Mathematics-I
Prof. S. K Ray
Indian Institute of Technology, Kanpur

Lecture - 32
Gauss Divergence Theorem

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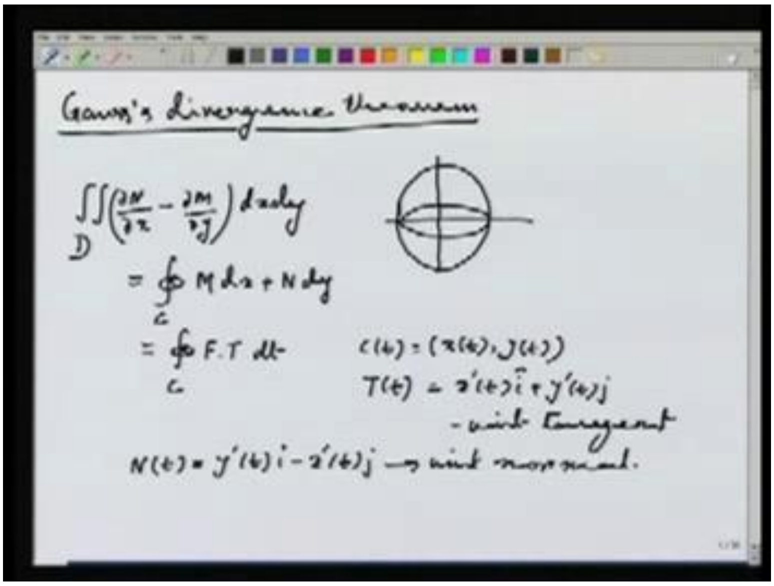

Today we are going to start with Gauss Divergence Theorem. It is the last generalization of second fundamental theorem of calculus, which we are going to do. So, far we have done Green's theorem, which talks about a plane region bounded by a curve. And then, we have seen that certain double integral comes down to the line integral on the boundary. That was Green's theorem. Then, we have looked at one more generalization of that. When it was not a plane region, it was a curved region. And then the generalization was the Stokes theorem; that the double integral over the surface, comes down to a line integral on the boundary of that region.

Now, this time, what we are going to deal with is the connection between triple integral and the surface integral. So, the situation is something like this. What we are going to do, so let me look at the solid sphere noise. And what we are going to do is I will have certain functions, whose triple integral over this solid sphere, which makes sense, because the solid sphere is a three dimensional object.

The triple integral of certain function on the solid sphere, will turn out to be the surface integral of some other function. Where, the surface in portion is the boundary of the solid sphere. That is the usual sphere. So, that is generalization of Green's theorem again. But, first, let us see, how do, we generalize Green's theorem in this detections for triple integrals.

So, let me write down the statement of Green's theorem again. So, the statement was double integral over D del n del x minus del m del y, this if I do the double integration with respect to d x, d y. It turns out to be integral over some boundary curve with proper orientation. And then, we call it M d x plus N d y.

Now, we can recall, that this last integral can also be written in tangential form. That is c F dot T d t, where t is the unit tangent to the curve c. So, if c t is x t, y t. Then, T t is a constant multiple of x prime t i plus y prime t j. Let us assume that this is unit tangent. If that is the case, then I can see N t. That is equals to y prime t i minus x prime t j is certainly unit normal, because if I look at the dot product of T t and N t i get 0. So, n t is orthogonal to the tangent. So, it is the normal.

