**Mathematics-I**
**Prof. S. K. Ray**
**Indian Institute of Technology, Kanpur**

**Lecture - 31**
**Stokes Theorem**

Today we are going to talk about a generalization of Green's Theorem, which is known as Stokes Theorem. So, first let us recall what is Green's Theorem?

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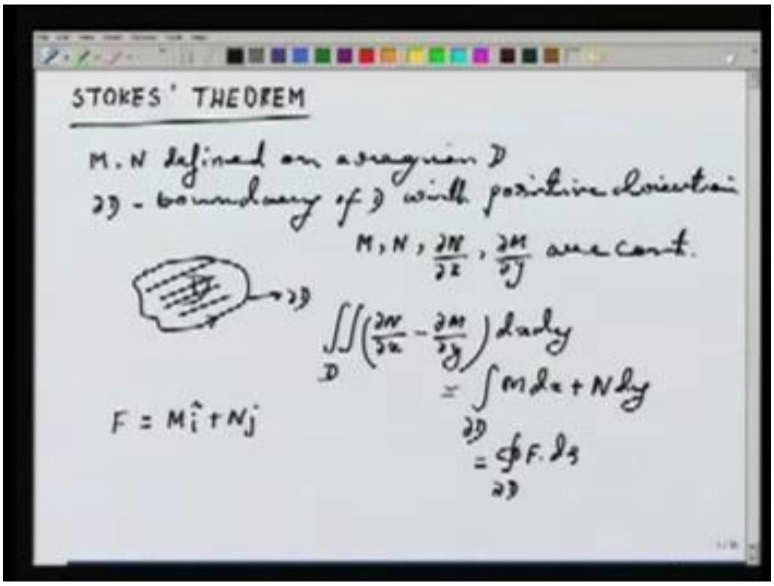


So, the theorem was that suppose I have the functions M and N defined on a region D. And let us say delta D, this denotes the boundary of D with positive directions. This positive direction just means that the orient is boundary in such way. That the area region D is on the left hand side of myself. That is, if this whole thing is my region D. And if I want to keep the region left I have to give this boundary this direction, with this direction this thing is called delta D. That is the boundary and the inside region is D.

Now, if I have further conditions on M and N such that, M and N. And the partial derivatives del N del x, del M del y these are continuous. Then Green theorem tells us that double integral over D del N del x minus del M del y d x d y. This is integral over delta D M d x plus N d y, where the right hand side denotes the line integral of the vector field F which I can write as M i plus N j. Then this is nothing but, integral over delta D F dot d x. This is the usual line integral.

So, from a double integral we are coming down to a low dimensional integral. That is the line integral, that is what is the essence of Green theorem? So, that way it generalizes the second fundamental theorem of calculus that we have already seen.

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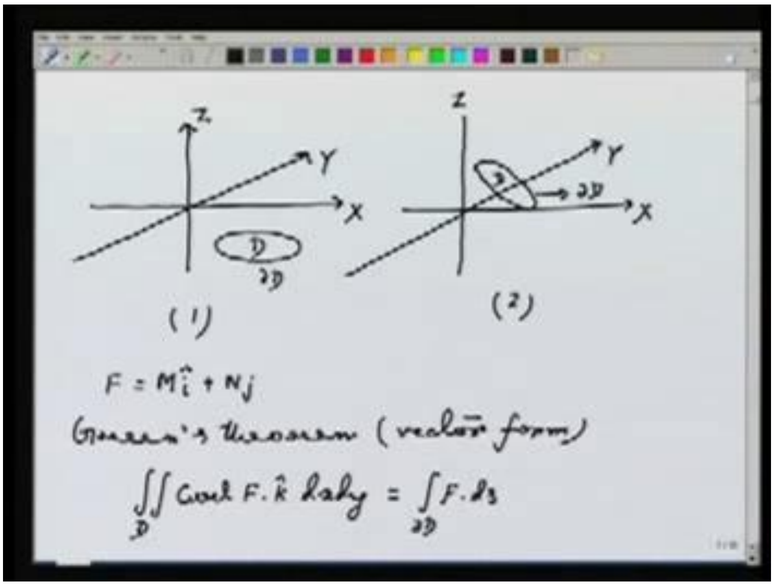

Now, today what I want to do is can actually explained by the following picture. So, what we have done in Green theorem is something like this. So, these are my axis, this is x, this is y, this is z and this, let us say is my region. So, this is outside boundary is delta D inside thing is D. And then, I had given a function whose double integral over D comes down as line integral of some other function on delta D.

Now, the question is, what are we going to do if I am in a situation as the following. Let us draw the picture again. So, this is x-axis, this is y-axis, this is z-axis. Now, notice that the region D which I have drawn in the picture one, that is lying in the x y plane. But, suppose this region D is not lying in the x y-plane. Suppose it is in some oblique region like this. Suppose this is my D, it makes sense to talk about delta D again. But, in that case what can we do, is there any analogous of Green theorem in this kind of situation. In picture 1 I know what to do, question is about picture 2. That means, D is not on the x y plane, it is on some oblique plane than what do you do.

Now, to understand that we needed a listed statement of Green theorem, usually known as the vector form of Green theorem, well it is just as follows. Consider the function F which is as I said before M i plus N j. Then I said Green's theorem states the following, this is called the vector form. So, what is the vector form, it is given as double integral over D curl F dot k d x d y. This k stands for vector, that is equals to integral over delta D F dot d s.

