such kinds. Now, these kinds of points are called saddle points. So, saddle points are points for which of course, f x and f y turn out to be 0. But, when I look at f x x f y y minus f x y square times x naught, y naught at x naught, y naught, that turns out to be negative. These are called saddle points. So, the quantity D, which I had written down at x naught, y naught that turns out to be negative, those are saddle points.

Fourth is what happens, if D is equal to 0. We will see in examples, that this means the test is inconclusive. That is, if this quantity turns out to be equal 0, we need just cannot say anything. So, what are the favorable cases then? That you look at first, so what is the strategy of proving getting maximum, minimum is simple. First, we look at the partial derivatives.

So, let me explain it here, what is the strategy. First, look at all x, y such that f x equal to 0, equal to f y, that is step 1. Collect the 0's, what you get are called the critical points. After getting those points, there are two things you need to do. First look at this quantity f x x f y y minus f x y square, see whether it is bigger than 0 or less than 0 at the critical points.

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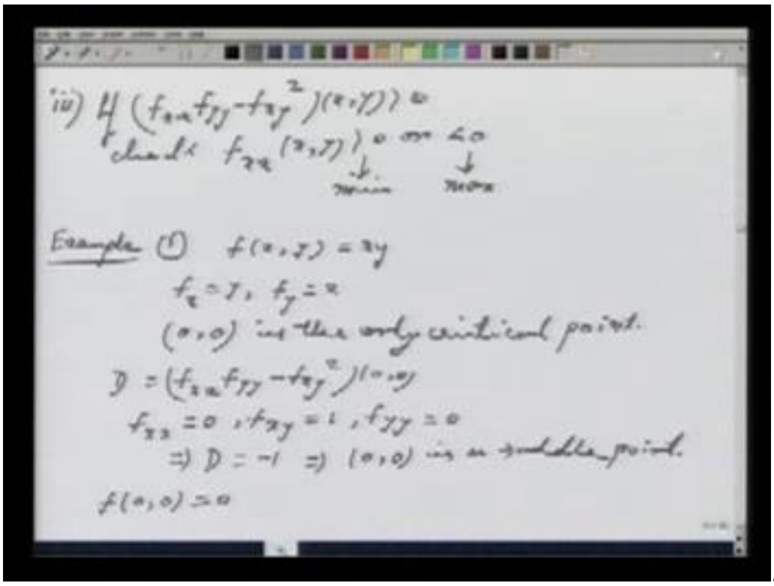

Third step is, if f x x f y y minus f x y square at x y. That means, at the critical points, if it is bigger than 0 check what about f x x bigger than 0 or less than 0. Depending on that, I will get, if it is bigger than 0, then I get minimum, if it is less than 0, I get maximum. While, checking D, if it is bigger than 0, then I will proceed, if it is less than 0, then saddle point.

If D turns out to be equal to 0, then none of these things will help you. You have to look at the function and try to see, whether you can find out maximum and minimum. The double derivative test essentially fails, if D is actually equal to 0. So, now slowly, let us turn on to some examples, where these methods will be described to you.

Let us look at our first example. Suppose, f x, y equal to x y. So, first I have to search for critical points. So, f x is y, f y is x. So, what are the points, where both f x and f y are 0. The only solution is the point 0 0, so 0 0 is the only critical point. Then, I have to search for D capital D, that is f x x f y y minus f x y square at the point 0 0. So, I need to calculate, what is f x x, f x x turns out to be 0. Because, f x is a function of y only.

What is f x y, that also turns out to be 0, no, f x y turns out to be 1 and what is f y y, because, f y is a function of x only. So, f y y is 0, this implies D is equal to minus 1. So, this implies in this case 0 0 is a saddle point. Can I see that, it is a saddle point, that is in arbitrary small neighborhoods of 0 0. The function takes value bigger than f 0, 0 and less than f 0, 0. Now, what is f 0 0? That is 0.