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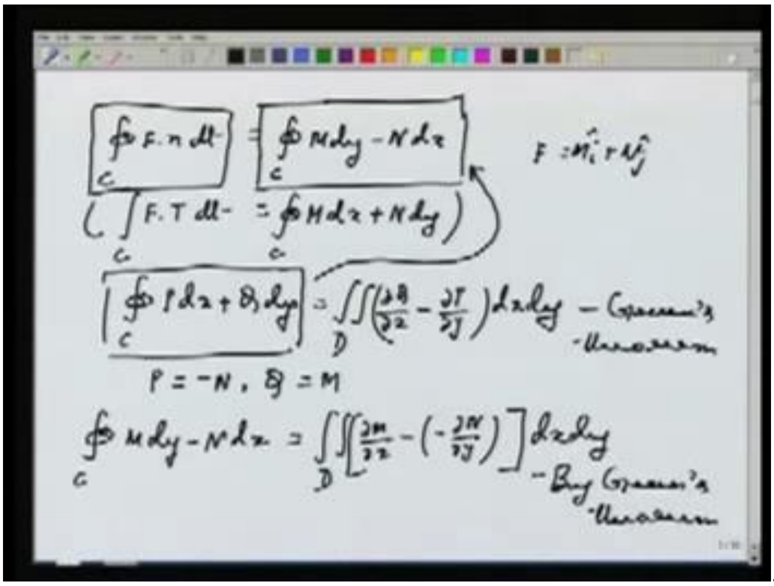

Now, let us look at this integral, integral over c F dot n d t. Instead of F dot T d t I look at F dot N d t. So, what does this mean actually, it means integral over c M d y minus N d x. Recall, that integral over c F dot T d t by that I meant, integral over c if M is equals to

M i plus N j. Then, F dot T was M d x plus N d y. So, with that notation integral over c F dot n d t turn out to be M d y minus N d x.

Now, can I write this as a double integral? So, this quantity, here I want to write as a double integral. So, given all the nice properties of the function M N and c, I can think about applying Green's theorem in this set up. So, in the usual statement of Green's theorem, what I have is, that integral over c P d x plus Q d y. That is double integral over D del Q del x, minus del P del y, d x d y. This is the situation.

Now, if I compare this side with this. Then, I get, what exactly is P and what is Q. So, I get that P is actually equals to minus N and Q equals to M. So, this is Green's theorem. Now, if I apply Green's theorem with P equals to minus N and Q equals to M. What I get is, that integral over c M d y minus N d x. That is double integral over D. So, first I have to write del q del x, but q is M. So, I get del M del x then minus del P del y, but p is minus N. So, it is minus of del N del y times d x d y this is by Green's theorem.

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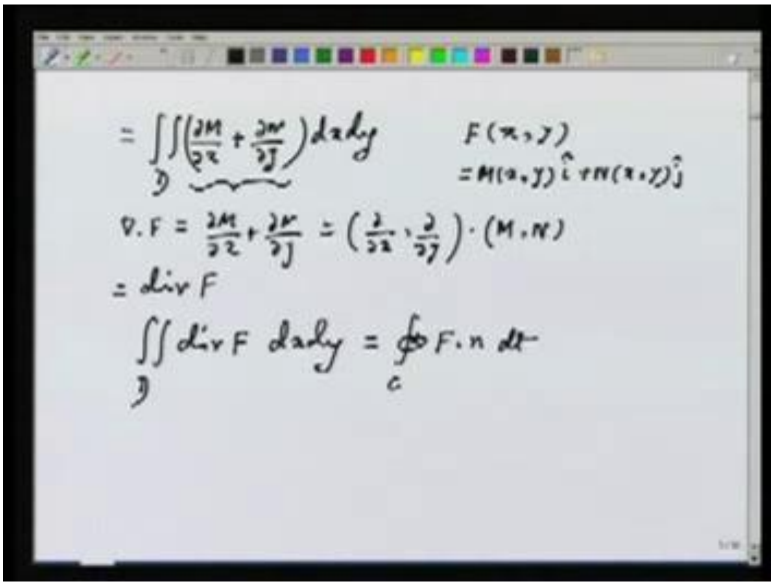

Now, this quantity is nothing but, if I simplify multiplying with a minus. I get del M del x plus del N del y d x, d y, where F x y equals to M x y i plus N x y j. Now, if I look at this integrant, this is then nothing but grad dot F by my definition. Because, grad dot F is nothing but del M del x plus del N del y to remember it, is just this del del x del del y dot M N. So, this is the quantity I get. Now, this is known as divergence of F, we call it div F.