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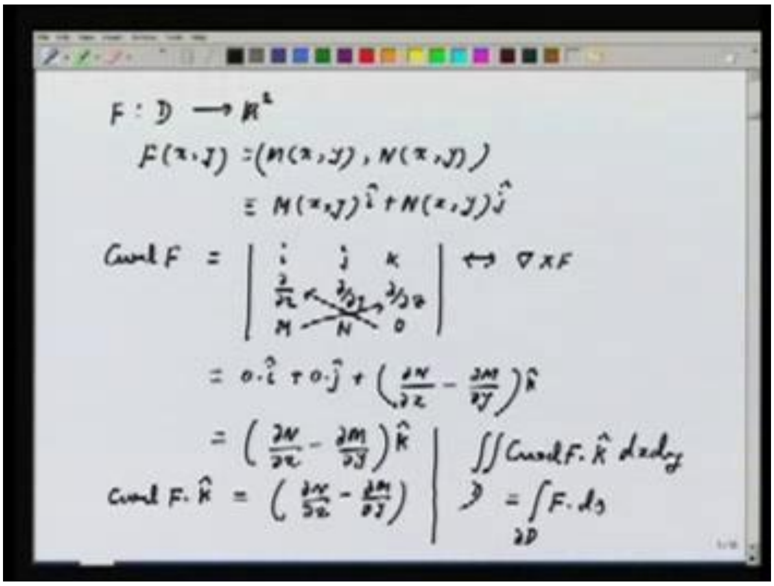


Now, for those who do not know the formal definition of the curl F, first let me say that. So, F is defined on the region D to R 2, so it is a vector field. I know already the definition of F, F of x y, that is M x y comma N x y. I can write it is in this form or equivalently in this form also. M x y i plus N x y j, then what does curl F stands for. Well, curl F means, just value of this determinant i del del x M, j del del y N, then k del del z 0.

It is also formally written as grad cross F. This symbol this inverted big delta. When we say this dot F, we mean gradient of F. And then F has to be a scalar valued which produces a vector field. But here in this case grad cross F formally means, this determinant i j k del del x del del y del del z, and then the components of the function, the x component, the y component and the z component.

So, if I calculate this, what does it come to, let us see. Del del y of 0 is anyway 0, now I have to look at del del z of N. But, N is the function of x and y, so del del z of N is 0. So, I get 0 dot i, that is the first quantity, then I look at j. Then I have to multiply, apply del d z on M. And then, 0 on del del x. Now, del del d z on M is again 0, because M is function of x and y, it is independent of z. So, that quantity is 0, so is del del x of 0.

So, I get 0 dot j, now what remains is k. So, that component I will write, that turns out to be clearly from the formula of determinants del N del x minus del M del d y k. So, curl F is actually the vector del N del x minus del M del y k. So, this is the particular case of the definition of curl of F. But, anyway let us continue with this particular case, then I will come to the general definition later.

Now, then what is curl F dot k. This turns out to be just del N del x minus del M del y. Now, if I want to use Green's theorem, the left hand side of the Green's theorem is double integral over D del N del x minus del M del y d x d y. So, that can be written as double integral over D curl F dot k d x d y. And then, by Green's theorem the right hand side is F dot d s. This is what I said the alternative vector form of Green's theorem.

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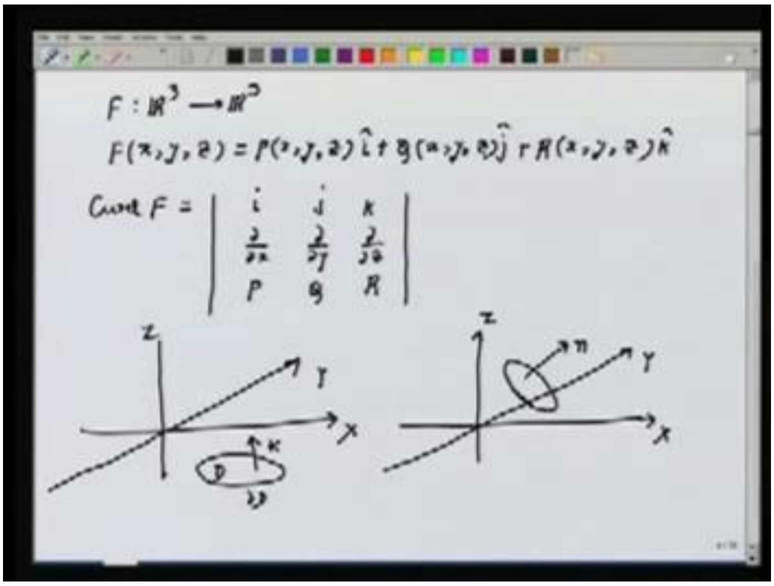


Now, in general given a function F from R 3 to R 3, it will be then of the form F of x y z that is equals to let say P x y z i plus Q x y z j plus R x y z k. In that case, curl F stands for this vector field i del del x P, j del del y Q, k del del z R, this is called curl F. So now, we know the general definition of curl. Now, how do we generalize this theorem of Green to Stokes, is actually given by generalizing the vector form of Green's theorem. So, look at the general form of Green's theorem. And try to see where exactly we are using the fact that my region is on x y. And then we try to generalize it for arbitrary surfaces.

So, stokes theorem, what it will do is that you have given a function and a surface. A very special kind of surface not an arbitrary surfaces, which is bounded. Let us say that the boundary of the surface is a simple closed curve. Then the surface integral of some function is connected to the line integral of the function over the boundary. That is what the Stokes Theorem says… So, when the surface integral actually in the x y plane, you know.

Then, the Stokes Theorem actually reduces to Green's theorem. So, that is a generalization. Now, how do we generalize, so again we draw the picture. These are my axis, this is my D, boundaries is delta D. Now, when you look at the statement of Green's theorem, there is some k which is coming. I am talking about the vector form of Green's theorem. So, it is curl F dot k, what exactly k has to do with this surface, that is what you want to understand.

Now, it is very simple, if you look at the normal to the surface in this direction this is k, this is the k direction. Now, in the general situation, this is x, this is y, this is z. Suppose, I have a very simple surface like this. This is D, the same d, but this time it is not on the x y plane, and I am putting it on some oblique position. Then, who should play the role of k, well the answer is natural, that you look at the normal to the surface N, which goes in this outside direction. That N is going to play the role of k. And the other thing which you have to be worried about is about this meaning of this outward normal.

