Mathematics-I

Prof. S.K. Ray
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture – 3
Sequences - II

Let us start with sequences again. In the last lecture, we have seen certain algebraic properties of the sequence.

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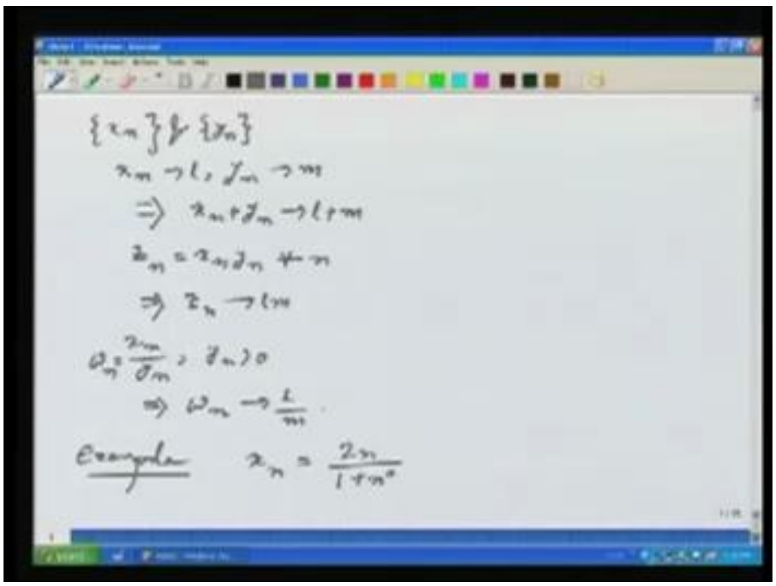

That is, let us say, I have a sequence x n and I have a sequence y n such that both the sequences are convergent and x n converges to l, y n converges to m, then we have that x n plus y n converges to l plus m. Similarly we have certain results about multiplication of sequences also. That is, given the sequences x n and y n, I can look at a new sequence z n. This is defined by x n into y n for all n and then we have seen that it implies z n also converges and it converges to l m. Similarly we have something about x n divided by y n, of course, under the condition that y n is bigger than 0. This implies that this new sequence, which you might call w n, this w n then converges to l by m.

Now the importance of these results actually is that given limits of sequences, we can certainly find out limits of some other sequences made out of the old sequences. For example, let us look at the sequence x n which is given by 2 n by 1 plus n squared, where n is bigger than or equal to 1. Now you want to know what is the limit of this sequence, if at all it converges. We will see very soon that it does.

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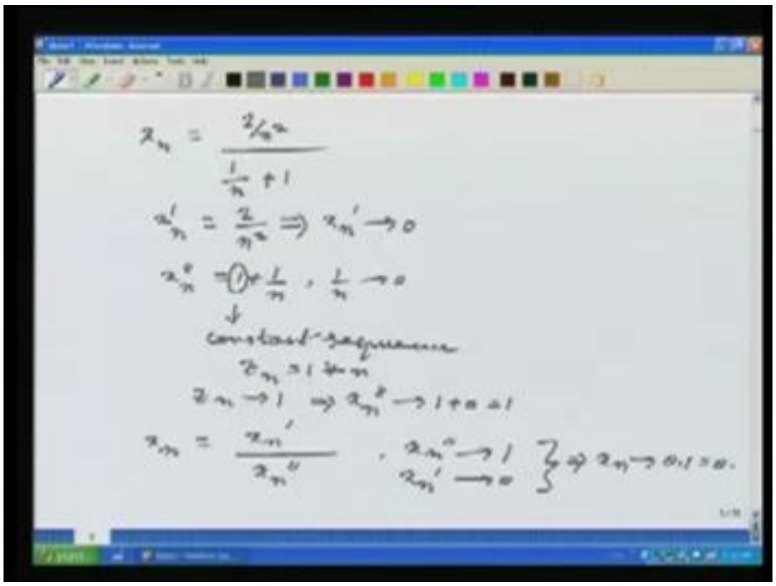

Well, what we do is, we write x n as 2 divided by n squared divided by 1 by n plus 1. That means, I am just dividing the numerator and the denominator by n squared. Now let us concentrate on the numerator, which I can think of it as x prime of n, which is 2 by n squared. What is the limit of this sequence? Well, by Archimedean property, I have that x n prime actually converges to 0. What about 1 by n plus 1? I call this sequence x n double prime which is 1 plus 1 by n. Notice that I already know that 1 by n converges to 0.

Then think of this 1 as the constant sequence 1. That is, it is given by z n equal to 1 for all n and then we know that the sequence z n converges to 1. This implies that the sequence x n double prime, using our previous result, that it converges to 1 plus 0 which is equals to 1. Now then I have that x n equals to x n prime divided by x n double prime and I know that x n double prime converges to 0, I am sorry, x n double prime converges to 1 and x n prime converges to 0. Then again use the previous result. This implies that x n converges to 0 into 1, which is 0. This is how, given old sequences you can produce new sequences which converge and we can find out our limit in terms of the old sequences.

But now the important question arises, that after all when does a sequence converge? Is it necessary that if I want to find out whether a sequence converges or not, is it really necessary to know what exactly is the limit? That is the question we are going to address but first we would like to have a necessary condition under which a sequence converges. Let us first observe one thing.