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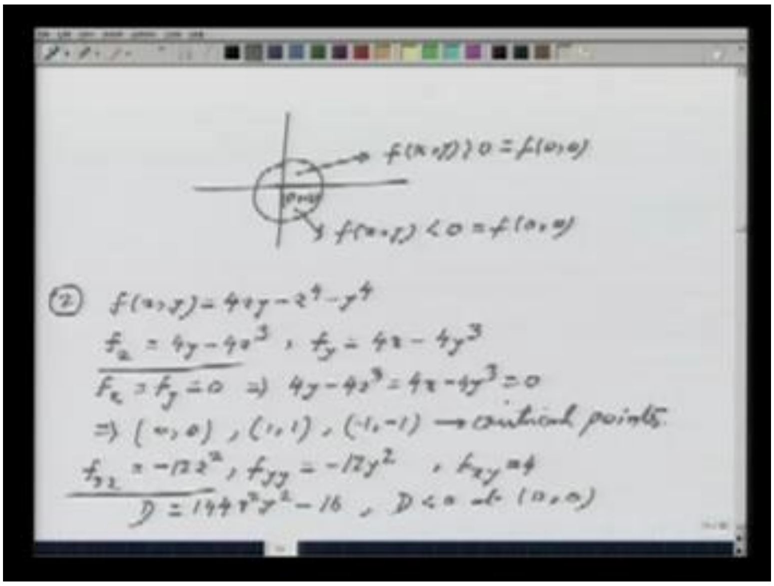

So, in two dimensions, this is the point 0 0, I take any disc around this. If I take a point here, then x and y both are positive, so f x y is bigger than 0. If I take a point here, then x is positive, but y is negative, so f x y is less than 0. But, this 0 is actually f of 0 0. So, you see that around any small disc about 0 0, whatever disc I choose, I can get x y, where the value of the function at x, y is bigger than f 0 0. And points, where the value of the function at 0 0 is at x y is less than f of 0 0. So, this justifies, why this is a saddle point.

Now, let us look at our second example. Now, let us look at this function f of x, y equal to 4 x y minus x 4 minus y 4. I want to check for maximum minimum of this function. So, I start again with partial derivatives. What is f x, it is 4 y minus 4 x cube. What is f y, that is 4 x minus 4 y cube. So, now I have to solve f x equal to f y equal to 0, this implies 4 y minus 4 x cube equal to 4 x minus 4 y cube. From this, it implies that the only critical points you can get at 0 0, 1 1 and minus 1, minus 1, these are the critical points, this is easy to check.

Now, once I got hold of the critical points, I have to go to the next partial derivatives. That means, now I have to calculate, what is f x x, well f x x turns out to be minus 12 x square. What is f y y? That is minus 12 y square. Then, what is D, that is f x x into f y y. So, it is 144 x square y square minus. Now, I need what is f x y, well that is easy to calculate f x y is 4. So, it is minus f x y square; that means, I get 16. See, f x is 4 y minus 4 x cube. So, what is f x y, I have to differentiate it with respect to y, what I get is 4. But, in D, what appears is f x y square, so I get 16. Now, I will check, now at 0 0, so this d is certainly less than 0 at 0 0.

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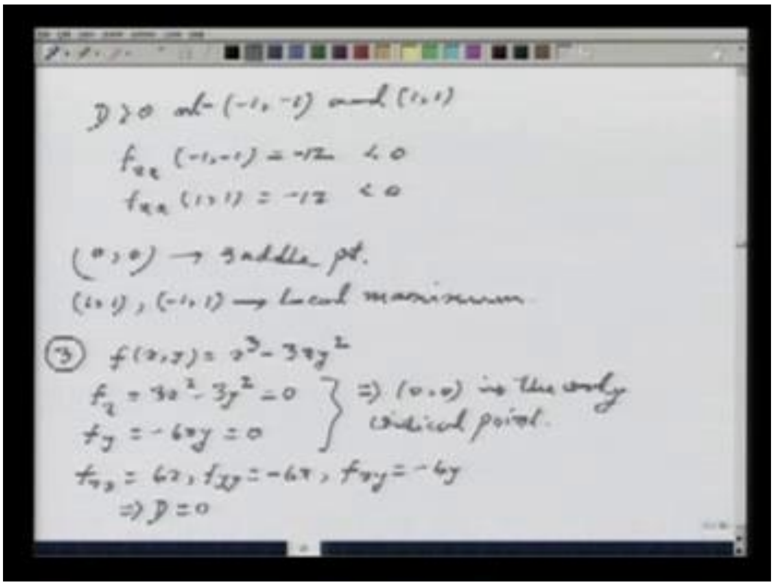