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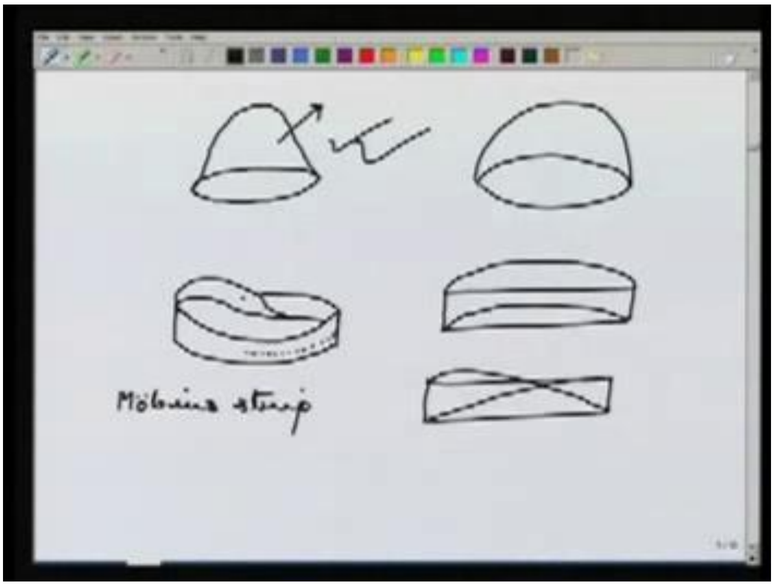

Now, intuitively given any surface, let me try to draw a surface like this. Suppose, this is the surface and this disc portion which I have written is the boundary of the surface. Now, when I am talking about this kind of a surface, I can certainly see, that this surface has two sides, one is outside other is inside. Now, what is outside and what is inside that can vary from person to person.

Well I call the outside direction is this, that is what I am going to assume. So, given any surface which has this kind of phenomena. That means, it has two sides, I can talk about the outward unit normal vector. And we can actually derive the formulas of the outward unit normal vectors also. Now, the only trouble is which is bit counter intuitive, but it can happen. That there exists surfaces which may not have two sides. It might happen that the surface has only one side.

Now, mathematically to determine, how do you exactly express side of a surface, is bit complicated. So, you try to understand in a intuitive fashion, what exactly is the side of the surface. For example, given this surface, suppose I what to paint one side of the surface. So, I can start painting one side of the surface and can finish it. Then the other side of the surface, there is no paint there, that is certainly true for this surface, it is like a hat actually.

So, you can always paint the outside of a hat or you can paint inside of a hat. Or if you like, suppose you have two colors red and blue. You use the red color on the outside of the hat and the blue color on the inside of the hat, that is possible. But, there are surfaces for which it is not possible. So, one such surface I will draw for you, it is called Mobius Strip. It is a very interesting surface, because of the following reason. And then, do you understand, what exactly I mean on this surface. So, suppose if you have band of paper.

So, that paper if you glue both sides, then it becomes like a cylinder. But, without doing that, what you can do is, given the band of paper just do not glue these two sides together, twist the paper and then glue. So, in the cylinder form what happens is, you have this, you glue this side to this side. And you glue this side to this side, it takes the form of a cylinder.

But, suppose you do not do this, what you do is the following, that you twist and glue this side to this and this side to that. Once you do that then there is a twist which comes in. Now, the problem is, if you start painting one side of the surface, you land up doing both sides. So, let us see why it happens. So, suppose I have start painting this side. So, I am going on painting this side, I go the other way. Then on the other side, I am painting the other side. But, then there is a twist here.

So, from the outside now, I come inside. So, I am painting the inside portion now. And then the whole inside also get painted, you see. So that means, this surface has got only one side. Since, there is a twist on one side of the surface, through that twist you are coming from outside to inside. So, there is exists only side, so this is called Mobius strip. For us and for stokes theorem, this kind of surfaces are bad surfaces. We are not going to deal with this kind of surfaces.

We will deal with only those kinds of surfaces which has got two sides two possible distinct sides like this picture. You know like a hat or let us say dome of a sphere, this has a well defined two sides with which we can work. So, these are usually called Orientable surfaces. But, let us not go for those technical terms, intuitively all I want to say is, we are going to deal with surfaces which has two distinct sides. And then, it would make sense for us to talk about outward unit normal vector. That means, I can talk about normal vectors in this direction.

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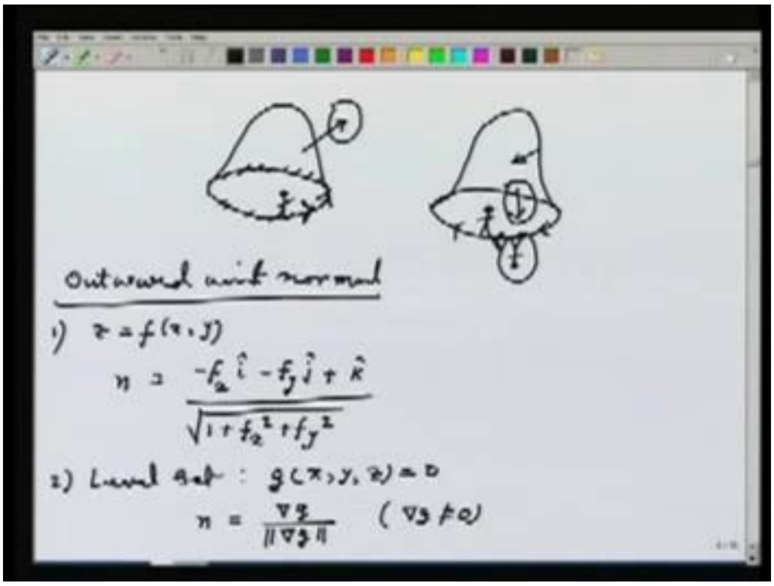


Now, again I have to talk about the direction of boundary. Because, it is involved in Green's theorem also. So, suppose this is my surface, so the boundary is this one. This is the boundary of the surface. I demand that the boundary of the surface should be a simple closed curve. On that boundary, suppose I choose the outward unit normal vector in this given direction, according to the picture. Then what is the direction on the boundary which you are going to work with.

Well, the point is if I stand here. Suppose some person is standing here, with his head towards the direction of the outward normal vector. Then, the surface should be on the left hand side of yours that is the direction. So, according to this picture, the direction is this, because if I am walking in this direction, my head pointing towards the outward normal vector the surface is on my left. To understand it, let me again draw this same picture, this is my surface, suppose I choose the normal vector to be inside. So, near this point normal vectors are pointing downward. Suppose some person stands here. Suppose now this guy has his head to the opposite direction of the normal vector. So, that is not the correct way to walk. So, the correct way to walk would be this way. That your head is pointing towards the direction of the normal. And then, you want to have your surface on the left hand side that means, this is the correct direction, is this clear now.

