Mathematics-I
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Lecture - 30
Green's Theorem

In today's lecture, we are going to talk about generalizations of the second fundamental theorem calculus for double integrals. And that is called Green's Theorem.

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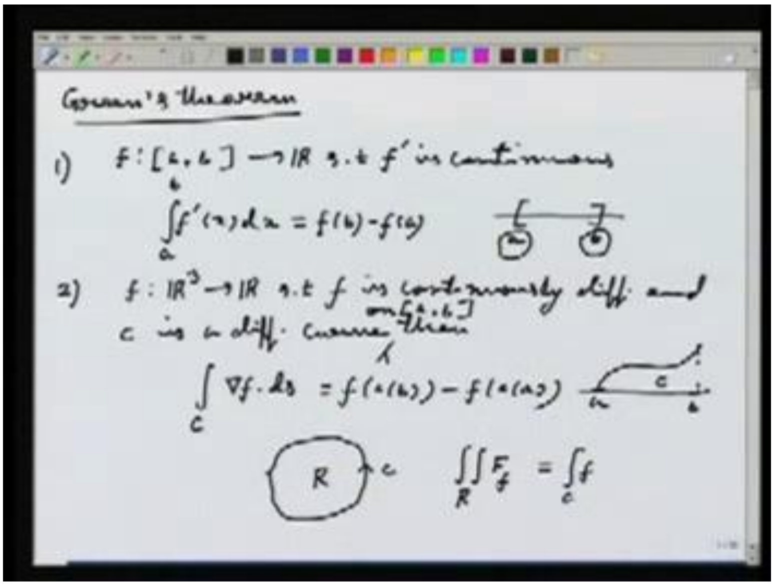

So, in today's lecture, we are going to deal with something called Green's theorem. So, let us recall first, what exactly was the second fundamental theorem of calculus. That suppose I have function f from closed interval a, b to R. Let us also assume, that f prime is continuous. Then we have proved that integral from a to b, f prime x d x, trans out to be f b minus f a. In all the lectures from now on, we will seek for the generalization of this.

When the left hand integral is generalized to a double integral or a triple integral or things like that. Today, we are going to deal with the double integrals. Now, you already have one generalization of this for line integrals. That is if I have a function f from R 3 to R. Such that, f is continuously differentiable, that is derivative of f is also continuous. And c is a differential curve, then we have proved that this is true. Let us see a differential curve.

Let us say on the interval a, b, then grad f which is a vector field, dot d s, integral over c. That trans out to be f of c, b minus f of c, a. This is the result which we have proved, it uses the first one, that is the classical and second fundamental theorem. In today's lecture, I am going to generalize further now, for double integrals. Now, if you look at the first one, that I have an integral over n, interval.

That comes to f b minus f a that is points. Now, if I try with some region like this, this is a region R, it is boundary is a curve c. Then if I want to generalize the first result, that is the second fundamental theorem of calculus. I should have something like double integral over R of some function, which depends on a give function F. I will right now call it F f, it should be given by, the line integral of function f over c.

The dimension comes down to one less. In the first one, there was an interval; I look at the end the boundary of that interval, which comes out to be two points. Here, I have the interval a, b this is a and this is b. Then the boundary of this interval, at these two points a and b. Finally, the integral of the derivative of f is given, in terms of the value of the function at the boundary points.

If you look at the second one, what had been said is, suppose this is the curve c. This point is a and this point is b. And I have integral of something the integrant depends on the function f. Then again it is value is given by the value of the function at the boundary points, that is c, a and c, b. Now, the same kind of thing, I want to do now for double integral. And that is green theorems. But before I start, I will need certain notations, so first I define something.

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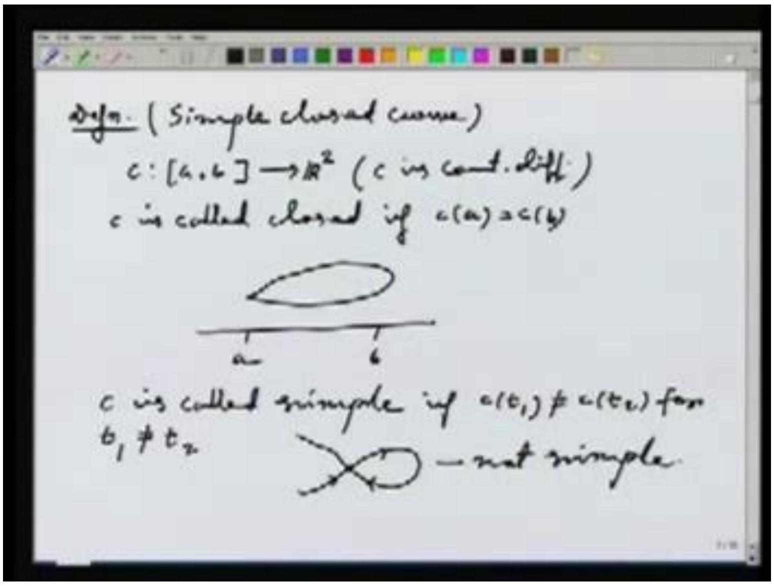

It is definition of a simple closed curve, that is what we will write. So, let us say, c is a curve, we will deal only with R 2. Now I always assume c is continuously differentiable. Because in most of the cases you will at least have differentiability, continuity may not be there. But, continuously differentiable is not really necessary, you can work with differential curves. Then c is called a closed curve, if c a equal to c b. That is, if this is a and this is b, suppose c starts from here, this is c a, then c a is same as c b.