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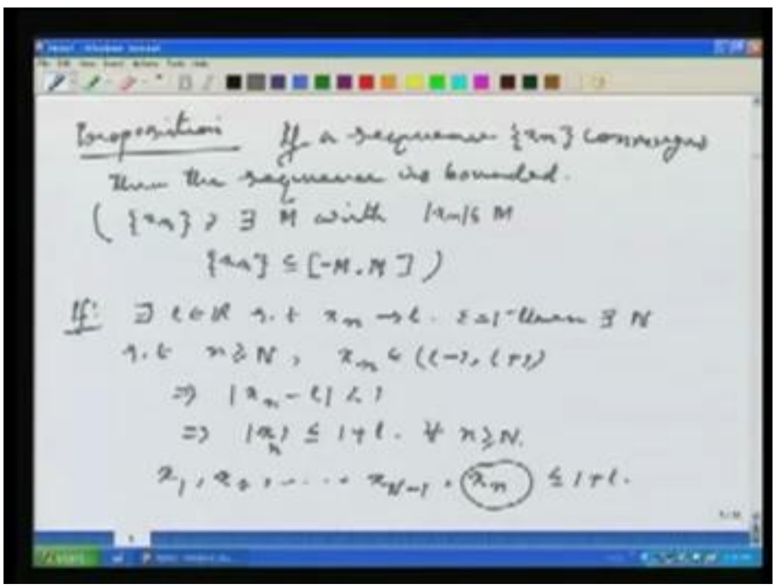

I will call it a proposition. It is really easy to prove; follows straight way from the definition. If a sequence x n converges, then the sequence is bounded. What does a bounded sequence mean? Bounded sequence means it is a sequence x n, such that there exist a real number M with mod of x n is less or equals to M. That essentially means the whole sequence can be thought of as a subset of the interval minus M, M. So let us see, how does the proof follow? Since I said the sequence x n converges, that means there exist a real number l, where the sequence converge. So there exists l, a real number such that x n converges to l.

Then I apply the definition. I take epsilon equal to 1. Then there exist a stage such that after this stage, the whole sequence x n actually is in the interval l minus 1, l plus 1. This implies that mod of x n minus l is less 1, which again implies mod of x n less or equals to 1. But this happens after certain stage. If I have to prove that a sequence is bounded, I have to show that there exist a number M, such that all the terms of the sequence are less than that number M.

I have shown it; that this is true after certain stage. What happens to the previous stages? Notice that there are only finitely many stages x 1, then x 2, it remains up to x n minus 1 and after this whatever x n I have, they satisfy the inequality that this is lesser equal to 1 plus l. Then to arrange the bound which I called M, what I do is, I just look at the maximum of the numbers x 1, x 2, x n minus 1 and then I look at 1 plus mod l. This certainly exists which I call M.

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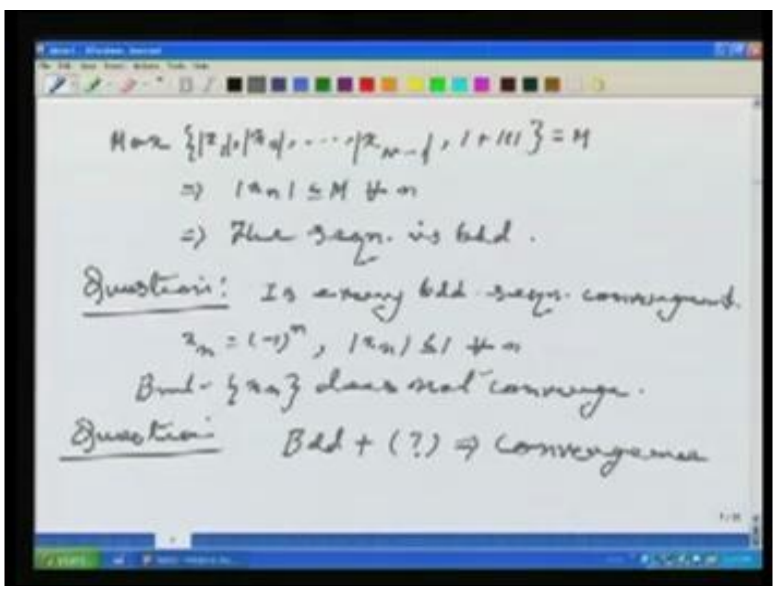

Then this implies that mod of x n, let me put mods here also, then it implies that this is less or equals to M for all n. This implies the sequence is bounded. Then the question remains, is the converse true? So the question is, is every bounded sequence convergent? The answer is no and we can easily see that. If I look at the sequence x n equals to minus 1 to the power n, then it is clear that mod of x n is less or equals to 1 for all n but we know that the sequence does not converge but x n does not converge. So the question then remains and that is the most fundamental question which we are going to take up now, that bounded plus what extra condition I need, which will imply convergence.

