Mathematics-I
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Lecture - 28
Multiple Integrals

In today's lecture, we are going to talk about change of variable in double and triple integral.

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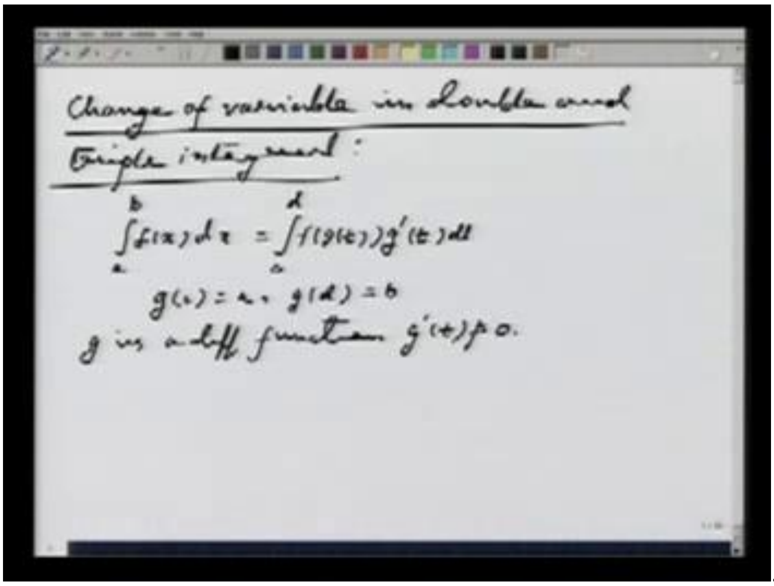

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Well, unlike one variable case to prove the formulas for the change of variable in double integral is conceptually difficult. So, what we would try to do is, we will try to explain, what does change of variable formula mean. And try to give you a neat formula, based on which we can calculate certain integrals and use the change of variable formula. Well, in the one variable case, we know that this formula is true, that in integral a to b, f x, d x is, integral c to d f of g t times, g prime t, d t, where g of c is equals to a and g of d is equal to b. And g is a differentiable function with g prime t not equal to 0. This is the classical formula of change of variable in one dimensional Riemann integration.

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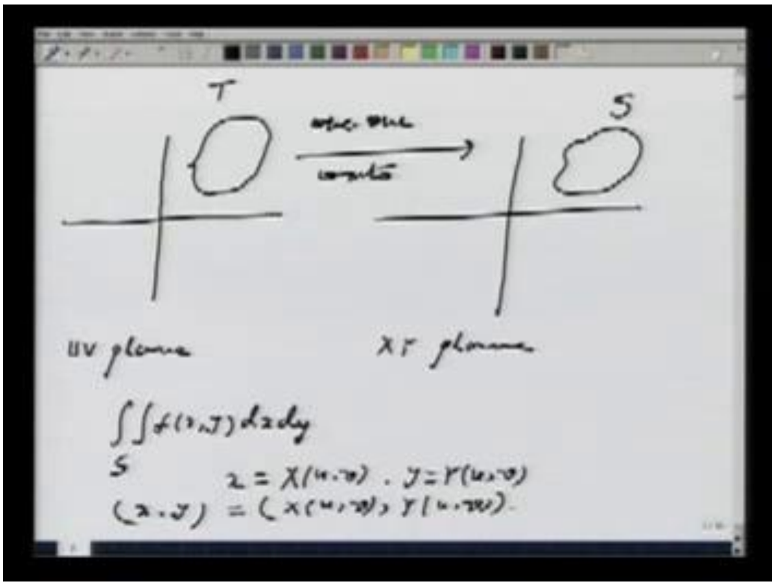

Now, you want to have a generalization of this, for double integral and the triple integral. So, the situation here is something like this. That let us say, this is the U V plane and this is the X Y plane. So, I would write this as X Y plane, it means points of coordinates little x and little y. This is U V plane, here points of coordinates, little u and little v. So, this U V plane is just a replica of the X Y plane.

Now, suppose on the U V plane, I have a domain t and on the X Y plane. Let us say, I have a domain s. And I am interested in this double integral, integral over S, f of x y, d x, d y. Now, suppose I have a 1 to 1 map here, it is 1, 1; that means, it is injective and on to map. Then, I can say, that this x is given by a function capital X including two variables u and v and little y is a function of two variables given by capital Y. And the variables are u and v. So, I can say that x y is capital X of u v. This would be very clear, when you look at examples, capital Y of u v.

Since, I am dealing with two variables. When, I change from one domain to the other. The new coordinates, which are appearing, that is x and y. They are functions of both the variables, which occurred previously. That is little u and v, that is the whole point of writing little x, as a function of u and v and little y, as a function of u and v. Once, I have this, then I look at the following.

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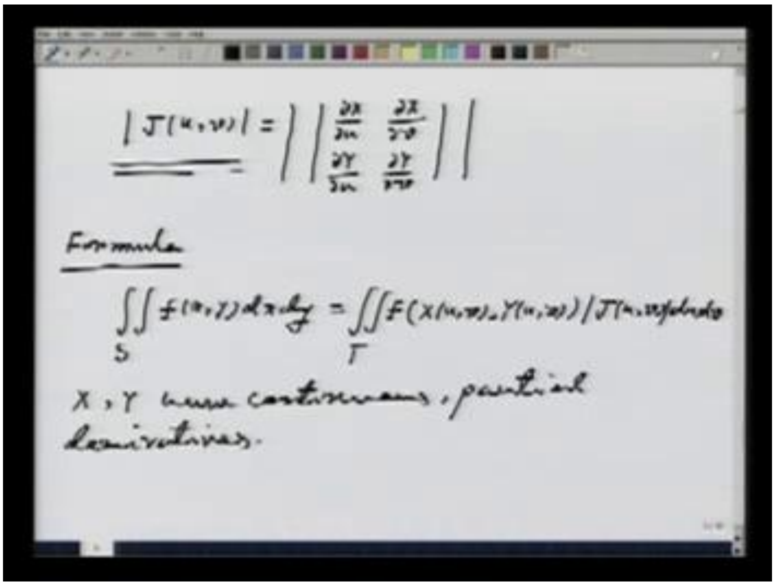

I will denote it by, mod J u v. What is this quantity? It is the modulus of the determinant of the following quantity, del x del u, del x del v, then del y del u, del y del v. That is look at the function capital X, it is function of two variables u and v. I look at it is partial derivatives. So, there are two partial derivatives, one with respect to u; one with respect to v. I put them in the first row of a two by two matrix.