That is it is closed, that means the initial point and the final point is same. And c is called simple, if there is no crossing in the curve. That is, if c t 1 is not equal to c t 2, for t 1 not equals to t 2, that is given to different points. Given to different parameters t 1 and t 2, c t 1 is not same as c t 2. That means, if I look at a curve of this kind, suppose this is my curve. Then this curve is not a simple curve this is not simple why, because for some value of parameter t 1.

Suppose, there is the detections of the curve, at for one parameter I come to this point, then I go along this curve. Then I come back and again I pass through a same point, you know the different values of parameter. That means, there exists t 1 and t 2 such that, c t 1 is equals to c t 2. That means, this curve is not simple. So, this kind of curve, we are not going to deal with. So, what is the simple closed curve, this is the curve for which the initial point is same the final point. That is the first condition.

Second one is the fact that curve is simple means, that given two different values of parameters t 1 and t 2. The corresponding points on the curves c t 1 and c t 2, they are different, they are not same. With this now that what we are going to deal with certain directions.

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That whenever I talk about a region R, this is a region R, whose boundary is a curve c, now this curve c has a direction. I am always going to work with the direction of c, in such a fashion, that the region in question, which I am going to deal with, is always on the left hand side of the curve. That means, suppose I am standing on the curve c at this point, then I get this detection. That means, I am working along the curve on the given detection.

If I do that, then you can see that the region which I am bothered about that is R. That is actually on the left hand side of myself. Suppose the situation was different, suppose this inside is R. And suppose I am working in this direction. Then if I am standing on this curve at this point and walking in the given direction. Then the region R is actually in right hand side of mine, which I am not going to work with.

So, this is the case I am going to work with. That you choose the direction of the curve c, in such a fashion, that the region prescribed. That is the region bounded by the curve is always on the left hand side of myself. With this, so when I am integrating on the curve, suppose I am talking about the line integral F dot d s. To prescribe the direction of the curve what I do is, I draw this here, then I draw an arrow. So, that gives me the direction of the curve, which way it is moving. So, here it means, that the region R is always on the left hand side of myself. Now, let us come to the statement of Green's theorem.

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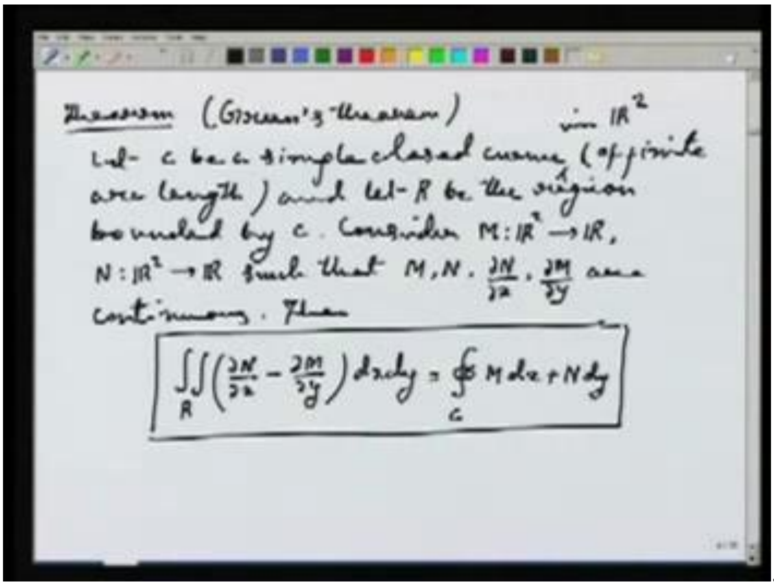

Let c be a simple closed curve of finite arc length. In applications you will always see that, this condition is given to us. And let R be the region, bounded by c. Consider the functions, two functions M from R 2 to R. Here what I mean is, Green's theorem on plane, that is c is a simple closed curve in R 2. Now, consider functions M from R 2 to R and N from R 2 to R. Such that, M, N, del N del x, that is the partial derivative with respect to x, del M del y are continuous.

That is I am assuming the existence of the partial derivatives also. Then the following is true, the double integral over R del N del x minus del M del y. Look at the double integral d x d y, so this is the double integral, if I want an analog of the second fundamental theorem of calculus. Now, the right hand side should be an integral of one dimension less. That is it should be a thinner one, so it should be an integral over the boundaries.

Well it is, it is integral over c, specified by this direction M d x, plus N d y, by. Now, you understand the meaning of this line integral. So, this is precisely what Green's theorem says. Now, we will go for the proof of this. Well for the proof what I do is, I will assume some condition on the curve c and prove the result under those conditions, because the general theorem would be very difficult to prove.

Although, I will show by some examples, that the conditions in which I am assuming on the curve c are not really necessary. Without those conditions also Green's theorem is true. So, let us go to the proof of this. So, once before the statement is this, that you look at double integral over R, del N del x minus del M del y, d x d y. That trans out to be a line integral over a boundary of the integrant is M d x plus N d y. So, let us go to the prove of this. First let me draw a picture for you, which would help you.