Look at the first picture. The outward normal is this one, the person is standing here, with its head pointing towards the unit normal. Then, the correct direction to walk is the direction. So, that if you walk in that direction, the surface remains on the left hand side of yours. In that case, the correct direction is this, what I have drawn here look at the picture again here, suppose I choose this normal direction which is going downwards. So, the person walks here with his head downwards, if it is possible to walk.

Then, the correct direction is such that if you look at the left hand side of yours. The surface is on the left hand side in that case, the correct direction to walk is this. Because, if you look at the right hand side of this person with his head downwards. If you look at the right hand side the surface is not there. So, this is the correct direction to choose, whenever we try to apply Stokes theorem. For that it will become necessary for us actually to draw the pictures and to understand which way we take the normal.

And if we choose one normal, then how do we choose the direction in the boundary, because you see from this picture, the orientation in the boundary. That means the direction in which I am going to do the line integral. Actually depends on the chosen direction of the normal. Now, let us come to the statement of stokes theorem. Well before that, let me tell you some thumb rules, which you always going to apply, that how to find the outward unit normal vector for certain given surfaces.

I will tell you just some formulas that will always work for you. So, I will just write out some formulas of outward unit normal for you. We always work with these normals. So, surface can be of the form z equal to F x y that is one case. In that case, the outward unit normal vector is given by, divided by the normal. So, that is 1 plus f x square plus f y square. Second case is, when it is a level set of some nice function g, that is it is of the form g x y z equals to 0. In that case, the formula for the outward unit normal vector is grad g divided by the normal grad g of course assuming that this is non zero. So that it would makes sense.

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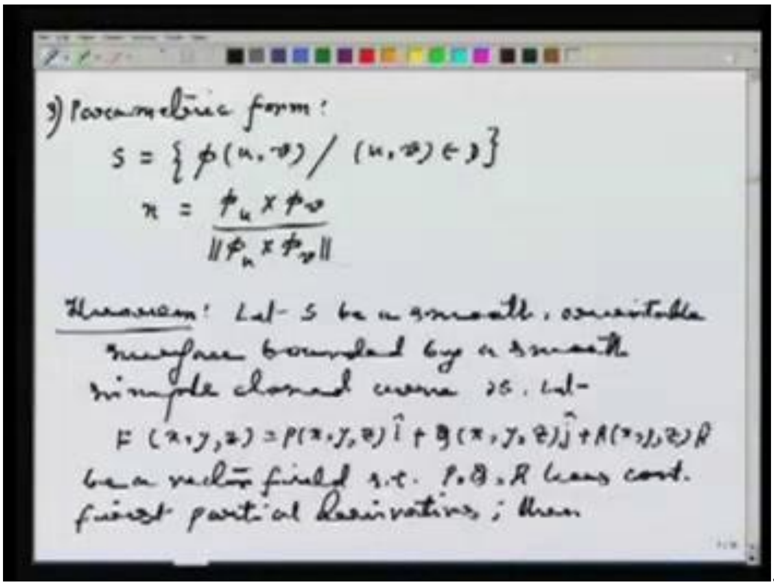

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Then comes to the third case the parametric form, so if the surface is given by, then n is we know how this formula comes actually this last one is the most general. That given the parametric form what you do is you fix d. Then it becomes a curve on the surface, differentiate with respect to the first variable. That gives you tangent in one direction. Then what you fix what you do is, you fix u it becomes a curve in the parameter v. That is the another curve on the surface, differentiate to get a tangent in the in that direction.

Take the cross product of these two tangents vectors. That gives you a normal, that normal we are taking as the outward unit normal vector. And then, you divide by the norm of that vector to make it a unit normal. So, these are the three formulas which we are going to work with now. Now, let us come to the statement of stokes theorem. So, let us be a smooth, I will explain the meaning of all these words. Smooth orientable surface bounded by smooth simple closed curve which we might call delta s. Let F x y z be a vector field, such that P Q R has the continuous first partial derivatives.

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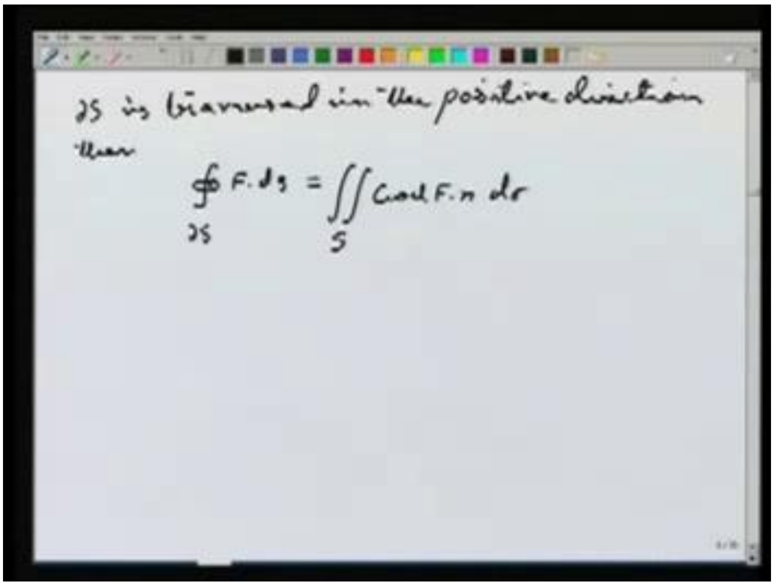

Then if delta is traversed in the positive direction, then we have the following formula, integral over delta s with the proper direction F dot d. This is the simple line integral this is equal to double integral over s curl F dot n then d sigma, where d sigma is the surface area of the surface s. So, you see the similarity with the Green's theorem. There this double integral over s that actually was double integral over d. Where d was the region in the x y-plane. And then, the measure we have used then the area of integration was just d x d y. And the role of n was played by the outward normal k that was towards z-axis in the positive directions, so here since the surface can be on in some other ways in the space. So, instead of k, we are going to take the outward unit normal vector.