So, then, what actually we got, if I look at the previous expression ((Refer Time: 08:23)) that I started with F dot n d t. This is quantity here. That is turning out to be double integral over D divergence F d x d y. So, this is integral over c F dot n d t. So, this is called the normal form of Green's theorem. It is actually this form, which we want to generalize to higher dimensions. So, that is divergence theorem.

So, what I will do is, I will just state divergence theorem for you. Then, for a particular example, I will verify the theorem. I will not go to the proof of the divergence theorem. Because it is long and it will take our time. And then we will show some applications of the divergence theorem. That certain integral becomes much easier to calculate, if you use divergence theorem.

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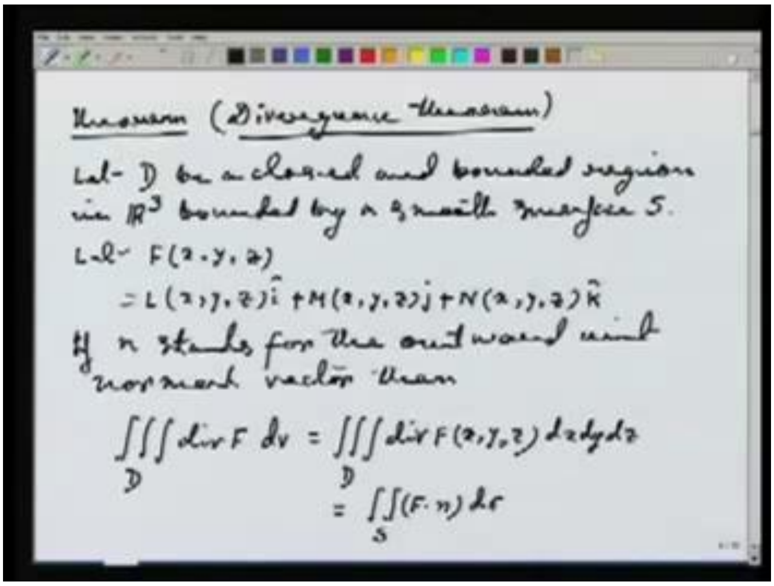

So, what exactly is the statement of divergence theorem? Let us go to that. So, let D be a closed and bounded region in R 3, which is bounded by a smooth surface S. Let F be a vector field. So, it is given by F x, y, z is equal to L x, y, z i plus M x, y, z j plus N x, y, z k. Now, if n stands for the outward unit, normal vector. Then triple integral over D divergence F d v, that is the three dimensional integral over the region d. So, actually what would you mean is, divergence F at x, y, z, d x, d y, d z.

So, you can write it in this form. Usually we do not write it in the form of x y z. Because, if I do that, then I have to tell you, what is the variation of the parameter x, y, z. But, this is just for understanding that, whose function this divergence f is and what is the area of integration. Area of integration is d x, d y, d z. That turns out to be double integral over S, actually integral over S is technically correct.

But, I want to say that it would be a integration of functions, which has 2 parameters. That is why, the double integration comes in, it is F dot n of this function with respect to the surface area d sigma. So, this is the statement, it is exactly analogous to the normal form of Green's theorem. So, divergence F given a function F, I can calculate the divergence of f of that function. Integrate that function with respect to the parameter x, y, z. The resulting integral is same as the integral of the function F dot n, with respect to the area of the surface d sigma on the boundary surface s. That is what divergence theorem says. So, to understand in a better way, well there are certain technical conditions involved, which you will certainly need for existence of all these integrals.