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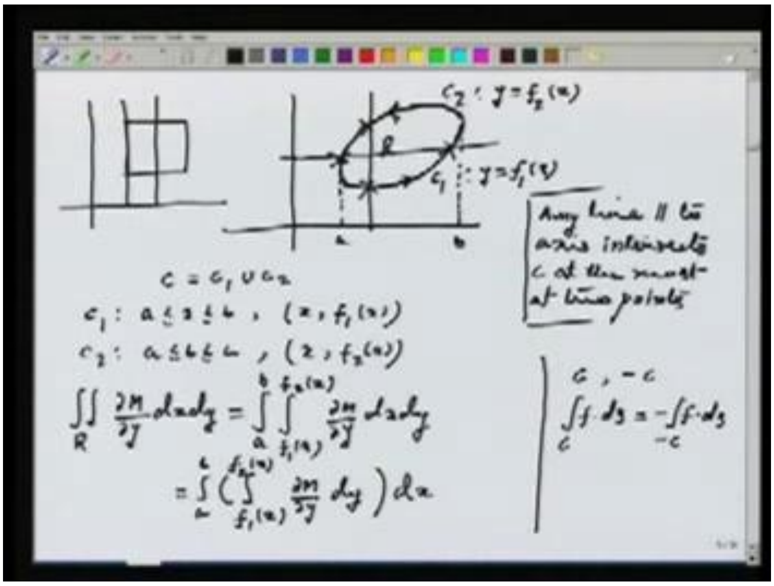

Let us say, a simple closed curve is this. This point is a, this point is b, R is the inside region, so I choose this direction. Now, from a to b, if I look at the first portion of the curve, this I am calling c 1. So, c 1 let us say is y equal to f 1 x. And the portion from b to a, which goes in the reverse direction I call c 2, which is let us say the graph of y equals to f 2 x. So, the whole boundary c is actually c 1 union c 2. Now, I make an assumption on the curve c, which this picture actually satisfies.

The condition is, I will write it here separately. That any line parallel to axis. It may be x axis, may be y axis, intersects c at the most at two points. For example, if I draw a line parallel to the y axis, it intersect the curve twice at this point, at this point. If I draw a curve parallel to the x axis, it intersects the curve at two points. So, this curve satisfies my property. But some curve which does not satisfy this, but you would like to work with perhaps is the rectangle.

Suppose I choose this rectangle. Then is the condition, that any line parallel to the axis, intersects this curve at the most two point, satisfy I say no. Because, if I look at this line parallel to the y axis, it satisfies the curve at two point. But, if I look at this line, this left hand line, it intersects the given curve at infinitely many points. So, a rectangle is not a curve, which satisfies my assumption. And hence the proof, which I am going to get for Green's theorem will not apply for rectangles, so will see differently later that what exactly happens for rectangles. So, let us start, what I do is, first let us say c 1 for this x varies from. And the curve is given by x, f 1 x, this is my c 1, c 2. Here x varies again from b to a, actually from a to b, but the direction is different. Here the curve is x, f 2 x. Notice, once which I have observed, while doing line integral. That you take a curve c in one direction of let us say, of increasing parameters and look at the line integral of the function.

Now, again you can look at the same curve with the decreasing value of the parameters. That is changing the direction. Then, how, those two line integrals are connected, that we have seen. It trans out, that one is negative of the other. That is, if c is the curve in one direction and minus c is the curve same curve in the reverse direction. Then integral over c, f dot d s is minus integral over minus f dot d s. This is something, which you have noted.

Now, let us start with the calculations. So, first I will see, what is double integral over R, del M, del y, d x d y. Because, if you look at the double integral, which I have written in statement of Greens theorem. It involves two terms del N, del x, minus del M del y. Now, it is the portion del M del y, which I am going to deal with first, we will see later. That del N del x can be tackled exactly the same way as del M del y. So, let me first start with del M del y.

So, I write it as an iterated integral. So, first the variation of x which is a to b. Then corresponding variation of y, that is from f 1 x to f 2 x. And then, I write del M del y, d x d y, actually to be very precise, I should write del M del y of x, y, because that is also a function of two variables. But that is what I mean, when I write del M del y. Now, I write this as integral a to b, I put a bracket here, I write integral f 1 x to f 2 x. So, this integral is going to be an y integral only, it is del M del y, d y then d x.

Now, I concentrate on the linear integral. I say this linear integral can be evaluated by second fundamental theorem of calculus. Because if I fix x, then as a function of y, the function M is differentiable. And what I am looking at, the integrant is nothing but, the derivative of that function, which I am integrating from the limit f x 1 to f x 2, where x is fixed, for a fixed x.

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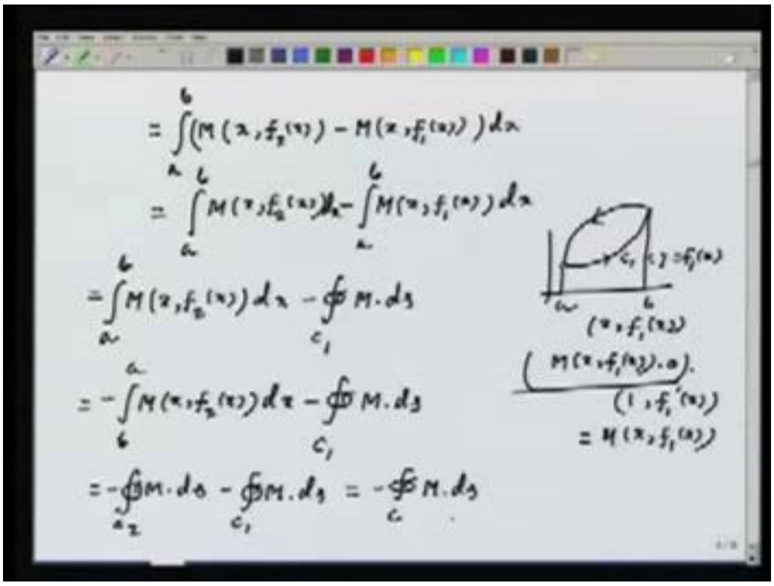