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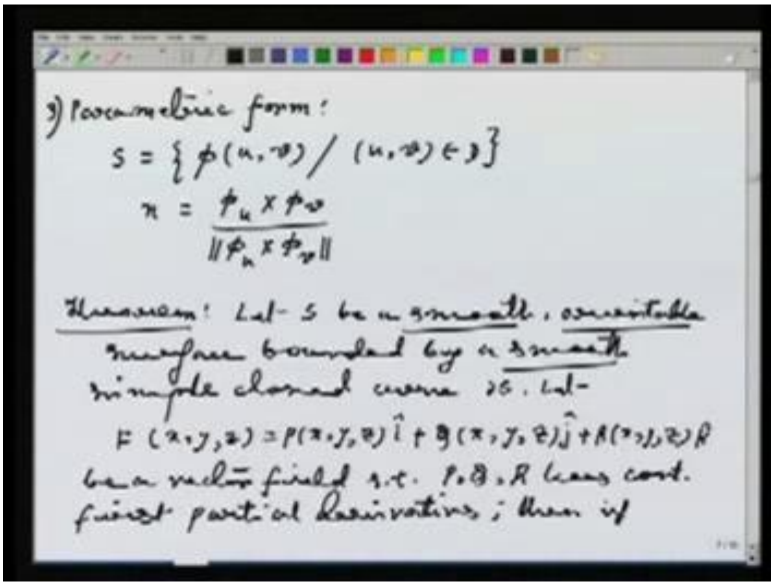


So, in the Green’s theorem k is actually analog of n. Only extra thing here is d sigma, there the analog of d x d y. Now, given any surface s, I can calculate what is d sigma? So, there is no problem in that. Now, the only the technical language here I have used is a smooth orientable surface. There is another smooth here, well orientable surface just means as I said that you have two well defined sides outside. And inside, if that is possible we call the surface is orientable.

It is not strictly speaking is correct in the terms of mathematical language. But, intuitively it is orientable means I just got two sides. I can talk about the outward unit normal and the inward unit normal. Now, smooth means what, well given any surface s it is either level set of functions. Or it is graph of a function or it is given by some parametric function phi. Well, smooth just means that all these functions has partial derivatives of all orders, it is just the technical condition which I am assuming. So, smooth means, if it is level set of a function g, g has partial derivatives of all orders. If it is graph of a function F, then F has a partial derivatives of all orders. Similarly with phi the parametric function that is what smoothness.

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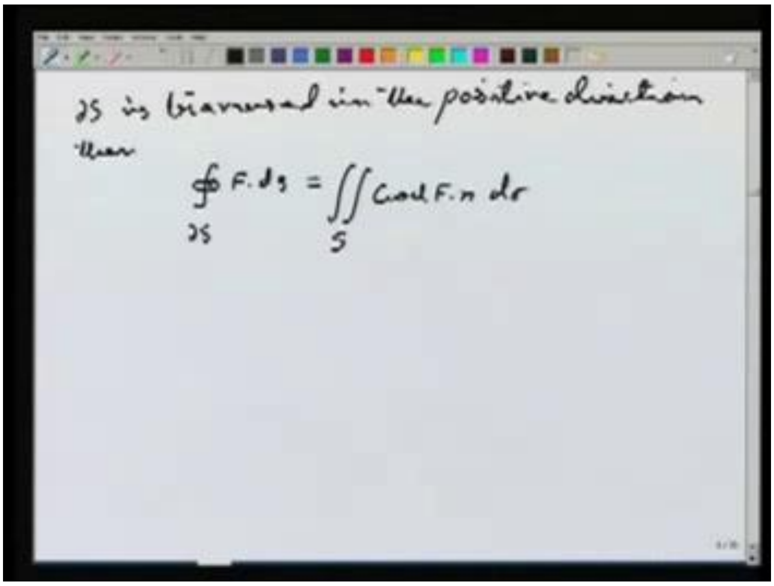


Now, to understand the theorem better what I will now do is. I will take an example, and in case of that example I will just try to verify stokes theorem. This theorem is not very difficult to prove. It just uses Green's theorem you can and change of variable formulas. Chain rule such elementary things, you can look at any standard calculus book for the proof of stokes theorem.

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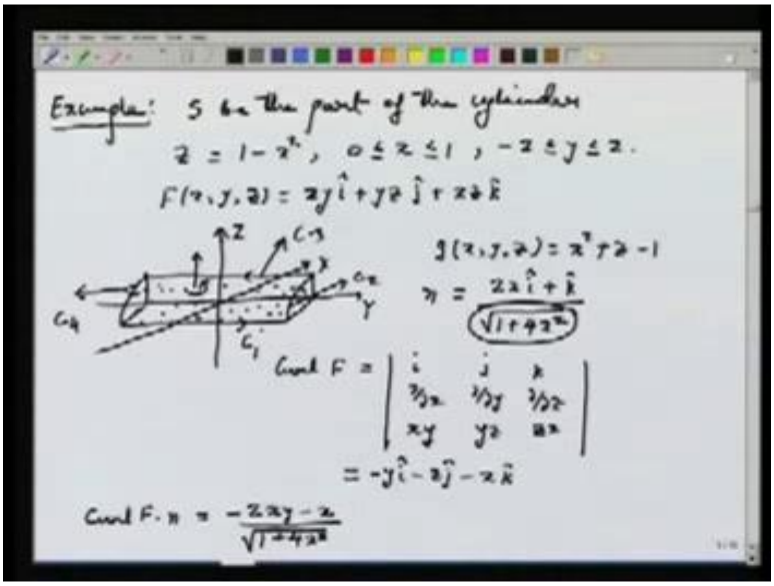


But, we are going to assume the stokes theorem statement. And then I am going to verify it, on a particular example and then apply it in certain situation. So, to verify stokes theorem let us look at the following example. So, first we need the surface. So, let s be the part of the cylinder, z equals to 1 minus x square, the range of x is and y lies between minus 2 and 2. So, this is the surface which is given to us. And the function that is the vector field. Let us say that is given by very symmetric kind of x y i plus y z j plus x z k.

So, here first we have to understand what is the outward unit normal? And hence the direction of the boundary of that surface. So, we try to draw the surface first. So, I draw the surface for you, so this is y axis, this is x axis, let us say this is z axis. Then the surface looks like this. So, the surface is this curve portion. So, here obviously, the outward unit normal vector is this. It goes in the top side and hence, it is easy to determine what is the direction in the boundary, that is this, then this, then this, this.