So, this implies the point 0 0 is a saddle point. Now, what happens at 1 1 and minus 1 1? Now, if I put x and y both to be equal to 1, then D is positive. So, D is bigger than 0, actually at 1 1 and minus 1 1 both. Since, in the expression of D I have a square in x and square in y, it does not matter whether I put plus 1 or minus 1. So, in any case I get D to be positive. Now, what remains to check is, what happens to f of x x at 1 or minus 1.

Now, you look at the expression of f of x x, there is no y involved only x and that is square. So, if I put x equal to 1 or minus 1, it does not matter. Whatever, you put you get that f x x is equal to minus 12 and this is also minus 12, both are less than 0. So, D is positive, but these quantities are less than 0. So, finally, what is my result, the result is 0 0, this is a saddle point, because D there is negative and 1 1 and minus 1, 1 these are local maximum fine.

So, this is the method is slowly getting clear. What, exactly you need to do, you first look at the partial derivatives, find the critical points. At the critical points, you check the D, the sign of D; first if it is negative, then you already know that, it is a saddle point. But, if it is positive, then you have to check f x x and try to see what happens there. Now, let us look at our third example, the function here is f x y equal to x cube minus 3 x y square.

So, again we start with f x, f x is 3 x square minus 3 y square, which is equal to 0, I have to see and then f y, that is minus 6 x y that is equal to 0. See, if I try to solve this, this implies the only critical point, which comes here is 0, 0 is the only critical point. Now, I have to see the behavior of the critical point 0, 0, so for that first I have to start with D. So, I need to know, what is f x x, f x x turns out to be 6 x. What is f y y, that turns out to be minus 6 x and then I have to look at f x y, that is minus 6 y.

Now, if I look at the point 0, 0 then all these quantities are actually equal to 0. So, this implies D; that is equal to 0. So, let us check our calculation again, f x x means what, I have to differentiate the function f x with respect to x, I get 6 x. Then, I look at f y y, so this is the function minus 6 x y, I have to differentiate it with respect to y, I get minus 6 x. Then, I have to look at f x y, so for that what I do is I look at the function f y and differentiate it with respect to x, I get minus 6 y.

So, this implies d is equal to 0, because, f x x into f y y at the point 0, 0 that is 0 minus f x y square, but at the point 0, 0 f x y is 0, so it is square is also 0, so is d is 0. So, the double derivative test cannot be applied. So, question then is, what the behavior of the point 0, 0 is.

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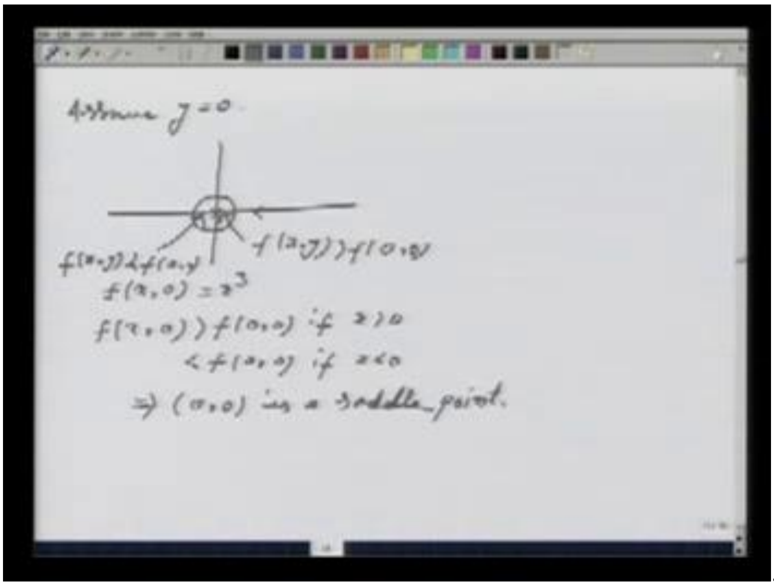