So, I can apply second fundamental theorem of calculus to get that, this is integral a to b, m of x, f 2 x, minus m of x, f 1 x. Whole thing d x, now I quickly separate the integral, it is integral a to b, m, x, f 2 x minus, integral a to b, m, x, f 1 x, d x. So, I have d x here, I should have d x here also. Now, first I concentrate on the first integral. What is this, I had a curve c 1, this is my c 1, this is a, this is b, I am going through the increasing value of the parameter, this is c 1. That is y equal to f 1 x. So, the curve is, x, f 1 x. I am trying to look at the line integral of a function M, which is real value.

So, I should look at M of x, f 1 x, dot. The second coordinate of M is 0, so I put 0 here, dot, the derivative of the curve. That is derivative of x, which is 1, times f 1 prime x. If I calculate, what I get is nothing but M, x, f 1 x. So, once again I have the function M, that function I am considering it. As a function from R 2 to R 2, given by M, x, y; x, y going to M, x, y comma 0. So, it is a function on R 2 that is what I have written here. Then I take the dot product with the derivative of the curve, the curve is x, f 1 x.

So, if I look at the derivative, it turns out to be 1, comma f 1 prime x. Then I look at the scalar product of, that this product of these two vectors. What I get then is, M of x, f 1 x. So, it is very clear then, I keep the first integral as it is, this one is minus, in this direction over c 1, M dot d s. Now, for the first integral again notice, I will be now looking at c 2, which goes in the reverse direction, from b to a. So, write it, from minus integral b to a, M x, f 2 x, d x, minus integral over c 1, which I am not changing. I already got which I want M dot d s. Then by the analogous fashion, if I apply the construction of line integral. Then this is nothing but integral over c 2, m dot d s. So, the end result is, minus integral over c, in this direction M dot d s.

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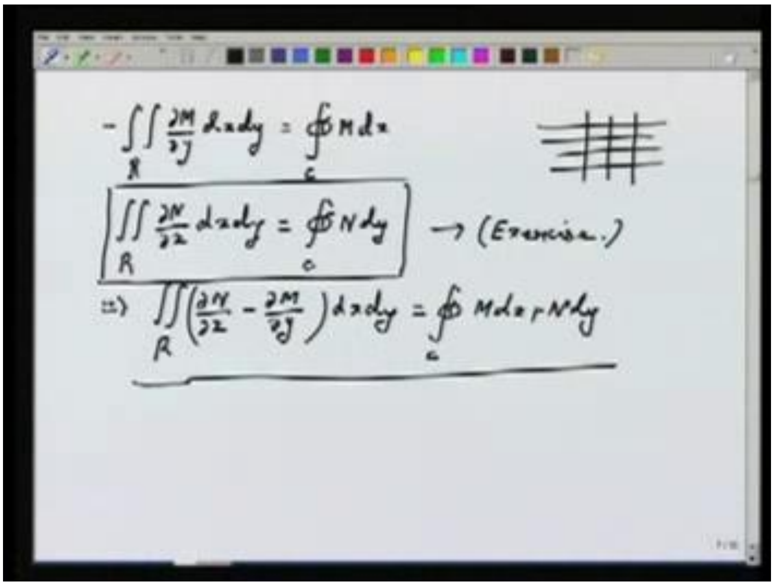

Now, if you calculate, the other one, so what we had proved so far is that double integral, over r minus del M del y, d x d y. That is integral over c, in this direction M, d x. Now it is exactly the analogous calculation. That would tell you that double integral over R, del N del x, d x d y, that is integral over c, N d y. Adding this I get the final result, that double integral over R, del N del x minus del M del y, d x d y, that is integral over c, M d x plus N d y.

Now, this is the equality which I did not prove. I will leave it as an exercise for you, what you have to do is, you use the same technique which I applied for del M del y. But, there I was looking at lines parallel to the y axis and then, I got two curves, f 1 x and f 2 x. Here, what we will do is, we will draw line parallels to the x axis and look at two functions g 1 y and g 2 y. And apply the second fundamental theorem. And the procedure is exactly same as the previous one, there is absolutely, no complications.

But, I am using the fact, that you take any line, which is parallel to any one of the axis. Then it intersects the curve at the most, at two points right. This lines I was using, which I have work with, similarly I have to look at this lines also. So, lines parallel to any of the axis intersects the curve at two points becomes necessary for this proof. But, in general, for the statement of the theorem, it is not needed without this condition also, the theorem goes through.

That is, if you take any simple closed curve c and R is the region bounded by that. You take the functions M and N satisfying the conditions which I said. Then this relation is true, fine. Now, let us see some examples of and applications of Green's theorem. Let us look at the first one.