Because, if I am walking in this direction, putting my head towards the normal direction. Then the surface is on the left hand side of myself. So, the surface is this portion, this top curve portion is the surface. So, first I need to find, then what is the expression of normal. So, I will use the fact that this is the level surface, given by f of x y z that is equals to x square plus z minus 1. Then n that is grad g; that means, twice x i plus k divided by the norm. That is square root of 1 plus 2 x whole square. That is 1 plus 4 x square. This is the outward normal you see the positive side appearing in k.

Then, the next thing I need is curl F. So, I will calculate, what is curl F, then i del del x, j del del y, k del del z and then x y, y z, z x. If I want to calculate, what do I get, I get i times del del y of z x which is 0. Because, z x has no y minus del del z of y z. This has 1 z, so it is minus y i. Then comes j, with j I first apply del del z on x y which is 0. Then minus del del z of z x which is minus z j. Then k I first apply del del x of y z which is 0, because it has no x, minus del del y of x y which has 1 y, so it is minus x k.

From these two data I can certainly calculate. What is curl F dot n, so curl F dot n. This is the integrant since I have the formula of n and I have the formula for curl F. If I do that what I get is, minus twice x y comes from the i component. Then j component in n is 0. So, it is only the k component which remains it is then minus x. So, it is minus 2 x y minus x.

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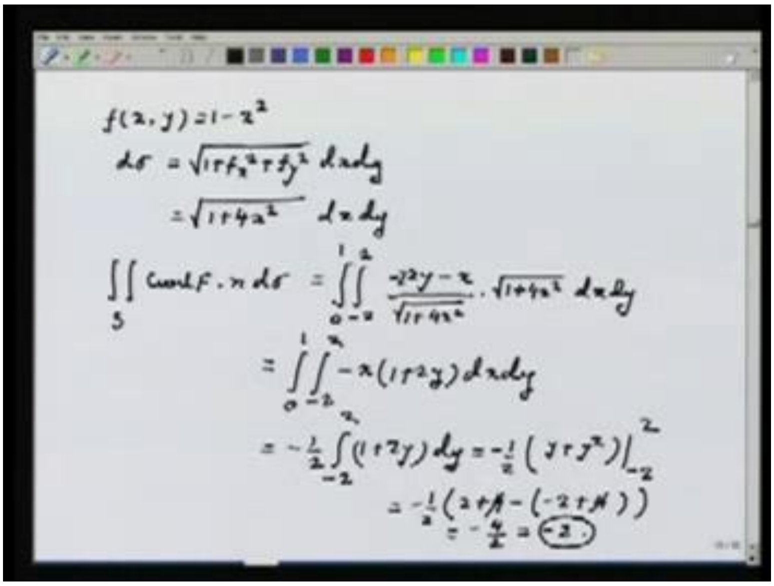

The next thing we need to calculate is the area of integration on the surface. So, what I do is, I look at the surface as f x y equals to 1 minus x square. In that case, I know the formula for d sigma is square root of 1 plus f $_x$ square plus f $_y$ square d x d y. Now, this is easy to calculate it is square root of 1 plus f $_y$ is anyway 0, because function has no y in it its independent of y. So, it is only f $_x$ square, but f $_x$ is minus 2 x. So, square is 4 x square d x d y. So now, I have to calculate the left hand side that is integral over s curl F dot n d sigma. So, this is now I know the parameterization. That is the variation of the x and y well the variation of x is from 0 to 1. Variation of y is from minus 2 to 2, curl F dot n anyway I have calculated, it is minus twice x y minus x divided by 1 plus 4 x square.

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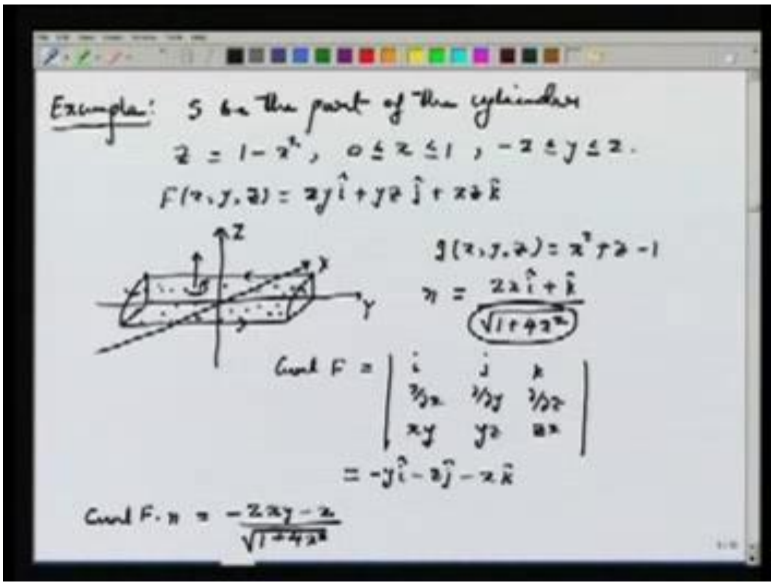


If I go back to the previous formula, what happens is well this formula is wrong, curl F dot n is minus x y minus x is correct, but one quantity square root of 1 plus 4 x square, because in n this quantity is sitting. Then, in this expression of curl F dot n d sigma double integral over s, I have to multiply with d sigma, but d sigma is square root of 1 plus 4 x square and then d x d y. So, what I get is integral from 0 to 1 integral from minus 2 to 2 minus x into 1 plus twice y d x d y.

Which then is minus of first I do the x integral, that is x square by 2. But, limit is from 0 to 1. So, I get half then minus 2 to 2 1 plus 2 y d y which is minus half into y plus y square from minus 2 to 2. So, I get minus half then 2 plus 4 minus minus 2 plus 4. So, these two fours cancel each other, what I get is minus 4 by 2 which is minus 2.