Then, I look at the function capital Y; it is a function of two variables u and v. I look at the partial derivatives; I put them in the second row. Then, I calculate the determinant and then, I take the modulus of that determinant, that is what is denoted by, mod J u v. And then, the change of variable formula is written as follows. The formula is, the double integral over S, f x y, d x, d y, this is double integral over t. That is the domain of the transformation f of x u v, y u v, then mod J u v, then d u, d v.

Notice here, that modulus of J u v, essentially plays the role of this quantity g prime t, which is appearing. This quantity is actually replaced by mod J u v. That is, what is, meant in this formula? Now, we are not going to prove this formula. Let us just assume this, well for these certain things, certain hypothesis I need. That is, that X and Y, they have continuous partial derivatives.

Otherwise, mod J u v will not make any sense. So, for any arbitrary kind of transformation, I cannot work the condition on the transformation is that the partial derivatives exist and they are continuous. Now, based on this formula, let me try to look at example.

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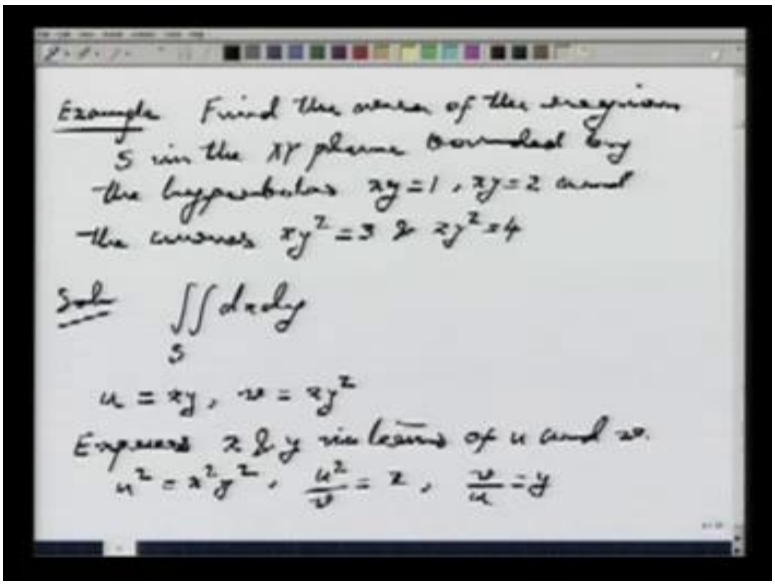

So, as an example, I would say, find the area of the region S in the X Y plane, bounded by the hyperbolas, x y equal to 1, x y equal to 2. And the curves x y square equal to 3 and x y square equal to 4. So, essentially, what I need to calculate is double integral over S, d x, d y. Now, you can see, that given the conditions of these hyperbolas and the curves, it might be complicated to find the limits of x and y, which describes the region S.

So, because of that, what we do is, to make it simpler. We want to pass on to the change of variable formulas. So, what I do is, I define, let us say, u equal to x y and v equal to x y square, suppose I just define this. Now, what I want to do is, express x and y, in terms of u and v, that is very simple. What I do is, I look at u square, which is x square y square. Then, I define, u square by v, what I get is x, from the above formulas. And once, I know this it is very easy to calculate y or straight away, you can just look at v by u, v by u, what you get is y.

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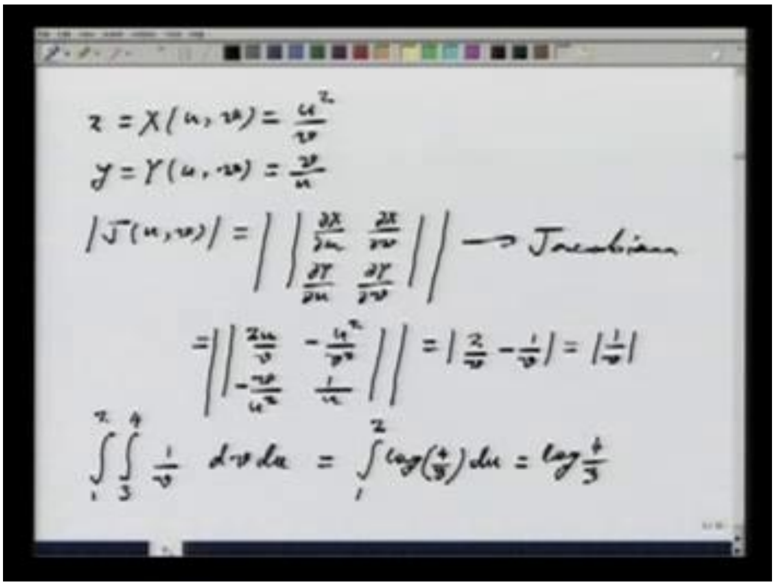

So, here, what is happening is, x equal to capital X of u v. So, what is this capital X of u v, I again look back at the formula, it is u square by v. So, this is u square by v. Similarly y, which is supposed to be capital y of u v, I can see from the previous formula, it is this quantity, so I get v by u. Once, I know this, I can calculate the Jacobian, that is the first thing. So, mod J u v, so this is modulus of the determinant of del x del u, then del x del v, then del y del u, del y del v, so it is modulus of.

Now, I can calculate, what is del x del u? That means, the partial derivative of this, that means, twice u by v, then minus u square by v square. Then, the second function, I could differentiate with respect to u now, so it is minus v by u square. Then, I have to differentiate with respect to v and then I get 1 by u. Then I have a modulus, I can try to calculate this, so what I get is modulus of twice 1 by v. Because, twice u by v into 1 by u, that means, 2 by v, then minus u square cancels, I get 1 by v. So, I get mod 1 by v, so this is the Jacobian.