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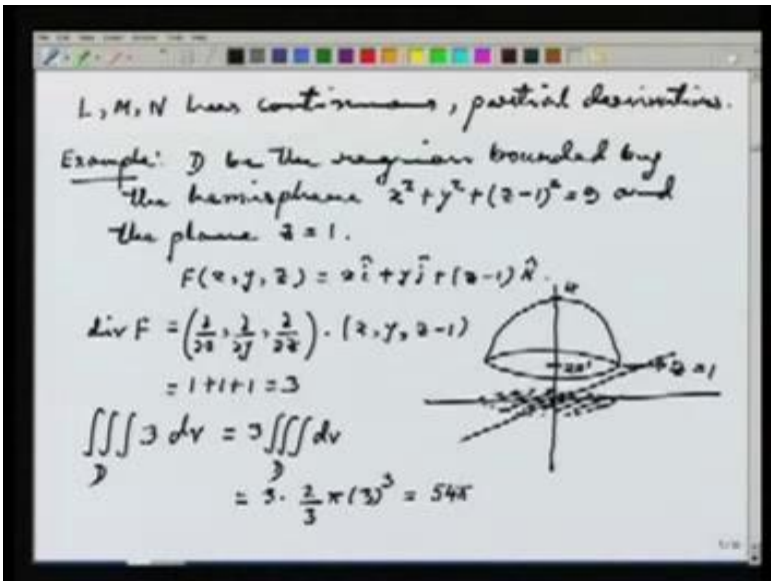

So, what we usually use is here that L, M, N has the following property. That L, M, N has continuous partial derivatives. So, those kind functions, we are choosing. Now, let us go to an example, where you can verify Green's theorem, that this is really true, without going into the proof of it. So, I will start with some simple example. So, let us say, D be the region, bounded by the hemisphere x square plus y square plus z minus 1 whole square equals to 9. And the plane z equals to 1.

So, you want to verify that the divergence theorem for some particular vector field F, which I am going to now define. So, F of x, y, z that is equals to x i plus y j plus z minus

1 k, so to understand the scenario, let me just the draw the picture, this is the z-axis, x-axis, y-axis. This is, let us say the height z equals to 1, this is the projection of that region here. So, my region is this hemisphere and this surface z equals to 1. So, this is the plane z equals to 1, which is intersecting the hemisphere.

So, this height any way we know is 4. Because, the radius of the sphere is 3 and this is z equals to 1 and this is the plane z equals to 1. That is, what I mean and this is the given function F. Now, L, M, N certainly satisfies all the conditions of being continuous and having continuous partial derivatives. So, all I do is, first I calculate divergence f. So, what is divergence of F? So, that is, I know it is del del x, del del y, del del z dot x, y, z minus 1.

If I calculate this, I get 1 plus 1 plus 1, 3. So, we are lucky here that divergence F becomes a constant function. So, all I have to do now is, triple integral over D 3 d v, but 3 being a constant comes out, all I am led with integral over d d v. That is the volume of the hemisphere given by the equation x square plus y square plus z minus 1 whole square equals to 9 and cut by the plane z equals to 1. That is actually passing through the center for this noise sphere.

So, I know the volume of the sphere of radius r. That is 4 by 3 pi r cubes and this is the half of that chap. So, the volume is also half. So, this is 3 times instead of 4 by 3, I will have 2 by 3 pi, then the radius cube. What is the radius? Radius is anyway 3, so 3 cube. So, what I get is 27 into 2, that is 54 pi, that is one side. Now, I have to look at the other side, I have to calculate for that, first the outward unit normal vector n.

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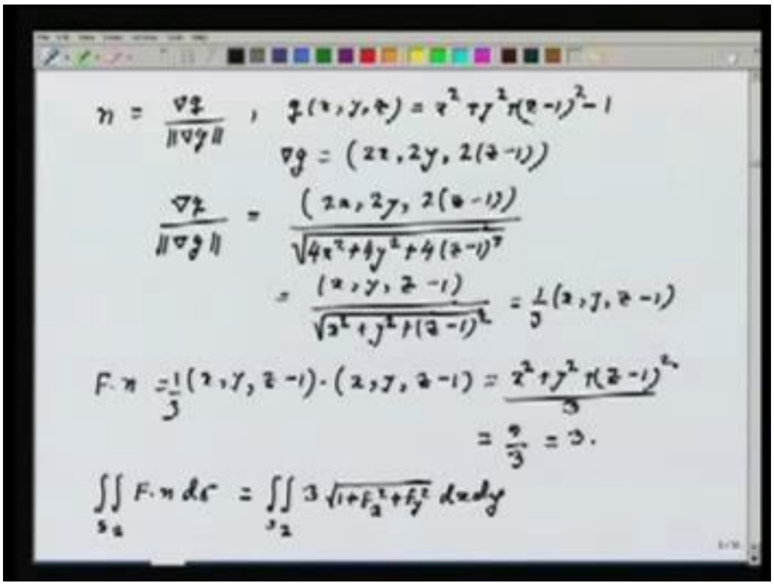