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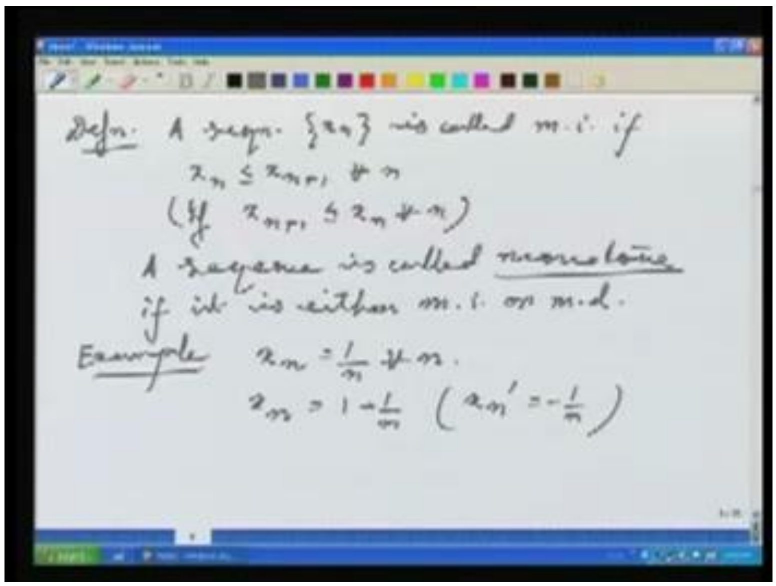

I will try to supply one such condition first. So let us see one such theorem.i.n this direction. For that first I need one definition. So the definition what I need is, of a monotone sequence. So what is a monotone sequence? A sequence x n is called monotone increasing, in short m.i, if x n is lesser equal to x n plus 1 for all n. That means, as you go towards infinity, your terms are slowly increasing one after other. Similarly the reverse of this is called the monotone decreasing sequence. That is, if x n plus 1 is lesser equals to x n for all n and a sequence is called monotone if it is either monotone increasing or monotone decreasing. So a sequence is called monotone if it is either monotone increasing or monotone decreasing and you will try to prove that given any sequence, if it is bounded and it is monotone then it actually converges and we have seen some such examples.

For example, if I look at this sequence x n equals to 1 by n for all n, we have already seen that the limit of this sequence is 0. Notice that this sequence is a bounded sequence. It is anyway bounded by 1 and in the below, it is bounded by 0 and certainly it is monotone because as you go on increasing this n the numbers actually decrease. So it is a monotone decreasing sequence. Similarly I can look at x n equals to 1 minus 1 by n since I know that the sequence 1 by n converges to 0. That means, 1 minus 1 by n, that converges to 1 and this sequence anyway is monotone increasing because as you go on increasing the ns, this 1 by ns are decreasing. That means 1 minus 1 by n are increasing and there is nothing holy about this one. In fact I look at x n prime equals to minus 1 by n. Then this sequence also converges to 0 and it is now monotone increasing. So let us try to prove the theorem now.

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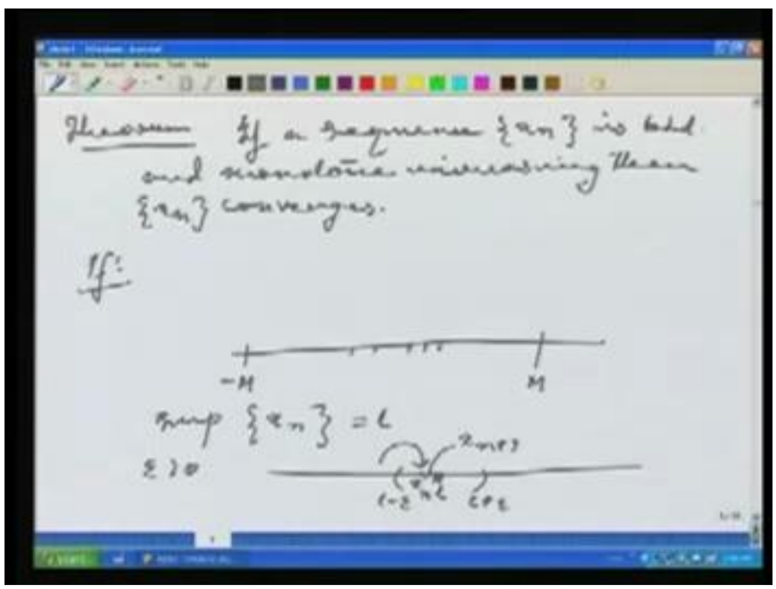

If the sequence x n is bounded and monotone increasing, then x n converges. I have stated the theorem just for monotone increasing sequence but you can prove the same result for monotone decreasing sequences also. Just look at the proof which I gave you. Then try to mimic the proof. The result will follow analogously and the proof essentially is just drawing the picture. It will follow that way. So before writing down the formal proof, let me just draw the picture of this.

So this is real line and the sequence x n points are here but there is a bound here, let us say M and there is a minus M here. Now the sequence has to converge somewhere. First, you have to somehow get hold of the candidate who can be the limit of the sequence. Who can be the candidate? First of all, once somebody gives you a bounded non empty subset of real line then the first thing you will think about is its supremum. You can think about it is infremum also. Let us get hold of the supremum. First, so supremum of x n let us look at and let me call it l since I am always denoting the limit is by l, now you can guess that this l is actually going to be the limit.

Let us try to see whether it is true or not. So I take any epsilon bigger than 0. That means I have l here and I am choosing an interval around this. Now the first thing I know is that this l minus epsilon, this certainly cannot be an upper bound of this x n s. If it is, then this l cannot be the supremum because l minus epsilon is less than l, means what? It means there exists a member of the sequence which surpasses this l minus epsilon. That is, it is sitting somewhere here.

At the same time you notice that all these x n s are actually less than l and it is monotone increasing. So what happens to l x n plus 1, if I want to look at x n plus 1, it has to be somewhere here. Then x n plus 1 is sitting here. Then x n plus 2 is sitting here; so is x n plus 3. That means after this x n, all the terms are actually between l minus epsilon and l by monotonousity of the sequence. Well that is the proof. So let us just try to write it down formally.