Well, what I do is I just assume y equal to 0. That means, I am looking at the behavior of the function only on the x axis. Because, y I am taking to be 0. But, then what is f x 0, that turns out to be equal to x cube. Then, I can certainly see that f x 0 is bigger than f of 0, 0. If x is bigger than 0 and it is strictly less than f 0, 0, if x is less than 0. That means, if I am concentrating around any disc and any disc around 0, 0 and then the points here gives me f of x y bigger than f of 0, 0.

But, if I am looking at points here on the left hand side, then f of x y is less than x 0, 0. What is the conclusion then it means 0, 0 is actually a saddle point. Because, in every neighborhood or a disc around 0, 0 there are points x, y where the value of the function is bigger than f 0, 0 and there are points, where the value of the function is less than f 0, 0. So, this implies 0, 0 is a saddle point, notice that this conclusion, we have drawn not by the second derivative test, by analyzing the given function.

Since, the function was of a particular form, the function is of this form I have exploited the form of the function to achieve that 0, 0 is a saddle point. The conclusion did not follow from the second derivative test because, second derivative test. If you want to apply then at the critical points D must be bigger than 0 or D must be less than 0, D bigger than 0 might give you maximum and minimum D less than 0 will give you saddle point.

But, if D equals to 0, then the second derivative test does not show anything. So, we have to use some trick, there when D is equal to 0 to check, what the nature of the point is. In this case, in particular we have gone through a particular axis and check that 0, 0 is a saddle point.

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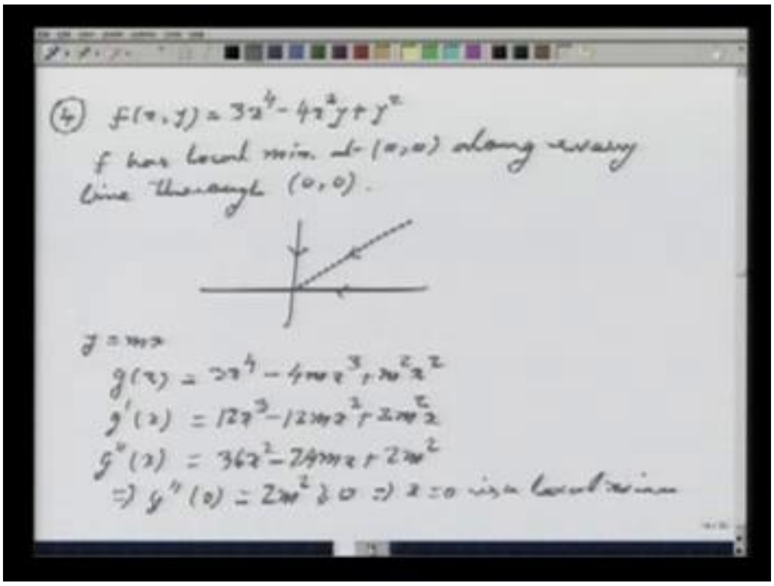

Let us go to our next example, the last one I look at the function f x, y equal to 3 x to the power 4 minus 4 x square y plus y square. The first thing I want to show is that f has local minimum at 0, 0 along every line through 0, 0 this is a line, this is a line; this is a line, so I am going to check. So, what I do is first I go through lines y equal to m x, the only line which I have here which is not of this form is the y axis. So, for that one needs to do separately the checking that 0, 0 is maximum that is you just put y equal to 0 and then check what happens.