Now, I look at the integral. I need the limits of u and v; that was the whole point of going to this change of variable. Because, the limit of x and y, which describes the region S turns out to be complicated. So, we were looking at the limits of u and v perhaps. That is simpler and describes the transformed origin. So, we look at the formulas of u and v here. Let us see, the hyperbolas are given by x y equal to 1 to x y equal to 2. But, according to my substitution, x y is equals to u, so u varies from 1 to 2.

Look at v, v is given by x y square and x y square varies from 3 to 4. So, u varies from 1 to 2 and v varies from 3 to 4, d v, d u. Now, what remains, is the function, but the function is the constant function unfortunately. So, I do not have to look at the function, but all I need to, now incorporate is the Jacobian. But, the variation of v is from 3 to 4, so v is always positive. So, modulus of 1 by v, is same as 1 by u. So, I have to integrate this way.

Apply Fubini's theorem, I can first integrate the v variable, that produces log v. That means, log 4 minus log 3. So, it is integral 1 to 2, log of 4 by 3, d u. But, in the integrand, there is no u. So, I take it out of the integral and what I get is integral 1 to v d u, which is 1, so final answer is log of 4 by 3. So, this is how on uses change of variable formula. Now, there are certain standard changes of variables, which you usually do in certain problems. For this, it is always good to know, what the Jacobian of that change is. This quantity mod J of u v, which I am writing here, this is called the Jacobian.

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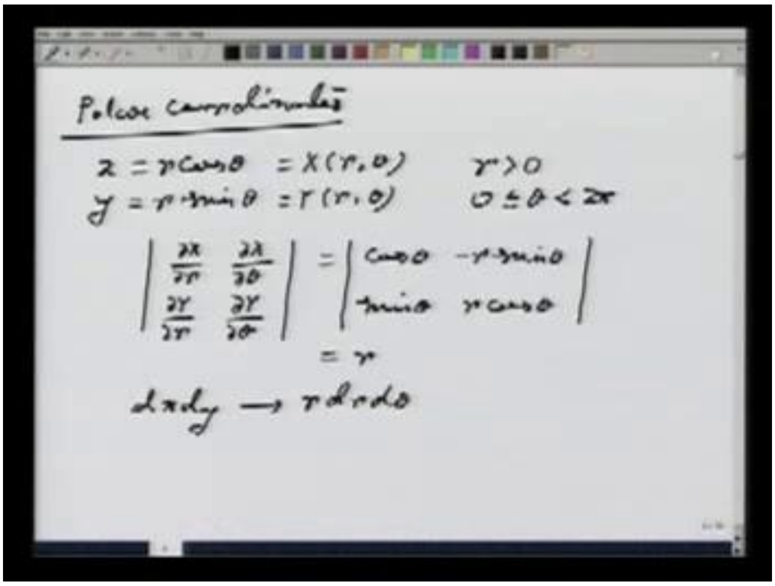

So, one such very useful transformation is changing from Cartesian to polar coordinates. So, we know, what it is, that x is given by r cos theta, y is given by r sine theta. So, that means, it is capital X of r theta, instead of u and v, it is capital Y of r theta. And then, to calculate the Jacobian, what I need to look at the determinant of del x del r, then del x del theta, del y del r, then del y del theta. So, I would write it down here.

So, if I look at del x del r, I get cos theta. Then, when I look at del theta, I get minus r sin theta. Then, in this case, here I get sine theta, because I am differentiating now y, with respect to r. Then, if I do with respect to theta, I get r cos theta. So, the determinant produces, r times cos square theta plus sine square theta. So, it is just r, do not need to take modulus. Here, I always take r bigger than 0 and 0 lesser equal to theta, strictly less 2 pi.

So, the Jacobian determinant under this change of variable d x d y, always changes to r d r d theta. Now, whenever usually, the thumb rule is, whenever you see the integral, where the quantities x square plus y square are appearing. Then it is better to shift things, to polar coordinates, you know…

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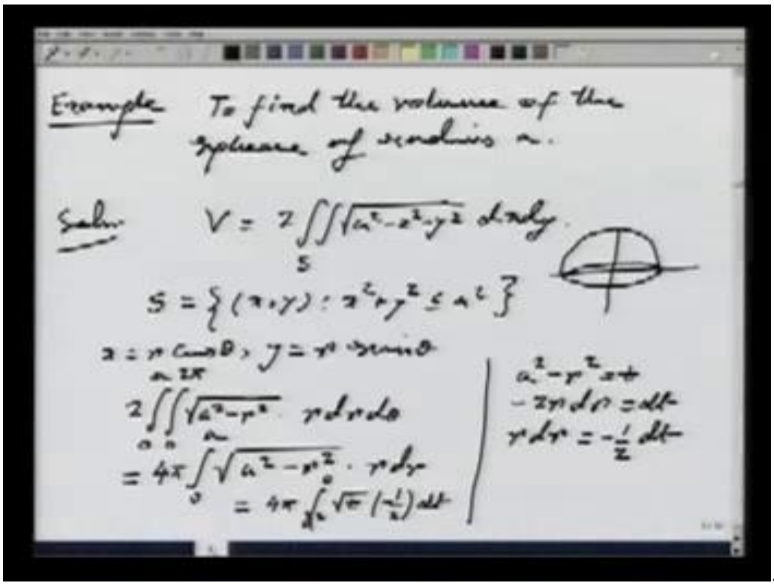

So, we show you, this by an example. So, here the problem is to find the volume of the sphere of radius a. Then, I know very easily, that this volume v, by symmetry is given by 2 times, double integral over S. Then, square root of a square minus, x square minus, y square d x, d y. Where, S denotes, the upper half of the sphere, you know. So, what is s, such that x square plus y square is lesser equal to a square.