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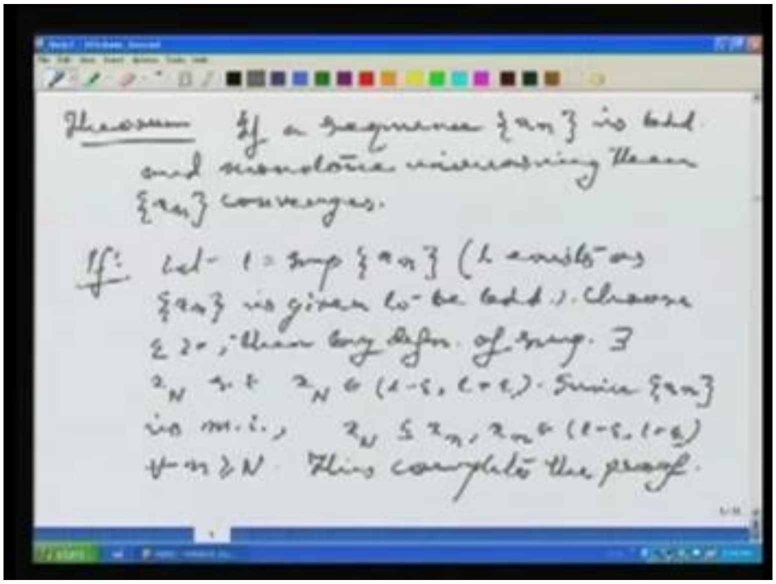

Let l be supremum of x n. In the bracket I will write l exists as x n is given to be bounded. Choose epsilon bigger than 0. Then by definition of supremum, there exist capital x n such that capital x n belongs to l minus epsilon, l plus epsilon. The rest is simple. Since x n is monotone increasing, x n is lesser equals to x n and x n belongs to l minus epsilon, l plus epsilon for all n bigger than or equals to N. But is not this the condition which you want for the convergence of the sequence? This completes the proof. If you want to prove an analogue of the same result with monotone decreasing sequence, then what is the change? The only change is, instead of this, the supremum rather we should look at the infremum. Then the rest of the proof is exactly same.

Now the next question we are going to address is, can we really talk about convergence of the sequence without referring to its limit, because as it happens that, given a sequence, it might be extremely difficult to find out exactly what the limit of the sequence is. So without referring to the limit can I talk about convergence of a sequence? If you look back at the definition of the convergence, it actually involves the limit. So is there any way to define convergence without referring to the limit.

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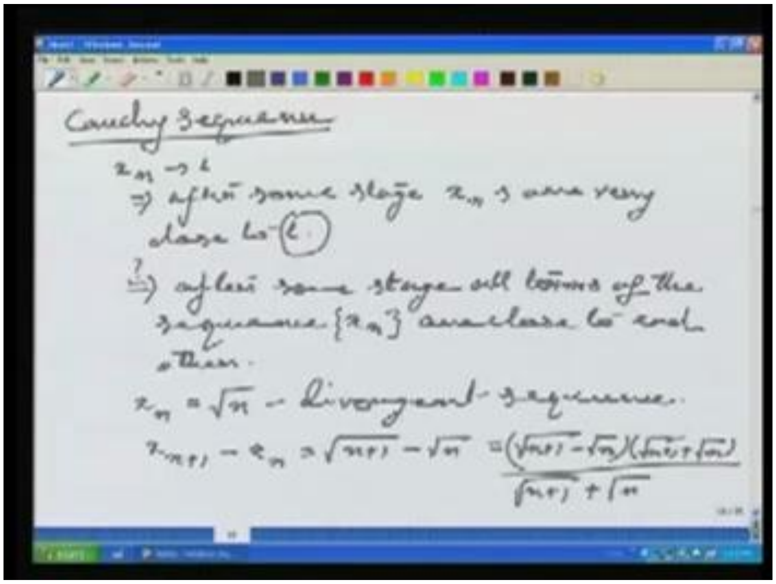

That brings us to the notion of something called Cauchy sequences. That is the thing we are going to take up now. It comes out from a very intuitive way of looking at the convergence that x n converges to l implies, after some stage all the x n s are very close to l. Now from this statement, if I want to get rid of this l, can I think of this way, that all the terms of the sequences are getting very close to each other. Is it same as saying that after some stage, all terms of the sequence x n are close to each other?

We want to see whether it is true or false. I would say intuitively it looks correct to me because if all the terms after certain stage are getting close to l certainly they are getting very close to each other but still some amount of quantification is actually required. Follows from this example, let us look at the sequence x n, which is given by square root of n plus 1 minus square root of n, sorry, let me look at this sequence x n equals to square root of n, certainly a divergent sequence but notice that if I look at x n plus 1 minus x n which is square root of n plus 1 minus square root of n which can also be written as the product of (square root of n plus 1 minus square root of n into square root of n plus one plus square root of n).