Now, since my surface is given as level surface of a function. I know that n is grad g by norm g, norm grad g. What is g? Well, g of x y z is x square plus y square plus z square z minus 1 whole square minus 1. Then, grad g is a vector, that vector is given by 2 x, 2 y, 2 times z minus 1. Then, what is grad g by norm grad g. That would be 2 x, 2 y, 2 minus z minus 1, divided by the norm of this vector. That is square root of 4 x square plus 4 y square plus 4 z minus 1 whole square.

So, I can take 2 out, I get x y z minus 1 divided by, but x, y, z is a point on the surface. And I know x square plus y square plus z minus 1 whole square is equals to 9. So, square root of 9, so it is 3. So, I get 1 3rd x, y, z minus 1. Now, I need to calculate, what is F dot n, well I know my expression of F, ((Refer Time: 20:09)) F is given by x y z minus 1. So, F dot n is x y z minus 1 dot n, which is 1 3rd, so that comes out x y z minus.

So, again I get x square plus y square plus z minus 1 whole square divided by 3. But, x y z is a point on the surface, so, this numerator is actually 9. So, it is 9 by 3. So, it is equals to 3. So, the function now, the integral on the boundary, if I look at the picture again ((Refer Time: 20:56)) this boundary is actually consisting of two curves, two surfaces. One surface is the part of the hemisphere and the other surface is the plane z equals to 1. So, the bounder integrals, now has two components. So, first one, I call S 2 and this one, I call S 1. So, S 1 is the plane z is equal to 1 and S 2 is the hemisphere.

So, first I like to calculate, what is double integral over s 2 F dot n d sigma. So, this then double integral over s 2 F dot n is anyway 3. And then, I can try to calculate, what d sigma is. Now, for d sigma what I do is upper hemisphere can be thought of us. So, I will write it as 1 plus f x square plus f y square, I will tell you shortly what I mean by F d x d y. So, what is f? Well the upper hemisphere can actually thought as graph of a function.

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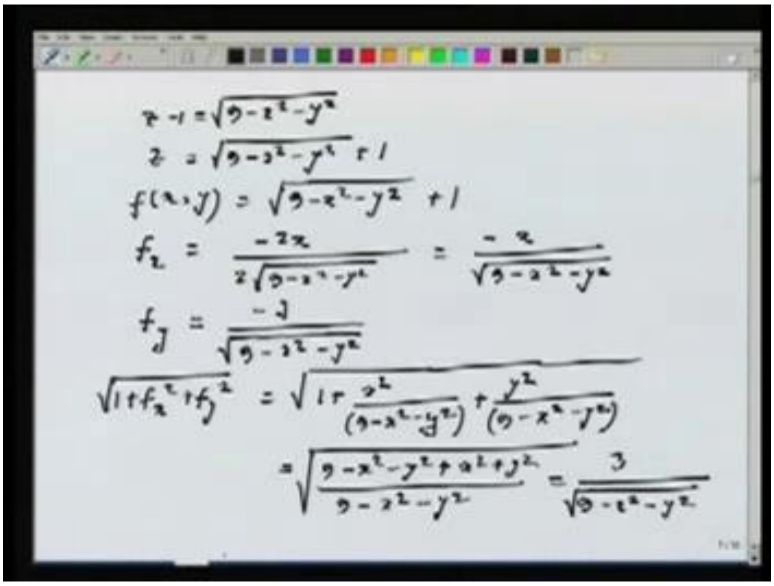

Because, I can write it as z minus 1 equals to square root of 9 minus x square minus y square. So, z equals to 9 minus x square minus y square plus 1. So, my f x y actually is square root of 9 minus x square minus y square plus 1. So, what is f x that means, I have to differentiate. So, it is minus twice x divided by twice square root of 9 minus x square minus y square. That is minus x divided by square root of 9 minus x square minus y square.