Well, you look at the function, if you put y to be equal to 0 the function you get is f x y equal to 3 x to the power 4, which has certainly minimum at 0, 0. So, it is the other lines which I am looking at, so if I put y equal to m x, then my function if I call it g x that is 3 x to the power 4 minus 4 m x cube I am putting y equal to m x plus m square x square, to check that 0 is the minimum. Now, I can use the one variable theory which I have at my disposal.

So, g prime x I look at that is 12 x cube minus 12 m x square plus twice m square x, so this certainly means that 0 is a critical point, because I can take x common. So, then I look at g double prime x, that is 36 x square minus 24 m x plus twice m square, so this implies that g double prime at 0 is twice m square, which is bigger than or equal to 0. This implies x equal to 0 is a local minimum and hence 0, 0 is a minimum of the function along the line y equal to m x. As we have already noticed that if I go through y axis then; obviously, 0, 0 is a local minimum. Because, my function is thrice x to the power 4, the minimum value is always at 0, 0.

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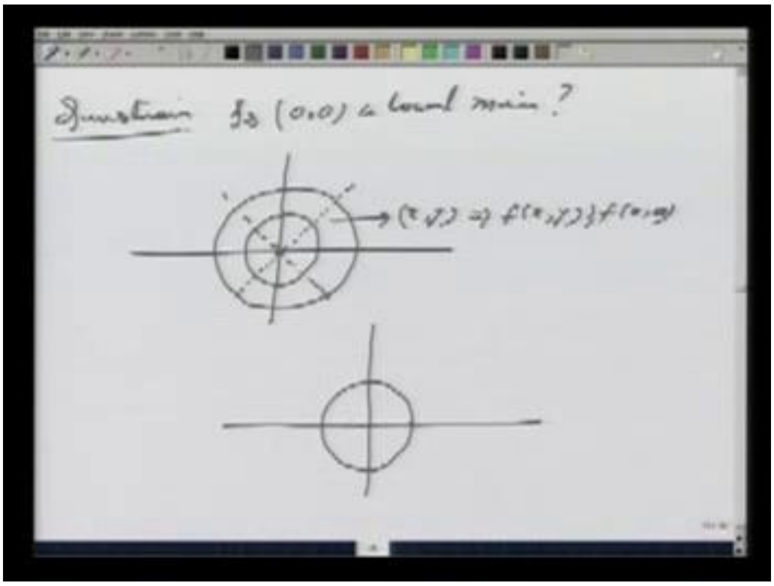

But, question now is 0, 0 a local minimum, what is the point of asking this question well the point is this, let me draw a bigger picture. Suppose I look at a disc around 0, 0 then I look at all the points in this dotted line, then I have checked that f of 0, 0. So, if x y is here this implies f of x, y is bigger than or equal to f of 0, 0. If I take this dotted line, then I might get another disc, where f of 0, 0 is less than all the points on this line the second dotted line.

But, the question is 0, 0 a local minimum; that means, can I say that there exist a disc around 0, 0 such that for every point in this disc f of x, y is bigger than or equal to f of 0, 0. So, do you understand the question here because, when I look at a line then 0, 0 is a local minimum means there exist an interval on which f of 0, 0 is the minimum value. but depending on the line the interval changes. But, here what I am asking is that can you find a disc such that in that disc for all the points x y f of x y is bigger than f 0, 0.