Since, I am taking, the positive square root. I am actually finding the upper half of the sphere. So, what I am finding out is this. But, since I am multiplying by 2, I am going to get the volume of the whole sphere, anyway. Now, looking at this integral, I can very easily see that, it is the situation, where I should use polar coordinates. So, use polar coordinates, x equal to r cos theta y equal to r sine theta.

Then, what do I get, then this integral in polar coordinates is 2 times, double integral 0 to a. Then, 0 to 2 pi, 0 to a, is the variation of r, 0 to 2 pi is the variation of theta, square root of a square minus r square. Now, comes the Jacobian, that is r, d r, d theta. Now, since there is no theta involved in the integral, it is only the integral in terms of r. Theta integral, I can calculate by Fubini's theorem. So, it is 4 pi times 0 to a, square root of a square minus r square into r theta square.

Now, this integral can be easily calculated. You put just, let us say a square minus r square is equal to t. Then, minus twice r d r equal to d t, that is r d r is equal to minus half d t. And, the limits of integration changes from, if r is equal to 0, then t equal to a square, if r equal to a, then t equal to 0. So, finally what I get is that this is 4 pi times integral a square to 0, root t times minus half d t.

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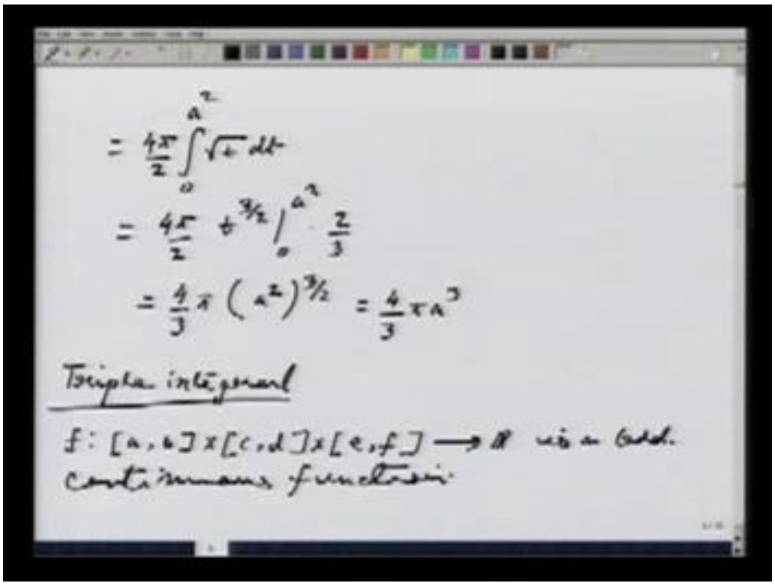

So, what I get here is, then 4 pi divided by 2, integral 0 to a square, square root of t, d t. Final answer, then is 4 pi by 2, t to the power 3 by 2, from 0 to a square into 2 by 3. So, what I get is, 4 by 3 pi, I get 4 by 3 pi, then t to the power 3 by 2, instead of that, I would put a square to the power 3 by 2, so a square to the power 3 by 2. So, it is 4, sorry, there is a, 3 here, so 4 by 3 pi a square to the power 3 by 2. That is 4 by 3 pi a cube. That is, we already know, it is the volume of the sphere, it comes out of double integration and it becomes very easy, just because of the change of variable formula.

Now, we will slowly turn ourselves towards the integrations of three variables. And again, we will try to go to change of variables formula for that. Now, we will be very brief about, the construction of Riemann integration in three variables. Because, it is very analogous to the two variables, so let us go by that, triple integral now. So, in the case of one dimensional integral, we started with intervals. In two dimensional integrals, we started with rectangles. In this case, we are going to start with cubes.

So, suppose, f is a bounded function, from the cube a, b is a bounded. Let us, let us not really demand continuity on these functions, because just for bounded functions, one can define Riemann integration. Because, but in most of the cases, we are going to deal with continuous functions. So, I do not think, there is any harm in putting the notion of continuity in the definition also. That is in fact the bounded continuous functions; we are looking at the construction of Riemann integration.

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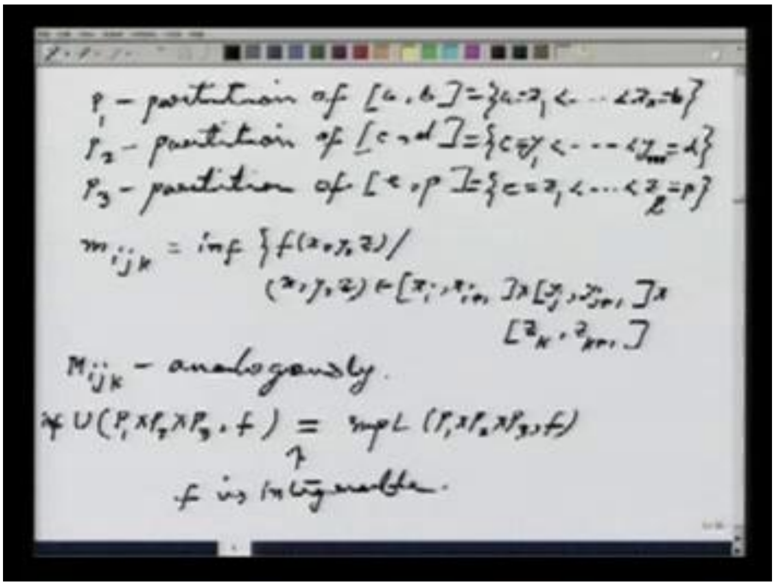

So, what you would do is, now first take P 1, this is a partition of a b. Then P 2, this is the partition of c d. Then, take P 3, this is the partition of e f. Then, I define, m i j k to be infremum of f x, y, z, where x, y, z belong to x i, x i plus 1, cross y j, y j plus 1, cross z, k, z, k plus 1. So, this partition let us assume that, they are given by, x 1, x 2, x n. Let us say, this is given by, y see y 1, let us say x 1, that is better up to y n. Let us say, this is, e equal to z 1, z k, z l, let us say.