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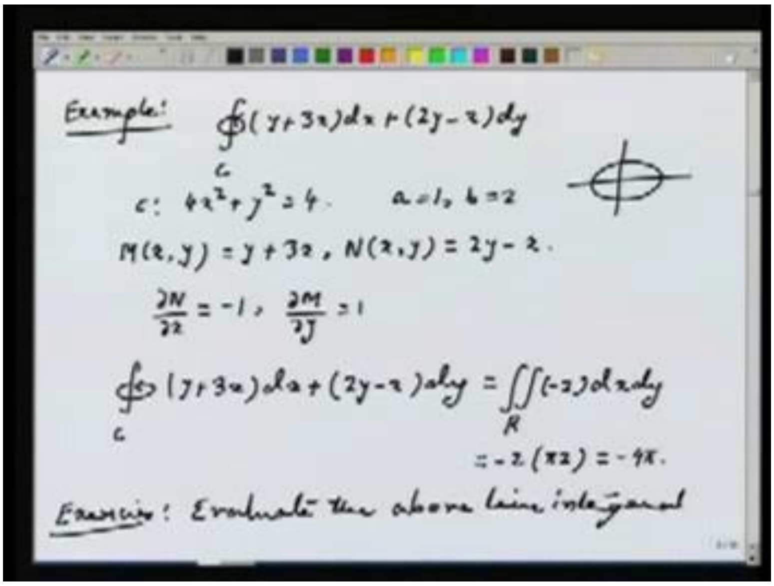

Let us look at this line integral. Integral over c, y plus 3 x, d x plus twice y minus x d y, what is the curve c, c is this curve, 4 x square plus y square equals to 4. That is it is an ellipse, over this curve I am looking at the integral. So, what I do is, to apply Green's theorem. Now, let me assume that M, x, y equals to y plus 3 x and N, x, y equals to 2 y minus x. Then what I will be needing, to apply Green's theorem is del N del x, which is equals to minus 1 and del M del y, which is equals to 1.

So, the given integral, if I call it I, let me take this direction. Then integral over c, y plus 3 x, d x, plus twice y, minus x, d y, that is equals to. Then by Green's theorem double integral over the region which is an ellipse I call it R. Then del N del x minus del M del y which in this case is minus 2, d x d y, that is minus 2 times area bounded by an ellipse of this form. But, if I look at the ellipse, x square by a square plus y square by b square equals to 1. Then we know that, area of the ellipse is actually pi, a b.

So, in this case a equals to 1, b equals to 2, so area is pi into 2. So, the answer is minus 4 pi, which you can check is correct. By evaluating the line integral just by hand, is in the parametric representation of the ellipse a t square twice a t. If you use that and evaluate this line integral. You see that you will get the correct answer, so this as an exercise, I will include, evaluate the above line integral.

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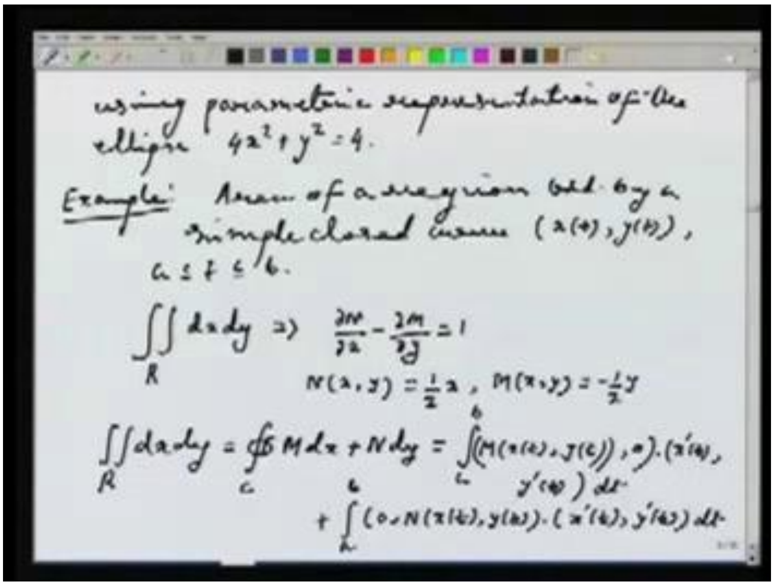

Using parametric representation of the ellipse and verify it is minus 4 pi. Now let us go to the second example, where I will get an expression for an area, area of a region bounded by a curve, well, bounded by a simple closed curve x t, y t, a lesser equals to, t lesser equals to b, let us say. Now, what is the area usually, it is double integral over R, d x, d y, so if I want to use green's theorem, that would imply, that del N del x, minus del M del y, must be equals to 1, otherwise there is no chance of doing this.

So, I choose N x, y, let us say to be equals to half x and M x, y to be equals to minus half y, if I do this, then I can calculate, I can check del N del x is equals to half and minus del M del x is also half, so del N del x minus del M del y equals to 1. So, by Green's theorem, then that would imply, the double integral over R, d x d y, that is, integral over c, that is the boundary M d x plus N d y, this is a line integral, so what does this mean, it means integral from a to b, then M of x t, y t.

So, I will write it in this form, M of x t, y t, then the dot product of the curve, so M comma 0, dot product of the curve, that is x prime t comma y prime t, d t, plus integral a to b, 0, then N, x t, y t, dot product, x prime t, y prime t, d t, now, I know the precise definition of M and N. Now M, x t, y t, sorry this is y t, now M x t, y t comma 0, dot x prime t, y prime t, suddenly, M times x prime t.