Similarly, f, y, that is minus y divided by square root of 9 minus x square minus y square. Then, what is square root of 1 plus f x square plus f y square. This is square root of 1 plus x square divided by 9 minus x square minus y square plus y square divided by 9 minus x square minus y square. If I calculate this whole quantity, what I get is square root of 9 minus x square minus y square plus x square plus y square divided by 9 minus x square minus y square. So, which is nothing but, 3 by square root of 9 minus x square minus y square.

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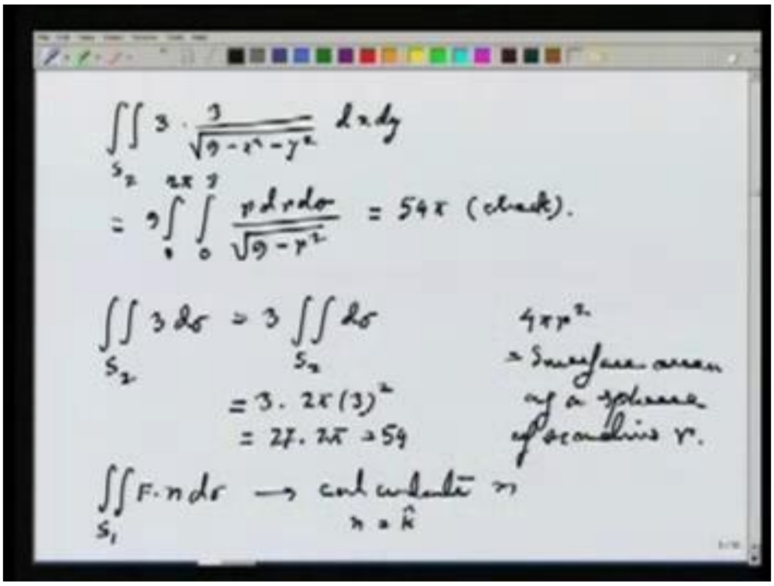

So, the integral, now I am looking at is double integral over s 2 3 times the function is 3 by 9 minus x square minus y square. So, it is 3 by square root of 9 minus x square minus y square d x, d y. Now, I can write this in polar coordinates as 9 0 to 2 pi, then the variation of radius vector 0 to 3 r d r d theta divided by square root of 9 minus r square. This integral, you can easily calculate and just check, that it is turns out to be 54 pi, but there is easier way of doing it, because ultimately you are interested in doing, what is the integral I am going to calculate, well the integral I am trying to calculate is, double integral over s 2 3 times d sigma. So, let us look at it directly, that double integral over s 2 3 d sigma. So, if I take 3 out, what I get is double integral over s 2, d sigma. But, then, what is d sigma, it is the surface area of the sphere of radius 3, then the half of that, because I am looking at the hemisphere.

But, I know, already from certain of our formulas, which we have calculated earlier as surface area of revolution and such thing that what actually is the surface area of a sphere of radius r. Well, I know the formula the formula is surface area of sphere of radius r is actually 4 pi r square, this is the surface area of a sphere of radius r. See, if I apply that formula, it is 3 times 2 p. Then, r square means 3 square.

So, this is then 27 into 2 pi, that is 54 pi. But, still one more portion of the surface is left, that is z is equals 1. So now, I have to calculate, what is double integral over s 1 F dot n d sigma. Notice that, S 1 is portion of the plane z equals to 1. So, on that, which function I want to evaluate. That is, what is F dot n? So, in this case, first I have to calculate n earlier the n, which I have calculated is for the hemisphere. Now, it is for the case z is equals to 1. But, in that case I know that n is actually equals to k. That is the outward unit normal vector.