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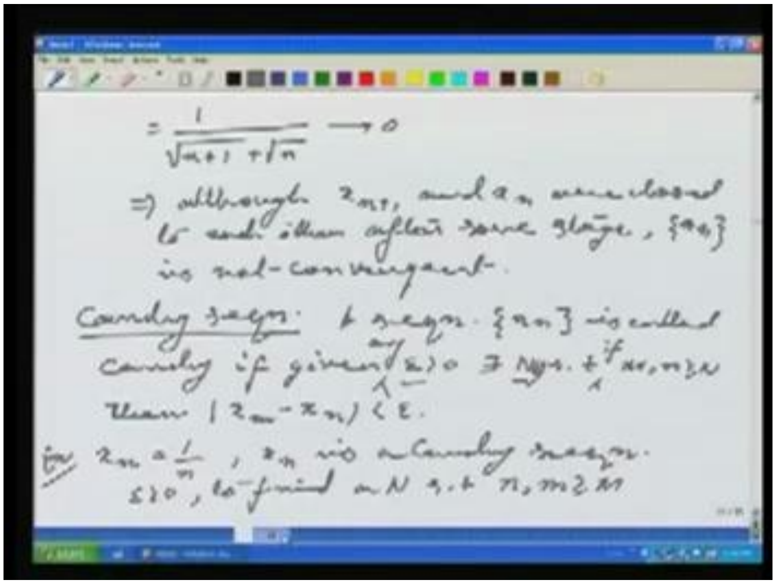

This is very simple now if you compute the numerator, what comes out that, this is equals to 1 by square root of n plus 1 plus square root of n. What does this show? It shows that this goes to 0 because the denominator converges to infinity and I am looking at something like 1 by infinity, you know. So that goes to 0. It implies that although x n plus 1 and x n are close to each other, after some stage x n is not convergent. That means, just two consecutive terms as you go towards infinity, they are getting close. That does not ensure the convergence of the sequence.

What actually you need is, all the terms after certain stage they can be squeezed together almost. That is the meaning of the, that will turn out to be the meaning of the convergence. So let us try to proceed in that direction. So first I define something called Cauchy sequence. We will see that the definition actually contrasts with the previous example in certain way. I say a sequence x n is called Cauchy, if given epsilon bigger than 0 there exist a stage such that after this stage, if I take two elements then the difference between x m and x n is less than epsilon.

Here one thing I have not written but you should understand that I actually mean, that means, whatever short length you assume after certain stage any two elements, they have the distance less than that epsilon. With this epsilon, N will vary. That is the only thing. So I can actually write N epsilon. Notice that if I have this sequence x n equals to 1 by n, I say x n is a Cauchy sequence. How does this follow? I pick any epsilon bigger than 0.

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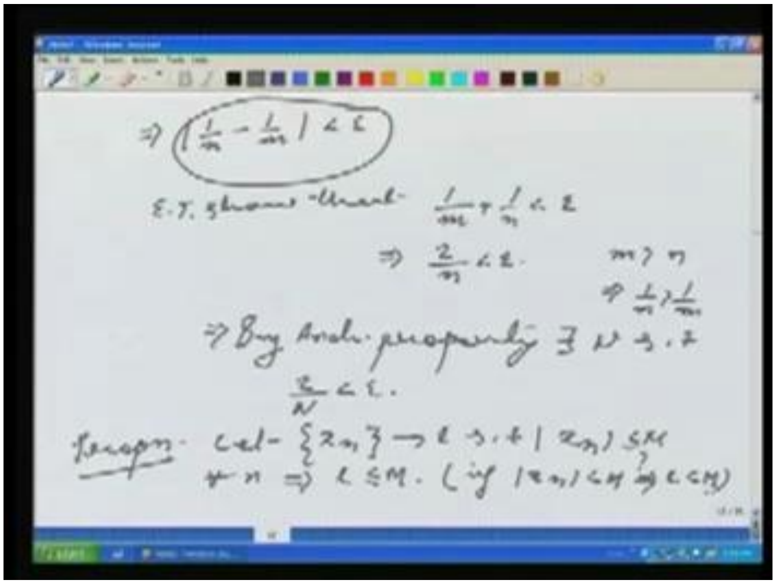

I have to find a stage capital N such that n m bigger than capital N implies mod of 1 by n minus 1 by m is less than epsilon. This is my job. To make this less than epsilon, I say it is enough to show that 1 by m plus 1 by n is less epsilon. This implies then if I assume that m is bigger than n, that 2 by n is less than epsilon but I can always find some such n again by Archimedean property. There exist capital N such that 2 by capital N is less than epsilon.

That finishes the job and I already know that the sequence x n converges. So it might happen that if a sequence is Cauchy, it does converge. To prove that, we have to prove certain things first. The first thing I need is, first an observation, let us write it this way: proposition. Let x n be a convergent sequence and converges to l such that x n is less or equal to m for all n. I say this implies l is also less or equals to m. You can take it as a little exercise, that if mod of x n is strictly less than M does this imply l is also strictly less than M. Think about it. Now, how to prove this?

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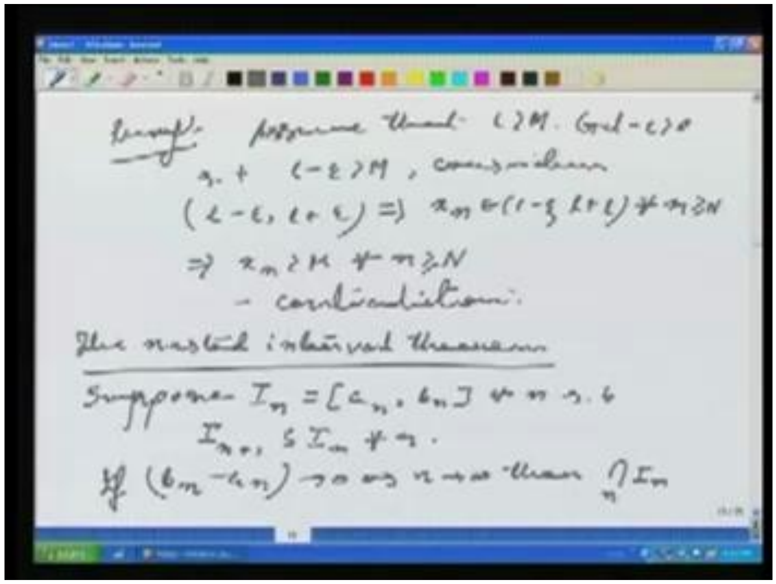