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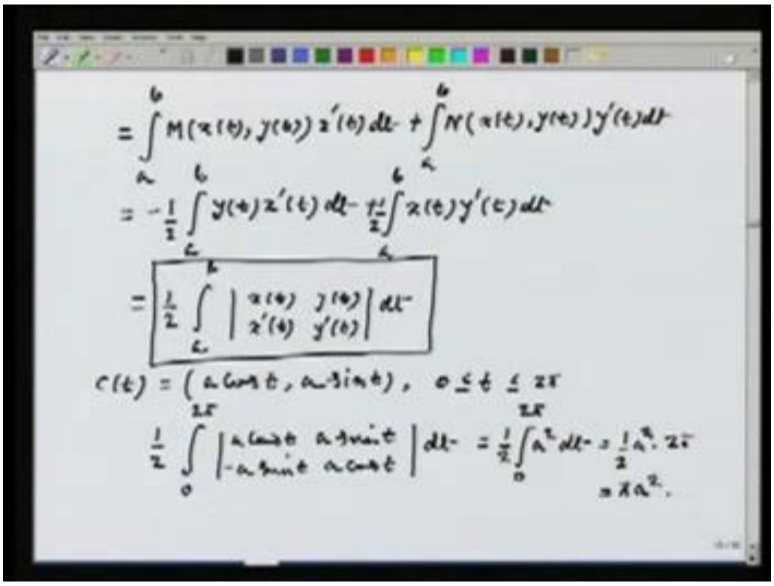

So, I will write it in this form now, this is integral from a to b, M x t, y t, times x prime t d t, plus a to b, N x t, y t, times y prime t, d t. Now, I will apply the definition of M and x, which I anyway know, that N x t, y t is half x t and M x t, y t is minus half y t, so I am going to apply that, so it is minus half, integral a to b, y t times x prime t, d t plus integral a to b, M x t, y t is half x t, so half here, then x t times, y prime t, d t. So, which now can be written as, half integral a to b, if you want to remember, what exactly the point it is.

It is x t, y t, look at this determinant, x prime t, y prime t, d t, if you calculate the determinant, this is precisely what we get. So, let us apply this formula for circle, let us say, so I take the curve c t, that is equals to, a cos t, a sin t, where 0 lesser equals to t, lesser equal to 2 pi. Then, the area according to the above formula is half, integral 0 to 2 pi, what is x t here, x t is, a cos t, so I write, a cos t, y t is a sin t, then the derivatives, that is minus a sin t, then a cosine t, calculate the determinant d t.

Now, calculating the determinant is very easy, it is a square cos square t, plus a square sin square t, so it is just a square, so this half, integral 0 to 2 pi a square d t, so a square comes out, it is half a square times 2 pi, which is pi a square. So, it matches with the formula of the area, for a circle of radius r, so in general, if you have a simple closed curve c, which is given by c t equals to x t, y t, then the region bounded by the curve c has an area, which is given precisely by this formula, with that what you do is, you form the determinant x t, y t, then x prime t, y prime t.

Calculate the determinant, you get a function of t, integrate that function with respect to d t, from the, on the range of the parameter set, that is from a to b, multiply it by half, what you get is the area. Now, we will show you, that there are curves which are not satisfies the criteria, which we have used in the proof of green's theorem, but for those curves, the statement of green's theorem is still true, one such example, is the rectangle.

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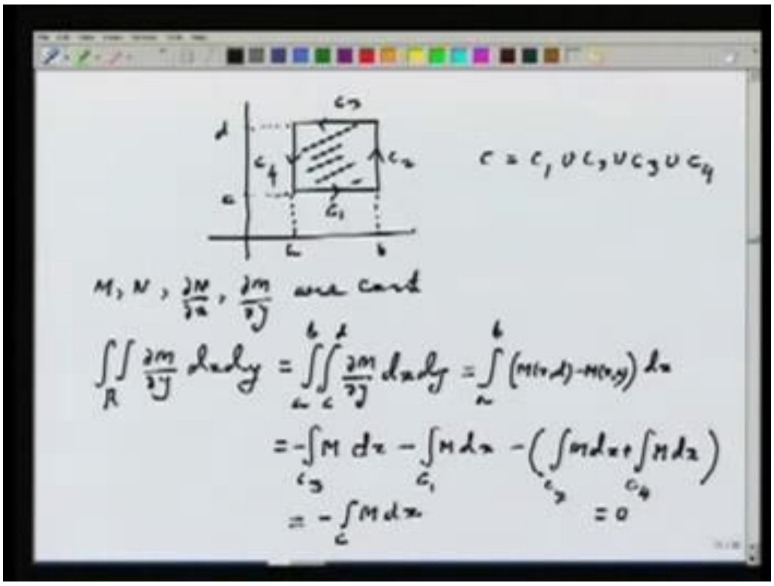

So, let me draw a rectangle, this is a, this is b, this is c and this is d, this is the direction, I break it into four curves, this is c 1, c 2, c 3, c 4, R is the region inside. Suppose, M, N, del N del x, del M del y are continuous, so I try to calculate first, the double integral over

R, let me see del M del y first, it is analogous for the other one, d x, d y, so I write down this double integral as the variation of x, which is from a to b, variation of y is from c to d, I have del M del y, d x, d y.

So, first I do the y integration, so it is a to b, then I have d x here, now the y integral, then is very easy to evaluate by again the second fundamental theorem of calculus, this is M x, d minus M x, c. Now this integrals, I can easily calculate and see, what exactly it is, if I look at a to b, M x d, d x, this is nothing but, integral over c 3 M dot d x, but with the reverse direction, so I will put a minus sign here, because the variation is from b to a now, then minus, what remains is integral a to b, M x c, d x.