Well this, f 1 should not confuse with the function f. So, let me change this, let me call it P. Then, one can define capital M i j k analogously. Then, you can construct, U P 1 cross p 2 cross P 3, f 1 can construct L, P 1 cross P 2 cross P 3, f, just by looking at these supremum and infremum. Then, I can look at the, infremum of the UPf's, I can look at the supremum of the UPf's varying over all partitions. If this quantities turn out to be same, then we say f is integrable, which is analogous to the double integration.

And as we have seen in the case of double integration, that it is actually the Fubini's theorem, which helps us, how to calculate integrals. So, similarly here, we will need a Fubini's theorem for triple integrals also. And we have defined it for cubes, as in the two dimensional case, we have defined it for rectangles. For general regions, general bounded regions in three dimension, what do you do. We will you find a cube, which contains that region.

And extend f to the whole of cube by making it 0 outside the region. And then, proceed with the construction, which I have given above. So, that defines the triple integral for bounded regions, which are more complicated than the cubes. And here, we are going to need some analog of Fubini's theorem also in the three dimensions. That I am going to explain now. And that is going to play the fundamental role, while working out triple integrals. And then we will shift to change of variable formulas.

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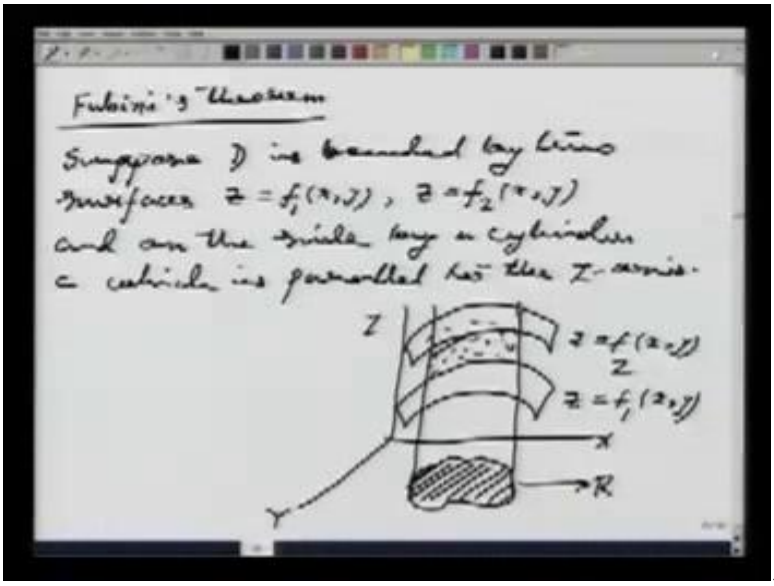

So, let us start first with Fubini's theorem. So, suppose D is region in three dimensions, it is bounded by two surfaces. Let us say, z equal to f 1, x y and z equal to f 2, x y, f 1, f 2 both are continuous functions of course. And on the side by a cylinder c, so on the side by a cylinder c, which is, parallel to the z axis, so in picture, what I mean is, something like this. So, this is the cylinder, this is x axis, this is y axis, this is z axis and there are some surfaces, you know this is z equal to f 1, x y.

Of course, it may naught be bounded it might go on, but as far as, my purpose is concerned, I am just looking at the intersection portion and let us say z equal to, let us call this f 2 x y, this is f 1 x y. So, this is the picture, so my region, of integration actually is, the in between portion, so to calculate integrals of functions over this region, what I do is, so this region is called D, the portion below, which is the projection of, that region on the X Y plane, I call it r. So, you get a portion on the cylinder, just project that portion on the X Y plane, it will produce something in the X Y plane, it is what I call r.

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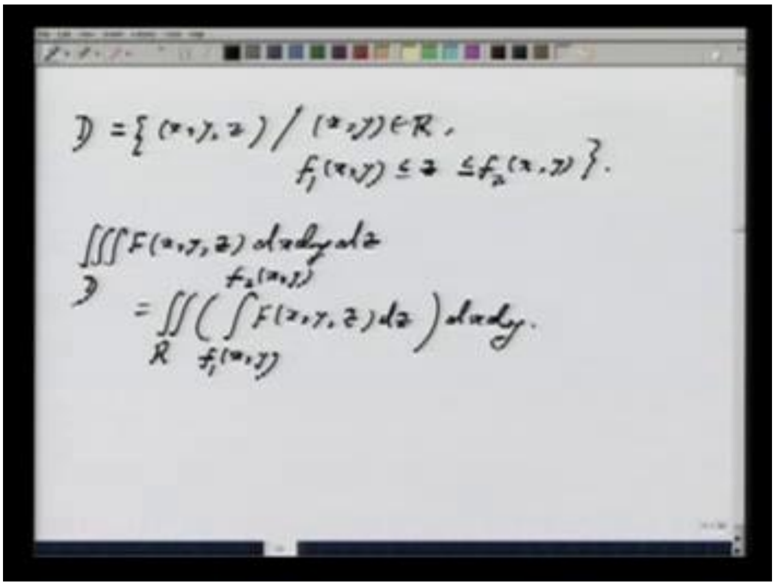

So, formally D, then is x, y, z such that x y belongs to r and z satisfies f 1, x y lesser equal to z, lesser equal to f 2 x y. Now, if I want to integrate a function over D. So, I would say triple integral over D, capital F x, y, z d x, d y, d z, how do I integrate this. The idea is, we will look at double integral of r of the function integral f 1, x y to f 2, x y, f of x, y, z d z, then d x d y. So, that is what you would do is look at the function, in the expression of function, you keep your x and y fix, just view it as a function of z.

Already, some limit is given to you, given on x y, once you fix your x and y, f 1 x y and f 2 x y are just two numbers. Now, you integrate the function f x y fixed, as a function z, put the limits, that gives you a function of x and y. Now, you are integrating, that function over r, that is the double integral. So, the triple integral, after first using one integral with respect to z, reducing it down to a double integral and for double integrals, we know how to compute things. May be we need Fubini’s theorem, may you need change of variable or something like that.