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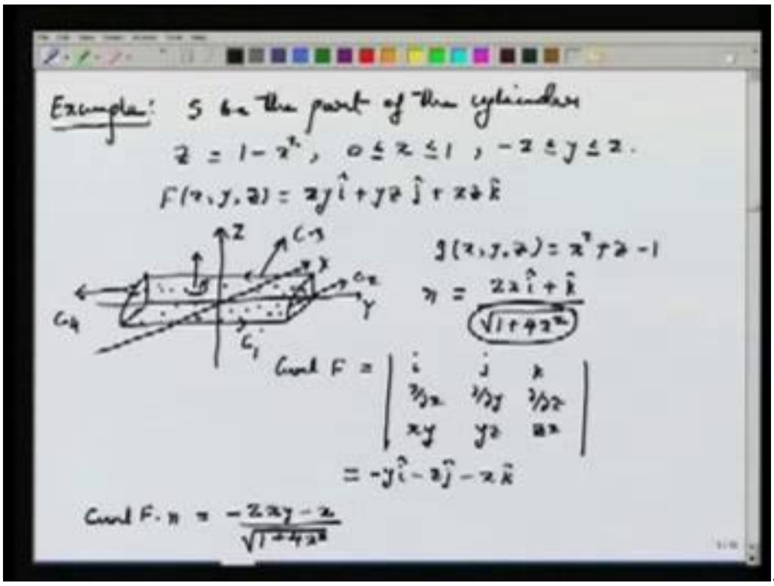


So, the left hand side in stokes theorem in this particular case turns out to be minus 2. Now, I have to look at the right hand side also. Now, before that let me go to the picture again. What is the boundary curve that is the portion of these two lines which are horizontal. And then, the curved line, these curves. So, I break it into four pieces, the first curve I call c 1, second curve that is this one, I call c 2. Then the third curve which I call c 3. Then the fourth curve which I call c 4. Now, to do the line integral of the function, I need to know that. What do exactly these curves represent, so let me note it here first that what is c 1 t.

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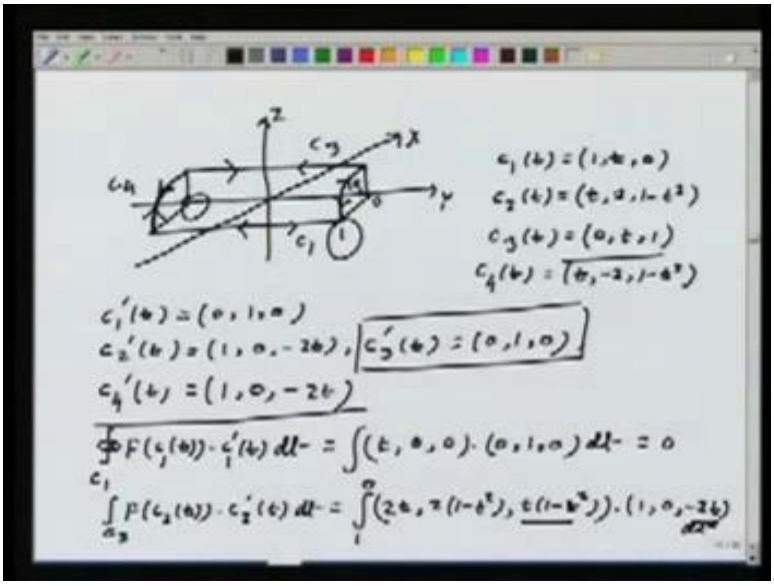


In c 1 I noticed that x is fixed, z is 0 only y is varying from minus 2 to 2. So, we draw the picture again, to understand the equations of the curves, this is y this is x this is z. So, this is c 1 this is c 2 this is c 3 this is c 4. So, what is equation of c 1, so c 1 t is x is fixed, it is 1 y is varying. So, I call it t z is 0 this is c 1 t. Then I have to go to c 2 t, in c 2 y is fixed it is always equals to 2. Only the other one is on the surface, so it is t 2 1 minus t square.

Then c 3 which is again like c 1 it is a horizontal line. But, in c 3 what is happening is x is 0 y is varying and z is 1, because x is 0 and z satisfies 1 minus x square. So, if x is 0 z is 1, so it is 0 t 1 then comes c 4 which is analogous to c 2. And that is t minus 2 1 minus t square, because I am on the other side of the y axis. Once I know all these, I can calculate primes also. So, c 1 prime t is 0 1 0, c 2 prime t that is 1 0 minus 2 t, c 3 prime t that is 0 1 0, c 4 prime t 1 0 minus 2 t.

It is analogous to c 2 prime t, now I need to know. What is F of c t and then integrate in a correct limit, so over c 1 when I am integrating. So, it is this over c 1 F of c t dot c prime t d t. So, what is the limit of c 1 what varies in c 1. I see t varies that is the y co-ordinate and how does it vary, well before going into that calculation. Let us first check, what is F of c t into c prime t.

So, what is F of c t, F is x into y plus then y into z then z into x. So, this is then, it is t that is x y, y z is 0, z x is also 0, dot c 1 prime t that is 0 1 0 d t with some limit. But, that limit I do not need to calculate, because I see the dot product inside is already 0. So, I get that this is 0 without even calculating the limit.

Now, I have to go to c 2, that is F of c 2 t dot c 2 prime t d t. Now, I need to know what is the limit of integration. Well, here it is very easy in c 2 the variation is only of x and it is varying from 1 to 0. So, this variation is from 1 to 0, see that depends on the chosen direction I am going in this direction. So, here x is 1 and here it is 0.

So, it is going from 1 to 0. So, let us calculate what is F of c 2 t that is 2 t that is x y, then y z that is 2 into 1 minus t square, then z x that is t into 1 minus t square times. Now, the dot product of c 2 prime t that is 1 0 minus 2 t d t, if I calculate this. What I will get is the second portion goes away because there is 1 0. So, it is 2 t times minus 4 t into 1 minus t square.

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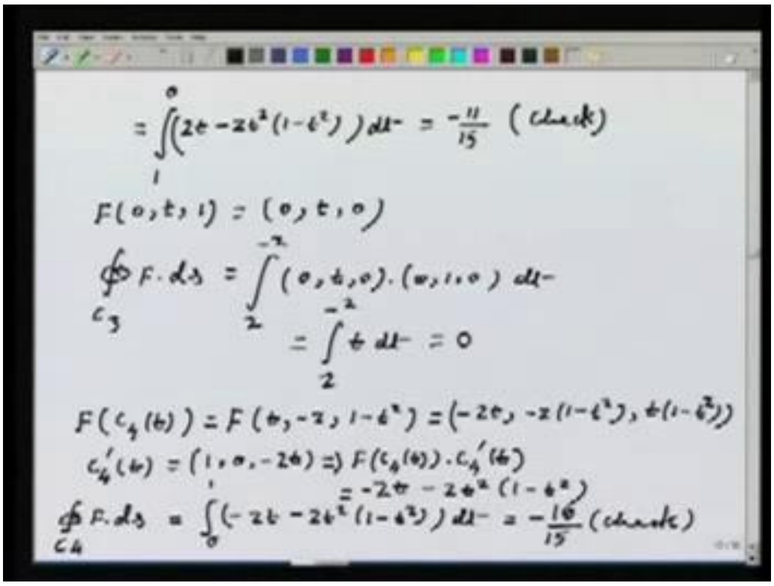


So, this is then equals to integral 1 to 0 2 t into. Let us check it is the product of this quantity into this quantity that is minus 2 t square into 1 minus t square. So, this is minus 2 t square into 1 minus t square d t. Now, this integration can be easily calculated and if you calculate this. This I leave as an exercise that it will turn out to be minus 11 by 15, check this.