That means, I am integrating on the curve c 1, when y is constant, that is c, so this integral is nothing but M d x, well I should not put dot here, this integral is certainly over c 1, but this is not exactly integral over c M d x, because two portion of the curve c 2 and c 4 are missing, so whatever is missing, I write here as, integral over c 2, M d x plus integral over c 4 M d x, ,but notice, that on c 2 there is absolute no variation of x and on c 4 also, there is absolute no variation of x.

That means, this integrals are actually 0, so the end result then, is minus integral over c, M, d x. Now, if I start with N, I can do exactly the analogous analysis and I will get integral N d y, that means, green's theorem is valid, even if I take the rectangular curve, given by here, given by c 1, c 2, c 3, c 4, so c is equals to c 1, c 2, c 3, c 4. Similarly, if I work on some annular region.

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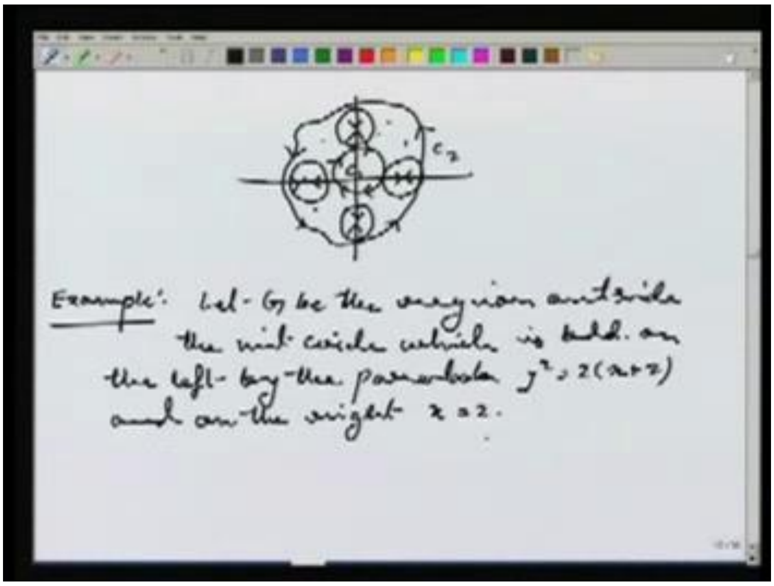

Let us say, this kind of and this is my region R, then it is boundary, let us say, this is c 2, this is c 1, I give this orientation, so where the region is always on my left and here, I give the other one, here what I can do is, I can actually break it into four regions and curves. So, I will use this you know, so this is one orientation of the curve, so this has four parts, where the area is always on the left, but here I will go the other way, then this, this region actually the line integrals cancel each other.

Similarly, if I look at this region, then my orientation is this, this, then this, this and if I and if look this region, it is this, this, so you see what happens is, that the line integrals on this region which I am circling, because of the reverse direction, they will cancel each other and you get, that green's theorem is verified and for this kind of regions also, which are annular region. So, let me show it by another example here. So example, so let G be the region, outside the unit circle, which is bounded on the left by the parabola, y square equals to twice x plus 2 and on the right, x equals to 2. I want to evaluate, so let me draw the curve, let me draw the region here first.

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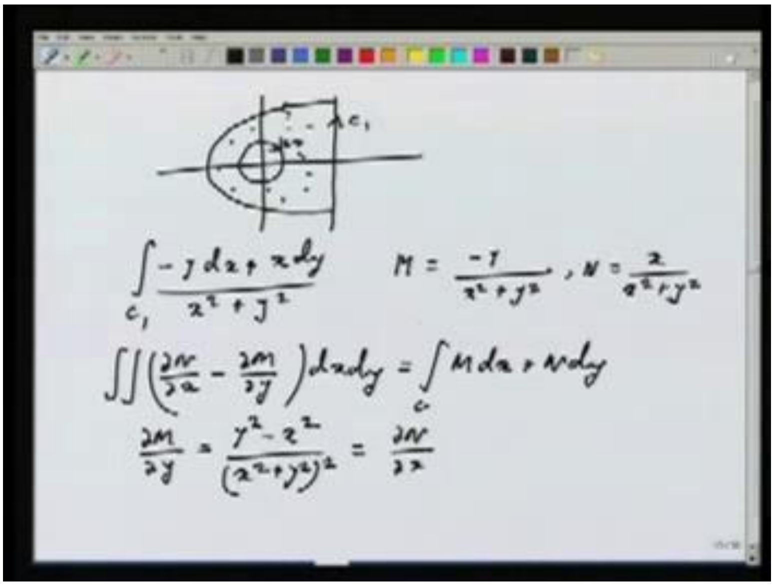

So, this is the unit circle, x equals to 2 and I draw this parabola, so my region is outside, let me call this curve c 1 and the inside curve, I call c 2. I want to evaluate, integral over c 1, minus y d x, plus x d y, divided by x square plus y square. So, obviously, I would choose M to be equals to, because in the green's theorem, the line integral N d x plus M d y, so whatever is way d s, I am choosing as M. So, it is, minus y plus x square plus y square and N equals to x by x square plus y square.

And I apply green's theorem for this region, so what do I get, I get double integral, del N del x, minus del M del y, d x, d y, that is integral over c, M d x plus N dy. So, c here is union of two curves, c 1 and c 2, I am interested only in c 1, by the way what is del N del x and del M del y, an easy calculation tells you that del M del y is actually equals to y square minus x square by x square plus y square, whole square and interestingly, this turns out to be same as del N del x. That means the left hand side integral is zero.