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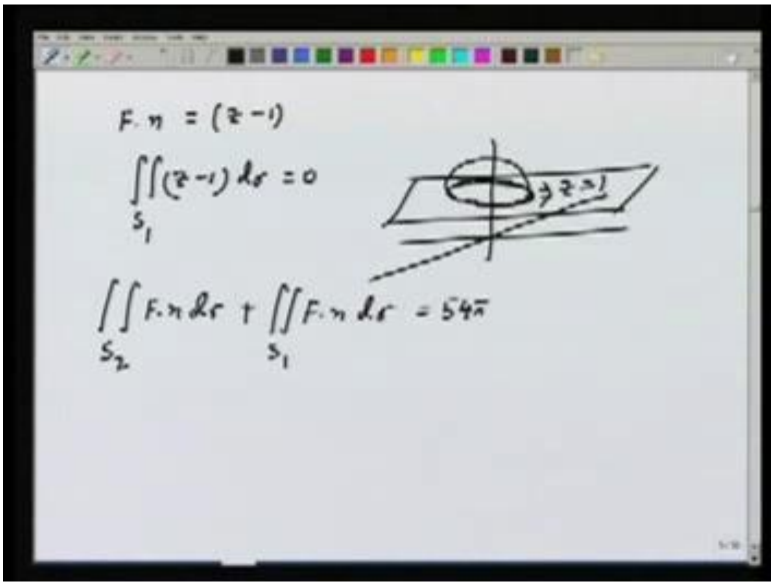

So, F dot n, since I know the value of the function is nothing but z minus 1. So, I have to integrate now, double integral over s 1 F dot n d sigma. That is z minus 1 d sigma, but notice that d sigma is the nothing but d x d y. Because s 1 turns out to be this plane region, when z is equals to 1. That means the integrant is any way is 0. So, this integral is 0. Because, on s 1, z is always 1, remember the picture, my surface is, this is z equals to 1 the plane. So, the integrant is 0.

So, the total integral over the boundary, that is the s 2 F dot n d sigma plus double integral over s 1 F dot n d sigma, that is 54 pi. But, if we recall, what exactly was the triple integral of divergence F, that also turns out to be equals to 54 pi. That means, those two sides are same and hence divergence theorem has been verified.

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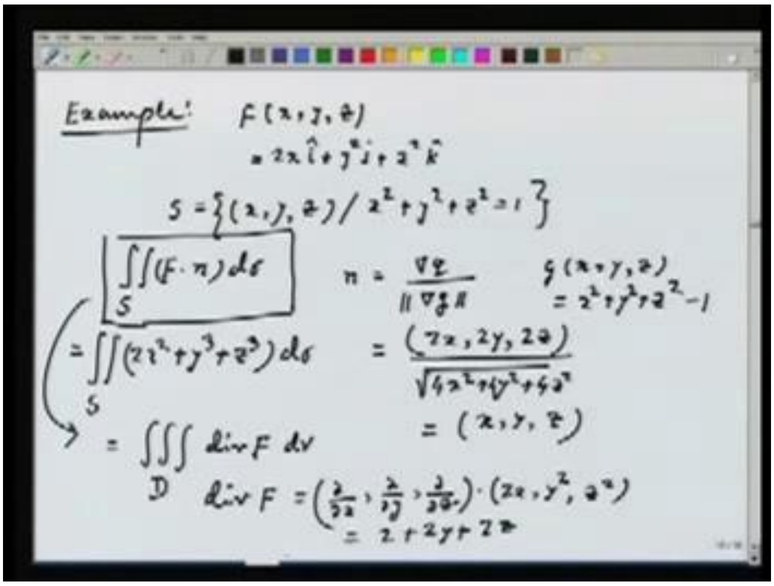

Let us look at the following example now. It is about evaluation of certain surface area surface integrals, let us look at this function F x, y, z. That is equals to twice x i plus y square j plus z square k. And let us say, S is the unit sphere and I want to calculate integral over S F dot n d sigma. In principle, I should be able to do that. So, let us try to see, what comes. So, what is n now, n again is given by the formula grad g by norm grad g. So, that if I calculate, I would get where g x, y, z for each point, I have to calculate the normal. So, locally I can think of it as a graph.