Proof is again simple. Assume that l strictly bigger than M, then anyway you can manufacture epsilon such that l minus epsilon is strictly bigger than M and then I look at.. Now I know since the sequence x n converges to l after certain stage, all the x n s are inside l minus epsilon, l plus epsilon but that implies those elements x n s are actually bigger than M, which they are not allowed to be. I say, by definition of convergence, this implies that x n belongs to l minus epsilon l plus epsilon for all n bigger than or equal to some capital N. But this implies x n is bigger than m for all n bigger than or equals to n which is a contradiction because all the x n s are supposed to be less than M.

The next thing I need is called the cantor intersection theorem or the nested interval theorem. It is again another statement which seems intuitively very clear but surprisingly it requires a proof. Well, the theorem is that, suppose I n is a closed interval given by a n and b n for all n such that I n plus 1 is contained in I n. That means as n increases, the intervals are getting smaller and smaller, one is contained in the other and that is the meaning of the word nested. Also assume that the closed interval a n, b n are getting smaller and smaller. That means, the intervals are actually squeezing. That means, their lengths are actually going to 0.

So that is the next statement. If b n minus a n, notice that this is nothing but the length of the intervals I n, if this goes to 0 as n goes to infinity, then what happens to the total intersection of the I n s? Does it become an empty set suddenly? The answer is no. It is still non empty but there is only one point in the intersection. Then intersection of n I n contains a single point.

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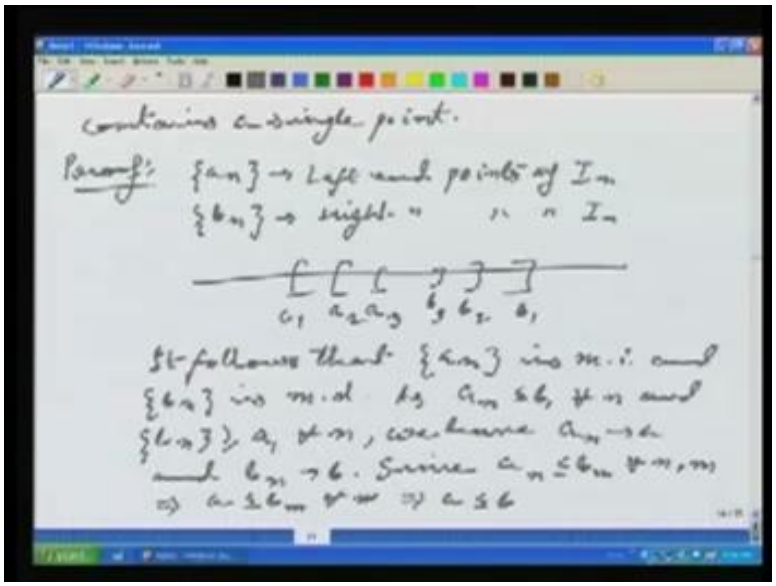

So let us try to prove this. First, let us consider the sequence a n. This is the left end points of I n. Similarly I can consider the b n s also. That is the right end points of I ns. Somehow I have manufactured two sequences and I want to talk about its limits. The question is, are these sequences convergent? I say yes because as I ns are getting smaller and smaller, this is my I so a 1, and b 1 then I look at I 2 which is a 2, b 2 then I look at I 3 which is a 3, b 3. It follows that the sequence a n is an increasing sequence. It follows that the sequence a n is monotone increasing and the sequence b n is monotone decreasing.

Now if I want to ensure that the sequence a n converges, what else I need? Every monotone increasing sequence does not converge. You can look at the sequence x n equals to n; it goes to infinity. So the condition you need is bounded that I have proved few minutes back. Let us see whether the sequence a n is bounded. Notice that all the a n s, which I am looking at, they are actually bounded by b 1 anyway. As a n is less or equals to b 1 follows straight way from the picture which I have drawn, for all n and the sequence b n s, it is actually bounded below by a 1 for all n. We have that both the sequences converge.

Now I need to know what is the relation between a n and b n. Now I apply the previous result which I have proved. All the a n s are less or equals to, since a n s, now notice that a n is lesser equals to b m for all m and n. That again follows from the picture. This implies then by our previous result, that a is less or equals to b l for all n. Then again reapplying the lemma it follows that a is lesser equals to b. If I can somehow show that a is equal to b then I am true. Now here actually the role is played by that the distances are actually getting to going to 0. So let us try to see why a equal to b.