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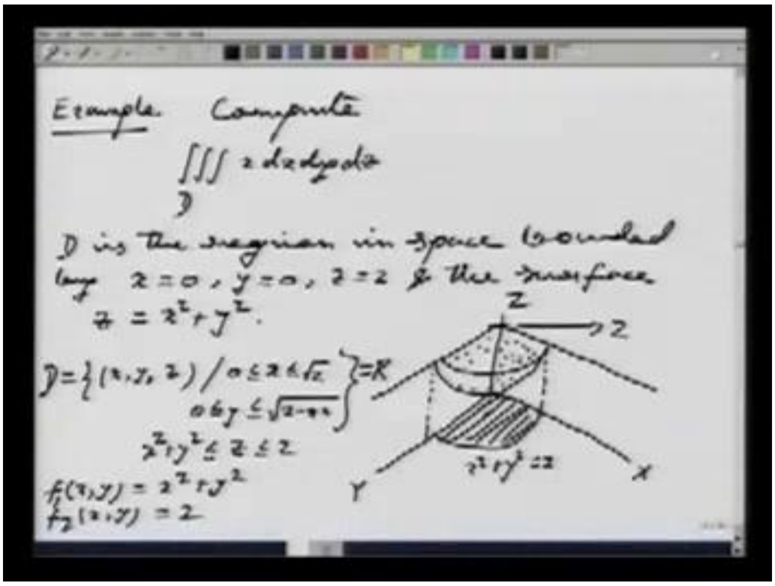

So, this is how, we are going to compute certain triple integrals. So, let me show it, by an example. So, let us try to compute, triple integral over D, let us take a very simple function x d x, d y, d z, the whole thing is what exactly is D, well I will describe, what is D, where D is the region in space. That means, in three dimension, bounded by the plane x equal to 0, that means, the y z plane, y equal to 0. That is x z plane, z equal to 2, that is parallel to the X Y plane and the surface, z equal to x square plus y square.

So, first let us try to have some idea about. So, this is Z axis, this is X axis, this is Y axis. So, let me write X here, Z here, Y here. So, this X axis actually denotes the plane y equal to 0. So, I will let us say this height is 2, so I will draw lines here, so my region, b I can very easily draw. Now, this is the intersection of z equals to x square plus y square. So, what is the region now, it is actually the inside portion. So, how do I, how do I draw it, well it is.

So, this is the region, on which I want to integrate my function x d x d y d z. Certainly, I am going to use Fubini’s theorem, for that what do I need is, the projection of this on the X Y plane. So, if I project it, I am going to get some portion of a circle, that you can very easily see, it is going to be, this circle, that is x square plus y square equal to 2. Because, this height is 2, so the projection is this that is going to remain r. So, what should be integral, so let me write down, what is D again from this picture.

D, then is x y z, such that how much x vary, then x vary from 0 to root 2. From here, x equal to 0, I can come up to here that means, i equal to 0, that means, x equal to root 2, i mean the first quadrant. So, 0 lesser equal to x, lesser equal to root 2, based on that, I get my y, that is 0 lesser equal to y, lesser equal to square root 2 minus x square, this portion actually gives me r, that is the parameterization of the projection. And then, what is the variation of z, it is x square plus y square, lesser equal to z, lesser equal to 2. So, in this example, my function f 1 is actually x square plus y square. So, f 1, x y is x square plus y square and my function f 2, x y fortunately is a constant function 2.

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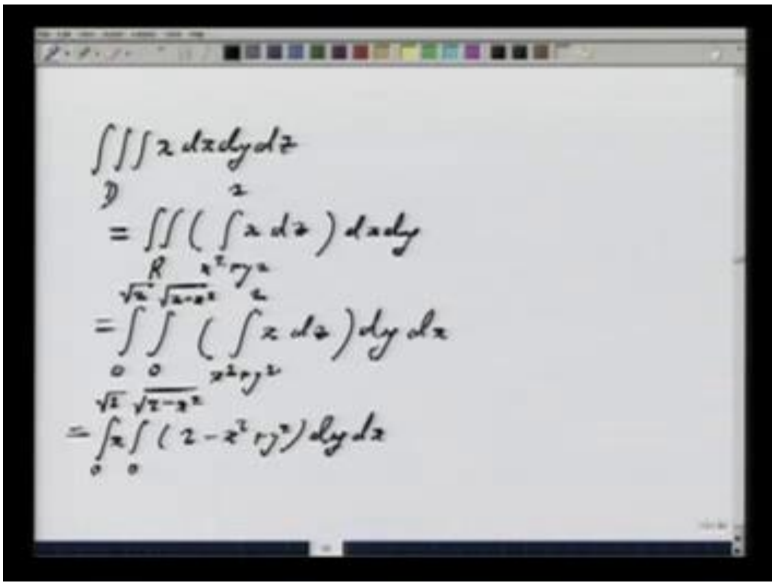

Now, once I know this, then I can try to calculate using Fubini's theorem. That means, triple integral over D x d x, d y, d z, this is same as double integral over R, then integral x square plus y square up to 2, x d z, then d x d y. I can write it more formally, saying what exactly R is R is 0 to root 2, then square root 0 to square root of 2 minus x square, then x square plus y square to 2, x d z first d y, then d x. Notice, once you fix your x and y, then look at the most inner integral, there is no x here. There is no z in the integrand.

So, this integral is z, just 2 minus x square plus y square. So, it then comes out to 0 to root 2, then square root of 0 to square root of 2 minus x square, then 2 minus x square plus y square. That means, z integral I have performed, then d y d x. This now can easily be computed by using the standard Fubini's theorem and two variables. So, this is the way, we usually try to go about triple integrals. If there is no change of variable required.

So, the method is just, if you have a body, of the form. That it is bounded by two surfaces, contained in a cylinder, whose axis is parallel to z axis, just try to find the projection of the body, that gives you a two dimensional integral. Then, calculate the variation of z. So, calculate the z integral first, by Fubini's theorem. These, then reduces down to a double integral, for which you already have the method. So, that here gives, you how to calculate the, usual triple integral. Now, let us try to go to more complicated triple integrals, where you will need the change of variable formula.