Then c 2 has been taken care of, I have to now go through c 3. So, what is F of c 3, I know the formula of c 3, this is 0 t 1. So, I have to calculate what is F of 0 t 1? This is x y which is 0, then y z then z x that is again 0. So, integral over c 3 in this direction F dot d s I have to look at, this is integral. Then I have to see what is the limit of c 3? c 3 varies from this way to this way. So, this is from 2 to minus 2 that is, from 2 to minus 2 F of c t that is 0 t 0 dot. Now, I need c 3 prime t, so this is c 3 prime t 0 1 0, so it is 0 1 0 then d t.

So, this is integral 2 to minus 2 t d t, this is an odd function whose integral I am doing from 2 to minus 2. So, this integral certainly 0, you can also calculate it is t square by 2. So, half comes out, so 2 to minus 2 t square which is 4 minus 4 that is 0. So, c 1 does not continuing anything I got 0, c 2 contribute anything I got 0, only integral over c 2 came out to be minus 11 by 15. Now, I have to calculate what is F of c 4, that is F of what is the equation of c 4 t this is t minus 2 1 minus t square.

So, t minus 2 1 minus t square, I can calculate from the formula of F, so it is x y then y z then z x. Now, I need c 4 prime t whose formula I know. I have calculated it once it is 1 0 minus 2 t. So, this is 1 0 minus 2 t this implies, I can calculate F of c 4 t dot c 4 prime t that turns out to be minus 2 t then minus 2 t square into 1 minus t square.

Now, I have to integrate to integrate. I need to know the limit of t since it comes this way the variation of t. That means, of x is actually from 1 to 0. So, this last integral, integral over c 4 F dot d s this is integral from 1 to 0. Let us look at it again, x varies from here to here this anyway I know is 1. This length is 1 as I have written here and this is 0. So, the variation of x in this case is the reverse of c 2. In case of c 2 it was varying from 1 to 0, in case of c 4 it will vary from 0 to 1. So, what I will get is integral from 0 to 1 minus 2 t minus 2 t square into 1 minus t square then d t. This integral is also easy to calculate and if you do this, you will get that this is minus 19 by 15.

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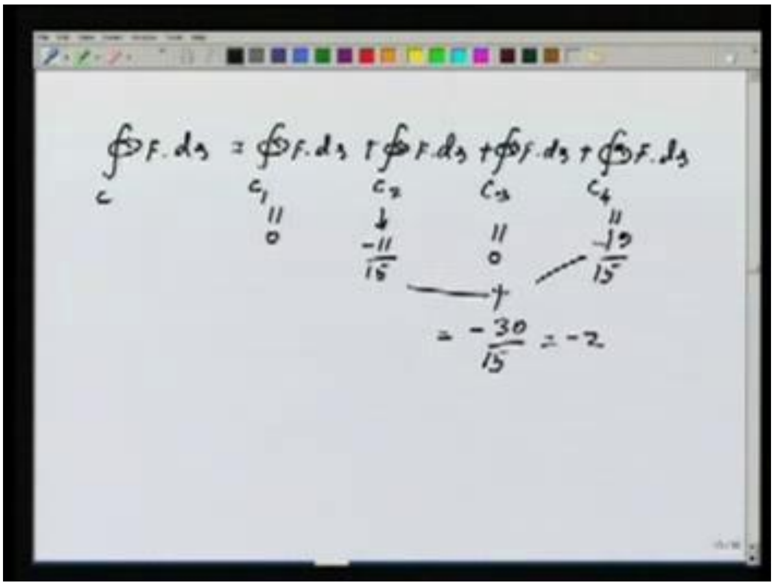


So finally when I look at the total line integral, that means integral over c F dot d s, I write it as integral over c 1 F dot d s. Plus integral over c 2 F dot d s, plus integral over c 3 F dot d s, plus integral over c 4 F dot d s. Now, this integral is 0 c 1 that we have seen, this is also 0, this turns out to be minus 11 by 15. And this turns out to be well c 3 turns out to be 0 c 4 turns out to be minus 19 by 15. If I add these two, what I get is this plus this, I get minus 30 by 15, which is equals to minus 2. So, the line integral portion turns out be minus 2. But, if I look at the surface integral portion which I have earlier calculated that also turns out to be minus 2. So, that means, curl F dot n d sigma double integral over s is turning out to be line integral of the function over the curve c.

But, note that for this it was really important to cease the direction properly. If in this ((Refer Time: 39:41)) picture I had chosen the direction. This way you can check yourself very easily that the integral will actually turn out to be equals to 2 not minus 2 which it should be. So, if you want to choose this direction, you have to choose the downward unit normal vector. In that case the surface integral will turn out to be equals to 2, and then they are same.

So, check it yourself the exercise is the normal which I have taken, take the normal same normal, but with minus sign; that means, the direction of the normal was reversed, and then change the directions of c 1 c 2 c 3 c 4 in the opposite direction. Then you will get that those two integrals are same. So, what does it say, it says exactly what I said about the choice choosing the direction of the boundary curve that you choose it in such a way.

That if your head is towards the normal then the surface should be on the left hand side of yours, with that convention we have verified stokes theorem. So, these lectures were about connecting the double integral with line integrals. In the next lecture, we will go to something called divergence theorem that will connect triple integrals with surface integrals. So, that is going to be the last generalization of the second fundamental theorem of calculus, which we are going to deal with. So, in the next lecture we will start with triple integrals and surface integrals.