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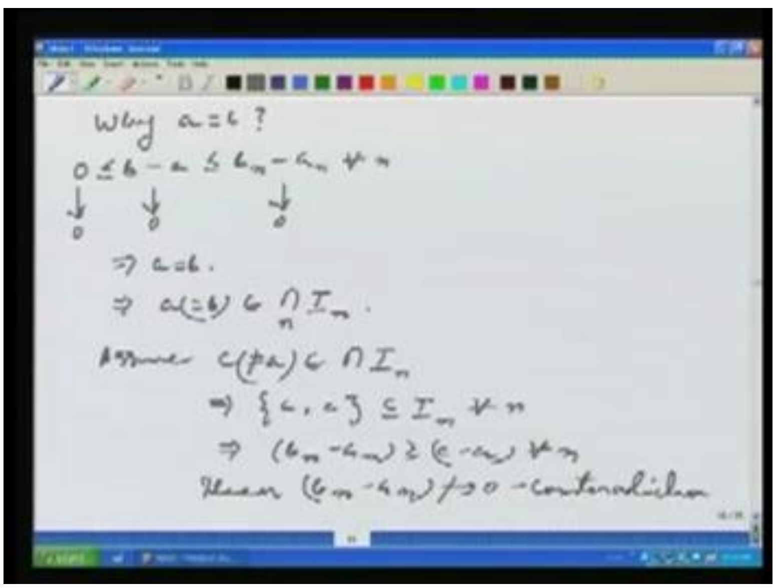

Now to say this is very easy, just notice one thing that b minus a is certainly less or equals to b n minus a n for all n and I also know that this is bigger than or equal to 0, that I have proved but as n goes to infinity, this converges to 0. This, I treat as the constant sequence which converges to 0 and then I use sandwich theorem. That means this converges to 0 but this converges to 0 means what? It is the constant sequence. This implies, a must be equal to b and that is it.

That is actually, it is in the total intersection but it still remains to prove. All I have proved is there is something in the total intersection. There might be some c n in the total intersection. Why there is a single point? Suppose there is some other element c in the total intersection. So assume c which is not equal to a is in intersection I n, means what? It implies that the set a and c. It is actually contained in I n for all n. This implies that b n minus a n, which is the length of the interval is certainly bigger than or equal to c minus a for all n and then the length b n minus an, which is supposed to go to 0 is not going to 0 which is a contradiction means what? It means, there is no such c and anyway there is a. So that means the total intersection must be a singleton set. So this completes the proof.

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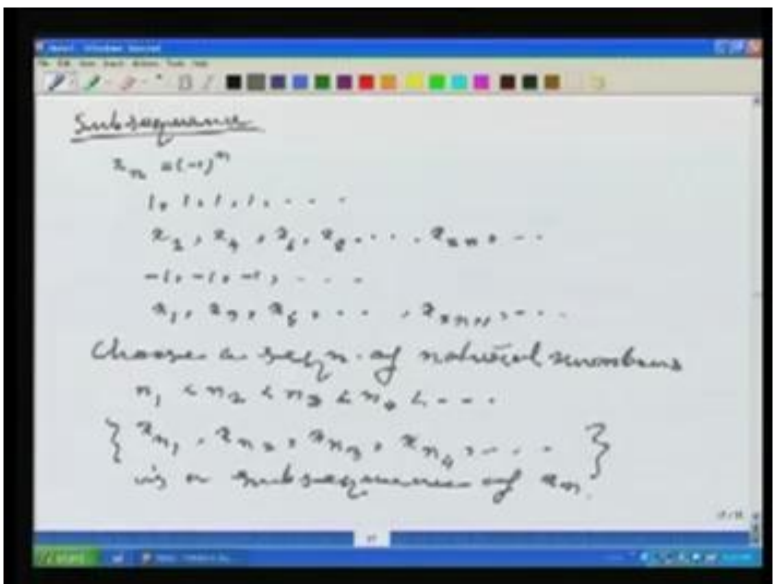

An important corollary of this nested interval theorem is this so called Bolzano Weierstrass theorem but before I come to that, let me explain certain notions first. So the notion I need is the notion of subsequence. What is a subsequence? It is something like you have given a sequence x n and you just forget certain terms of it. Whatever remains do not change the order. Look at them exactly in the same order. What you get is a sub sequence. So in particular, a sequence is certainly a subsequence of itself. First, let us look at examples and then I will give you the definition.

Let us look at this sequence x n equals to minus 1 whole power n and I say this sequence 1, 1, 1, 1 is a sub sequence because it is nothing but x 2, x 4, x 6, x 8, x 2 n and so on. You see, what I have forgotten, I have just forgotten the odd terms. Similarly I can look at minus 1, minus 1, minus 1, minus 1, the constant sub sequence which is nothing but x 1, x 3, x 5, x 2 n plus 1 and so on and notice that the sequence x n does not converge but the sub sequences which I have picked up, they converge.

That means, this sequence has at least got two sub sequences which converge. That means, even if the sequence is bad, it might have good sub sequences. So what I do is, first I choose a sequence of natural numbers n 1 less than n 2 less than n 3 less than n 4 and so on and then look at the corresponding terms of the given sequence x n, that is, x n 1, x n 2, x n 3, x n 4 and so on. It just means I have picked up certain terms from the sequence x n without changing their orders, the way they were arranged, exactly in the same way but I have deleted certain terms, that produces is a sub sequence. This is a sub sequence, a sub sequence of x n.