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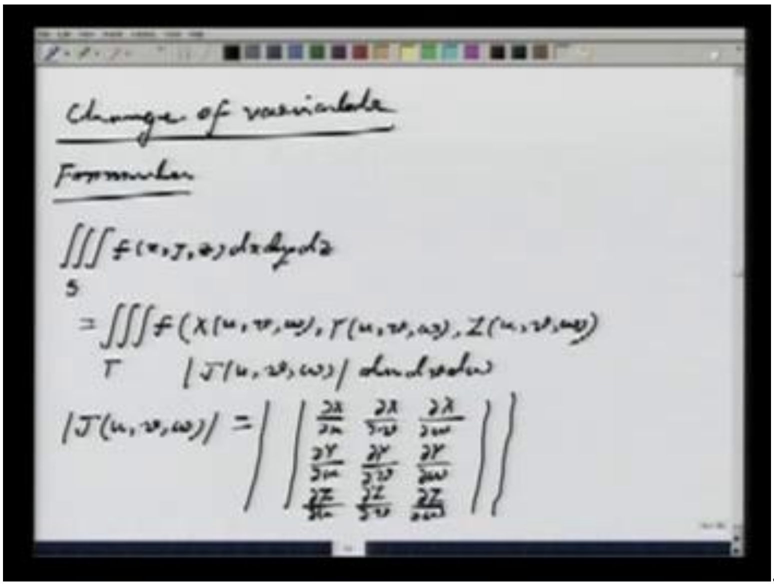

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So, let us, go to the change of variable formula, for triple integrals. Now, the formula is very analogous, to the two dimensional case, only to calculate the Jacobian, you have to calculate more, that is expected, you will get a 3 by 3 determinant now. So, the usual formula looks like, triple integral over S, f of x y z d x d y d z, this is same as, triple integral over T. Now, f of let us say x u, v, w in the two dimensional case, x was a function of u and v. Here, everything is going to be function of three variables, because I am in space.

So, the extra variable is w, then y of u, v, w, then z of u v w, then the Jacobian, which I denote as mod J u v w, then d u d v d w, then what is mod J u v. That is again, the modulus of certain determinant, analogous to two dimensional case, it is del x del u, del x del v, del x del w, then del y del u, del y del v, del z del u. So, this is the Jacobian, which one needs to calculate. You can take it, just in the spirit of two dimensions, it is just the analogous thing.

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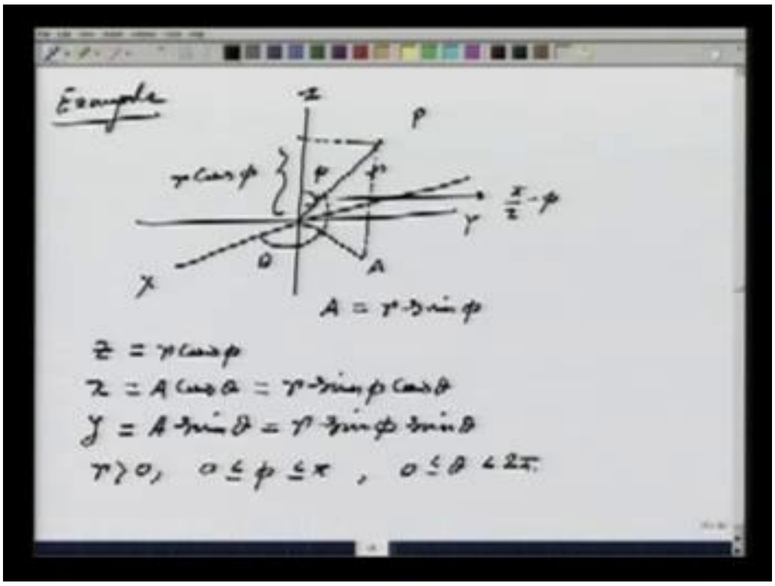

So, here, as an example, what I would like to do is polar coordinates. In three dimensions, which is slightly more involved, than the two dimensional case. So, here what are the parameters, suppose I take a point p here, then P has a distance r from the origin, this is let us say X axis, let me draw this. Let us say this is the X axis, this is the Y axis, this is the Z axis. Then, this point P, wherever it is, it makes at an angle with Z, so this is an angle call this phi.

Now, I project the point on the X Y plane, then this distance, I can certainly calculate, since this is phi. And then, if I call this point, some a let us say, then this is 90 minus phi, this is pi by 2 minus phi. So, this point A, then is clearly, r sine phi, what is the z coordinate of the point, that is this distance, it is very easy to calculate, it is r cos phi. So, the point it is, if I look at it is Cartesian coordinates, then it is Z axis is r cos phi. So, let me write that down first, that z equals to r cosine phi. That I found out.

Now, what is the x coordinates of this, that I need to find out, now I need one, more, variable I call this angle theta. So, theta is the angle, which this projection makes, with the positive direction of the X axis, then it is very easy to see, what the Cartesian coordinate is, then x is A cos theta, but what is A, A is r sine phi. So, it is r sine phi cos theta, then, y is certainly, A sine theta, that is r sine phi sine theta. So, what are the restrictions, that r is always bigger than 0, I will be using that, then for phi, I have 0 lesser equal to phi, lesser equal to pi, because it can come from top to the bottom only.

So, the angle, which it will be making is, up to pi by 2. And then, theta is our usual thing, that 0 lesser equal to, sorry this is phi. So, phi varies here clearly from 0 to pi, because the point p, it comes from the top to the bottom. So, the angle it traverses, it is 0 to pi, then I come to theta, theta is our usual thing, it varies from 0 to 2 pi.