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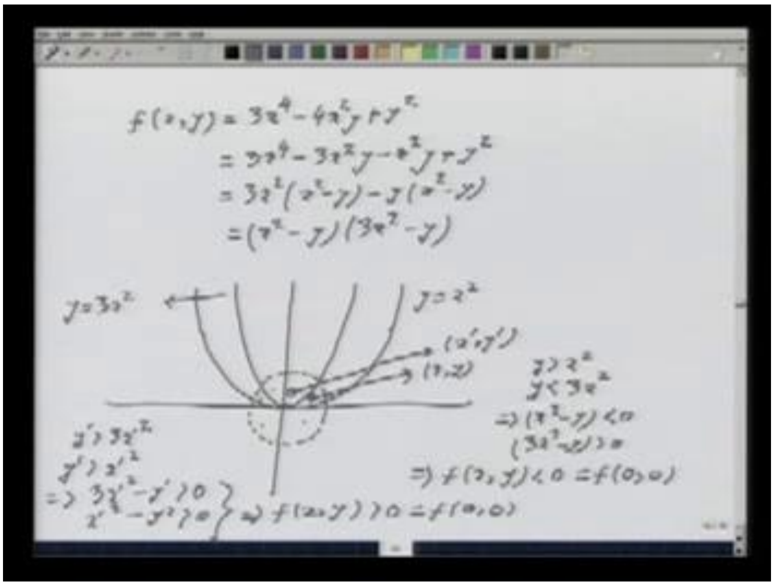

Well, we will see that the answer is not true that 0, 0 is not a local minimum. What I do is, I write down the expression of f x y again. I know the expression it is 3 x to the power 4 minus 4 x square y plus y square I write it in the form 3 x to the power 4 minus 3 x square y minus x square y plus y square, then I take 3 x square common I get x square minus y I take minus y common I get x square minus y. So, the function is x square minus y into 3 x square minus y.

Now, again let me draw a picture, what I do is, I draw the curve y equal to x square. So, when I draw the curve y equal to x square it looks like this, this is the curve y equal to x square. Now, let me draw the curve y equal to 3 x square that will look like this, this is y equal to 3 x square, now suppose I am searching for a disc, so let me draw a disc, a disc around 0, 0 looks like this. Now, in all this disc does not matter whatever radius you take this regions 1, 2, 3, 4, 5, 6 these six regions are always there in the disc.

Now, suppose I choose a point x here, suppose x y belongs here then I know that y is bigger than x square and y is less than 3 x square. This then would imply that x square minus y is less than 0 and 3 x square minus y is bigger than 0, this would imply then that f of x, y which is product of this one is positive other is negative is less than 0. But, notice this 0 is actually f at 0, 0. So, I get a point in the disc where f of x, y is less than f 0, 0.

But, what happens if I choose a point here suppose this is x prime y prime, so for that point what I have is that y prime is bigger than 3 x prime square and y prime is bigger than x prime square. This implies 3 x prime square minus y prime is bigger than 0 and x prime square minus y prime this is bigger than 0, this together would imply that f of x, y. Now, is positive bigger than 0, but that is f at 0, 0.

That means, what; that means, whatever disc I choose I get another point x prime y prime, where f of x prime y prime is bigger than f of 0, 0. That means, in arbitrary small disc around 0, 0 I get points x y such that f of x y is bigger than f 0, 0 and f of x y is less than f 0, 0, what does it mean, it means 0, 0 is a saddle point. So, notice something nice has happened here, that the point 0, 0 along every line then I can think of f as a function of one variable in that case 0, 0 is a local minimum for every line.

But, in the sense of two dimensions when instead of intervals you think about discs in that sense 0, 0 is not a local minimum. It is a saddle point this is happening simply because, instead of intervals now by neighborhood I mean disc. So, you are demanding more and that is why this problem is happening, so we will stop at this, this is the way one should go about finding the maximum minimum using the test of double derivative. The thing to remember here is that the most fundamental thing is first checking the critical points, which you get by equating the partial derivatives to 0.

Now, whether these critical points give you maximum or minimum depends on whether your D is positive. If D is positive then you can start checking whether it is maximum or minimum by going to f x x. But, if D is less than 0 you know certainly it is not maximum or minimum, it is a saddle point and the most problematic case is when d is equal to 0. Then, you have to look at the given nature of the function and you have to do something to see, what the nature of the function is… So, we have seen some examples where we have always exploited the nature of the function to determine what is the nature of the point. So, that is all for maximum and minimum. In the next lecture, we will go to another important problem called the method of Lagrange multipliers.