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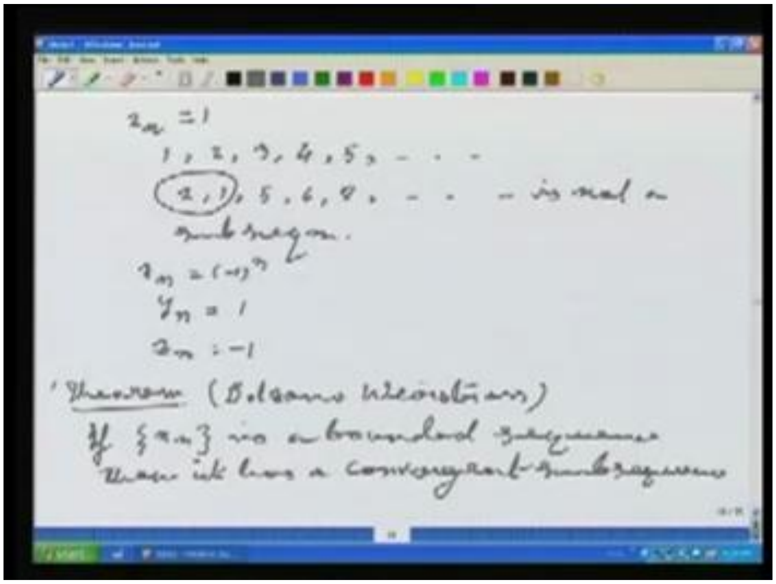

What is not a sub sequence? For example, let us look at this sequence x n equals to n, so the terms are 1, 2, 3, 4, 5 and so on. If I look at this now, 2, 1, 5, 6, 8, then this is not a sub sequence, why so, because I have changed the order here which I am not supposed to do and since the sub sequence is also a sequence, that means, there also has to be infinitely many terms in this. It cannot certainly stop. That means, it is not really just a subset of the sequence. It itself has to be a sequence.

Now I will try to show that what happens with the sequence x n equals to minus 1 to the power n. I could manage to get two sub sequences. Remember one was y n equals to 1 the constant sequence 1 and the sequence z n which is given by minus 1 to the power n. This is minus 1 to the power 1. These two sequences were convergent. We will try to show that this is not really an accident. Given any bounded sequence, we can always do that, at least we can always able to find a sub sequence of the sequence which converges, although the whole sequence may not converge. For example, this minus 1 to the power n, this whole sequence does not converge but I could extract a sub sequence y n which was convergent. This is precisely message from the Bolzano Weierstrass theorem.

So let us now state the theorem. Well the statement of the theorem says that if x n is a bounded sequence then it has a convergent sub sequence. We will see the proof of this will actually follows as a corollary of the nested interval theorem. Instead of writing down the detailed proof of the result, let me try to explain it pictorially what it means. Let us first draw the interval minus M, M within which my sequence lies.

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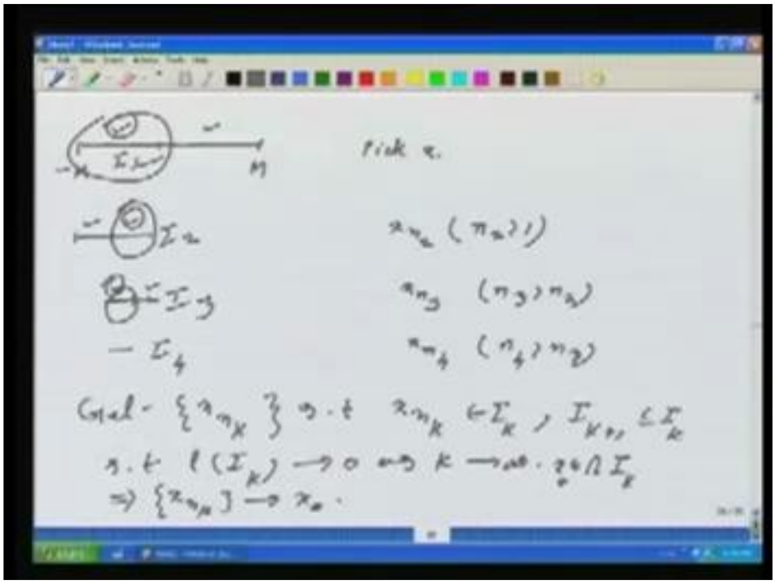

Let me first pick x 1, then I subdivide this, then either this contains infinitely many elements or this portion infinitely contains many elements. Let us assume this portion contains infinitely many elements. Let me draw it separately here and from this I pick x n 2. Now what I do is I subdivide this interval. Now again either this part contains infinitely many elements or this part contains infinitely many elements, since the sequence is infinite. Let us assume that it belongs to this part. Now what I do is, I write this separately here and I write it in a slightly magnified way because I have to subdivide further and let me pick x n 3 from this part.

I will make a choice, the choice is I will always assume that x n 2 is with the condition that n 2 is bigger than 1 and here I make the choice that n 3 is strictly bigger than n 2. That choice I can make because I certainly have infinitely many elements at my disposal. So I can anyway choose them. Now again I sub divide this fellow into two parts, two equal parts. Now either this has infinitely many or this has infinitely many elements. Let us assume this has. Then again I will write it here and I will pick an element x n 4 from there with the condition that n 4 is strictly bigger than n 3.

Now from this we can see that get we can get sub sequence x n k such that, if I call this subinterval i 1, this sub interval i 2, this i 3, this i 4, what I get is that x n k belongs to I k. It is also clear just from the construction that it is a nested sequence. That is, I k plus 1 certainly contained in I k and the way I am doing it certainly the lengths are increasing, that you can see from here.

First length was this portion, then it was this portion. Then it was this, then again getting half. So it is going down to 0. That is length of I k by l of I k, I mean, the length it goes to 0 as k goes to infinity. Now I am.i.n a position to apply the nested interval theorem. That is, I look at intersection of I. Then by the nested interval theorem, there is some x not sitting inside this and I say this implies that the sub sequence x n k converges to x not. Well, this is what we are trying to find out.