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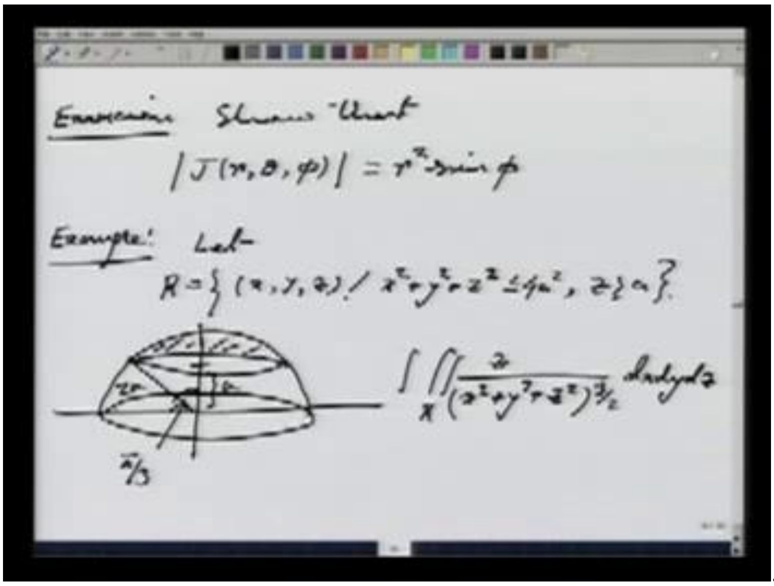

Now, one used to calculate, the Jacobian of this, which I leave as an exercise to you. Show that, modulus of J r theta phi, this turns out to be, r square sine phi, once I take the modulus. So, in the two dimensional case, it was just r d r d theta, the angular variable did not matter at all, but in three dimension. That is not going to happen you see, that r square comes certainly, but sine phi also comes, what does not come is theta, but phi comes.

Now, let me go to an example, of a problem, where change of variable formula is needed. So, I just say let, let us say, R is this, it is all x y z, such that x square, plus y square, plus z square is lesser equal to 4 a square that means, I am inside a sphere of radius 2 a and I take z bigger than or equal to a. So, the region, I am looking at is this, so the above portion of this smaller sphere, that is what I am looking at. And let us say, I want to integrate, triple integral over r, z by x square plus y square plus z square whole to the power 3 by 2 d v. That is d x d y d z, that is what I mean the volume integral.

Now, once you look at this quantity, you immediately understand that you need to go to the polar coordinates. Because, x square plus y square plus z square is appearing here. So, once you go to the polar coordinates, here the only thing is, you have to calculate the variations of phi theta, Now, look at the region, which I am integrating, theta is certainly going from 0 to 2 pi. There is no problem with that, but phi starts from 0 and stop at here.

So, you need to know, what is this phi, well that is not very hard. Because, you know, what this height is, this height is actually a and this portion is 2 a. So, that will tell you, that this phi, which I am looking at this is actually pi by 3, because cosine of phi is turning out to be half. So, once I know that, then I can get the variation of r also.

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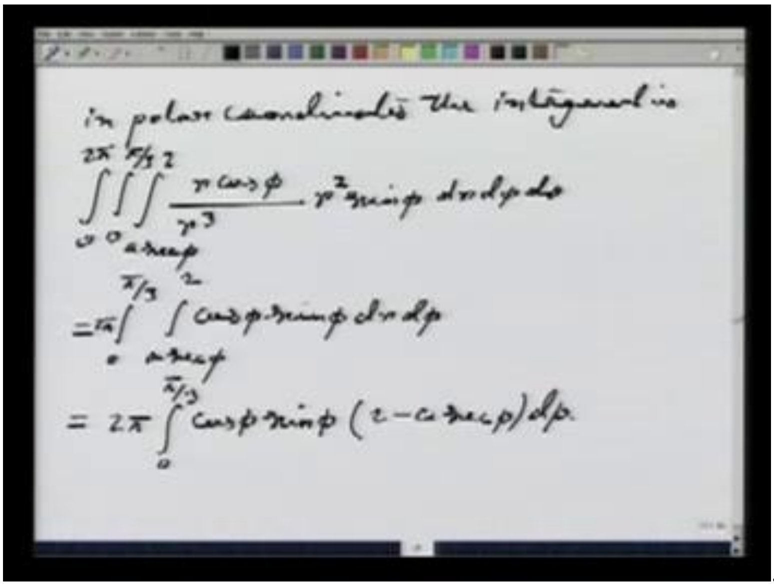

So, in polar coordinates, now in polar coordinates, the integral is let me write the three integrals, I will put the limit later. Then, z means, r cos phi, I know that, divided by x square plus y square plus z square to the power 3 by 2. That means, r cube here, then r square sine phi, d r d theta d phi, so I can put theta here, 0 to 2 pi. This is the variation of theta, so d phi d theta, then the variation of phi, which as we have seen, it is from 0 to pi by 3.

Then a little calculation regarding r is needed, which I urge you to do it yourself, that turns out to be a sec phi to 2, it is not really very difficult. Just look at the picture, then you see the variation of r, which comes out. Once you do this, then actually, this r goes away; you are finally, landing up with 0 to pi by 3, theta integral also I can manage. So, it is 2 pi, so it is 0 to pi by 3, integral a sec phi to 2, what you get finally is cos phi sine phi d r d phi, in the integral there is no r.

So, what you are landing up with now, 2 pi integral 0 to pi by 3, cos phi sine phi into 2 minus a sec phi d phi, this is the integral, you need to calculate, which is typical, typical trigonometric integral, which can be calculated very easily. So, in most of the cases, when we use to use a, when you have, the quantity x square plus y square plus z square in the triple integral, you should go for, the polar coordinates. All you have to be careful is that, given the region. What is the variation of r theta and phi, that you have to calculate.

And for that it is always good to visualize, the given region, in a geometric way. So, that, you can see from the picture, what it is going to be. So, this is more or less, what wanted to, say about change of variable formula, double integral and triple integral. From, now onwards we slowly shift towards, the proof of the fundamental theorem of calculus, in case of double integrals and triple integrals. That means, those are the famous theorems called Greens Theorem and Stokes Theorem and Divergence Theorems. Those are going to come in the next few lectures, as the analogs of the fundamental theorems of calculus.