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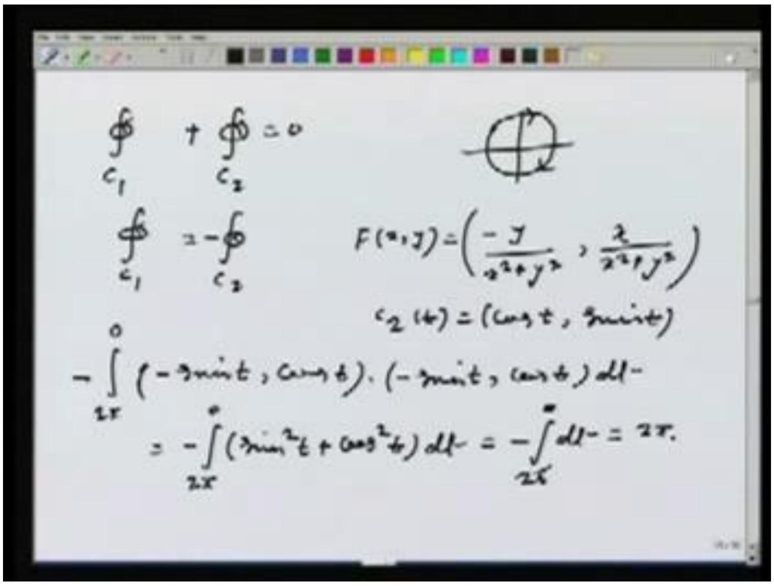

So, that would then mean, that integral over c 1 of the integrant, which I am not writing, plus integral over c 2 of the integrant, that is equals to 0, I am interested only in the integrant c 1, that, then is given by minus integrant over c 2. Now, c 2 I know, is an unit, is the unit circle, so I have to evaluate the line integral of the function, f x y, that is equals to, minus y by x square plus y square, then x by x square plus y square, so I have to do the line integral of this over c 2, where c 2 is the unit circle.

So, I can take c 2 t, actually equals to cos t, sin t, now this line integral is easy to evaluate, I look at minus, then 0 to 2 pi, f of c 2, that is minus sin t, then cosine t, I am not writing the denominator, because cos square plus sin square is 1, dot. Now, c 2 prime t, which is minus sin t, cosine t, d t, but this is minus, well because of the reverse direction, this integral actually is not 0 to 2 pi, it is from, is 2 pi to 0, because if you remember the direction of the circle was, this way. So, it is start this way and goes to the other way, that is why, the minus sign 2 pi to 0, so it is minus of sin square t, plus cos square t, d t that is equals to minus integral 2 pi to 0, d t. That means, it is just equal to 2 pi.

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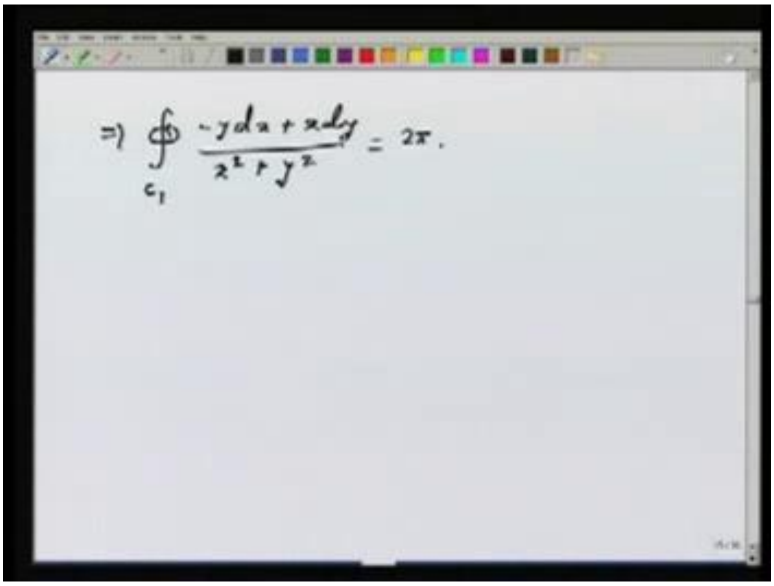

So, that would mean, that integral over c 1, in this direction, minus y d x, plus x d y by x square plus y square, is the integral over the c 2 portion, which I have just calculated is 2 pi, so this is, all I want to say about green's theorem. That you have a function, whose double integral over a region is connected to an integral, which has something to do with those given functions, over the boundary of that region. Now, we have given a proof of that result, where the boundary curve satisfies sudden condition, except simple closed.

It is there in the statement of green's theorem, that we always take boundaries, which are simple closed curves, but I have assumed something additional and that I have given you the proof, but I have shown you one example, which shows that those conditions, which I am assuming on the curve, c are not really needed. In fact, you can assume, that if c is just a simple closed curve, then green's theorem is true and using that I can show that given annular region greens theorem can be applied.

And using that, I could show that, given an annular region, green's theorem can be applied and using that, I have calculated certain line integrals, which usually looks complicated, but if you use green's theorem, then it can come down to a simpler integral and another application of greens theorem, which I had shown is that how to get an explicit expression, of the area of the region in terms of a boundary of the region.

Using that, we could calculate again the area of the disc of radius r in terms of it is boundary, which is the circle of radius r. So, that is it, for this lecture and in the next lecture, we will go to a triple integrals and try to come down to surface integral and some double integrals, so that would be the generalization of the green's theorem in higher dimension, that is in three dimension.