So, it is, what I get is then 2 x, 2 y, 2 z, so this comes out to be just x y z. So, then the integrant will turn out to be double integral over S, then F dot n is twice x square plus y cube plus z cube. And then d sigma, which I can calculate, now even if you calculate d sigma and put that in the integral, you would say this integral actually turns out be a very mess integral. It becomes very complicated to calculate that integral.

Because, the proper parameterization, you have to use, so that your integral becomes much easier and that is going to be very difficult here. Instead, suppose we invoke to the divergence theorem. Because, it is in that form actually the right hand side of the divergence theorem, which appears in the divergence theorem is double integral over S F dot n d sigma, this is that is given here.

If I apply divergence theorem, this is actually going to be equal to integral over the solid sphere, which I call d then divergence F d v the volume integral. That is the triple integral. Now, what is divergence of F that I can very easily calculate from this given function? So, what is divergence of f that is grad dot f this is del del x del del y del del z dot a function 2 x y square. The z square this then turns out to be 2 plus 2 y plus 2 z.

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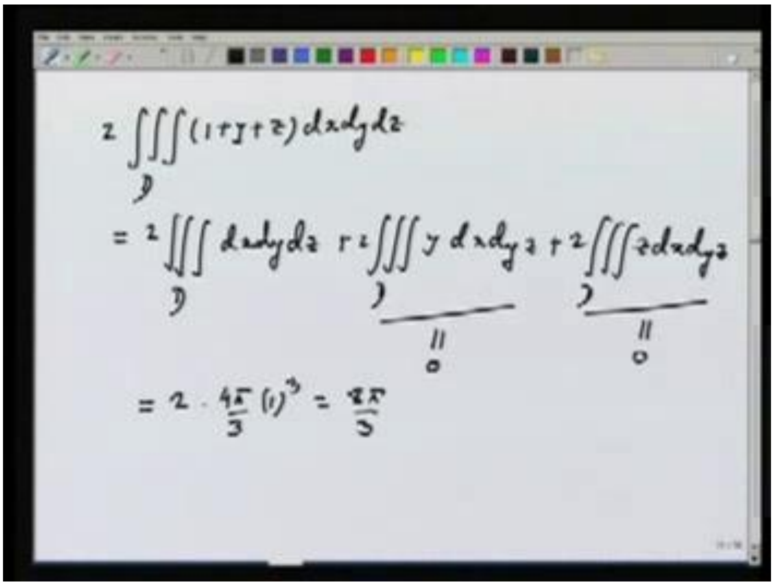

That means, what I need to calculate then is triple integral over D 2 times 1 plus y plus z. It is simply d x d y d z that. Now, this integral I want to evaluate now, this I can write as 2 times triple integral over D d x, d y, d z plus 2 times y d x d y d z plus z d x, d y, d z. Now, you can check very easily that this integral they are very symmetric and you can easily show that this integral is actually equal to 0. So, the end result is 2 times volume of the sphere of radius 1.

But, sphere of radius 1, this has got the area 4 pi by 3 r square r cube that is 1 cube here. So, it is 8 pi by 3, which is actually the left hand side of the divergence theorem, that is divergence of the function, you integrate on the triple integral over region d. What I get is 8 by 3. So, by gauss divergence theorem this quantity is exactly is same as this integration, which I have written here which usually very is difficult to calculate. But, if you use divergence theorem of gauss, then it becomes much easier to calculate.

So, this is, how many surface integrals, which are usually bit more complicated to compute can be transformed to a triple integral by using gauss divergence theorem. And then, it might be easier to calculate. So, this is what we wanted to do about generalizations of fundamental theorem of calculus. So, we have done it for plane regions, which was Green's theorem. Then we have done it for curved regions, which was stokes theorem. Then, we have done it for solid region, whose boundary then is a surface. That is divergence theorem. So, these are all about the generalization of second fundamental theorem of calculus. For all the other analogs of integrals, which we have done, namely surface integral, volume integral and line integral.