Mathematics-I

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Lecture – 4
Sequence- III

In today's lecture I will first try to justify why we started with Cauchy sequences at all. So first let me recall the definition of Cauchy sequence once again. The definition was this.

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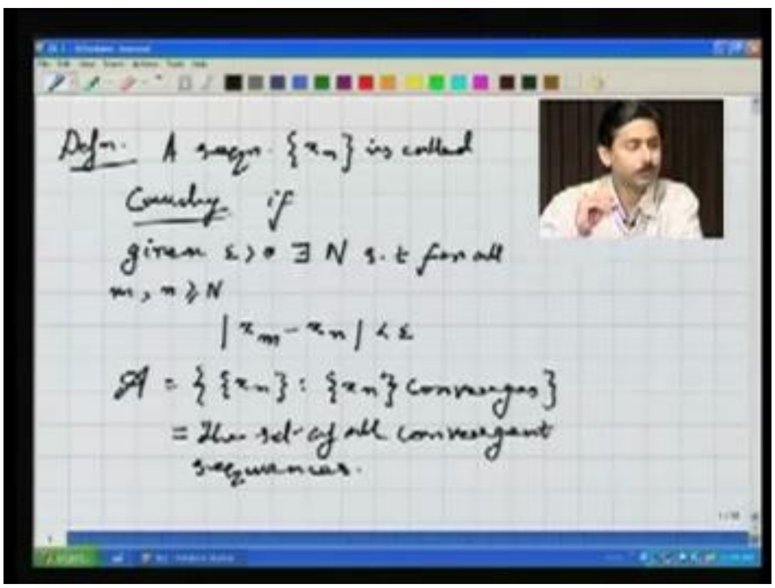

A sequence x n is called Cauchy if the following happens. The intuitive idea was that, however small epsilon you take, after some stage all the terms of the sequence are actually epsilon closed to each other. I am just going to write it in mathematical language. That is, given epsilon bigger than 0, there exist N which always we refer to as a stage, such that for all m n bigger than or equal to capital N, modulus of x m minus x n is less than epsilon. The first thing I will like to see, what Cauchy sequences has to do with convergent sequences.

So let me start with certain notations. I define A: this is collection of all sequences x n such that x n converges. In language, this is just the set of all convergent sequences. So I write it as the set of all convergent sequences and then I define a set B as the set of all Cauchy sequences.

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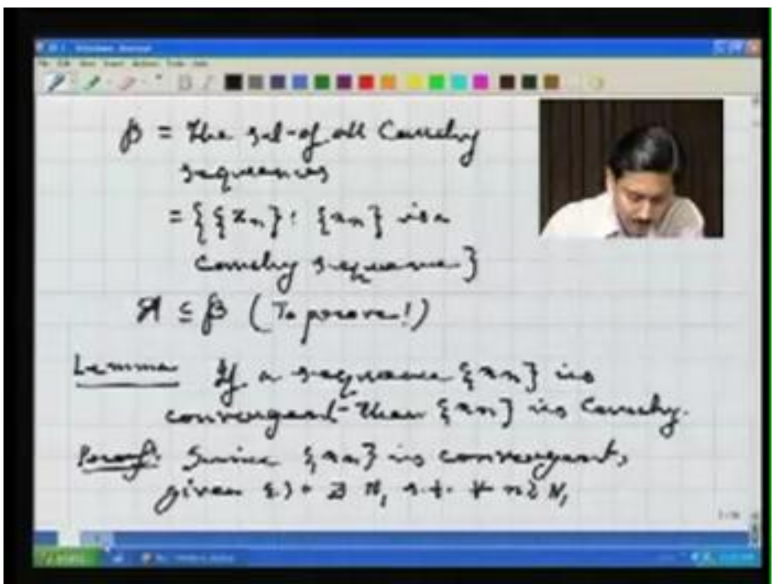

In symbol, it will look like x n such that x n is the Cauchy sequence and we are interested about the connection between A and B. The first thing I will like to show is A is actually subset of B. So this I start as a little lemma. If a sequence x n is convergent, then x n is Cauchy. So let us try to prove this. What I have to prove that given any epsilon, I have to find a stage so that after which any two terms are epsilon distance among each other.

Now since the sequence is given to be convergent, there must exist a limit, let us call that l. So I will write since x n is convergent.

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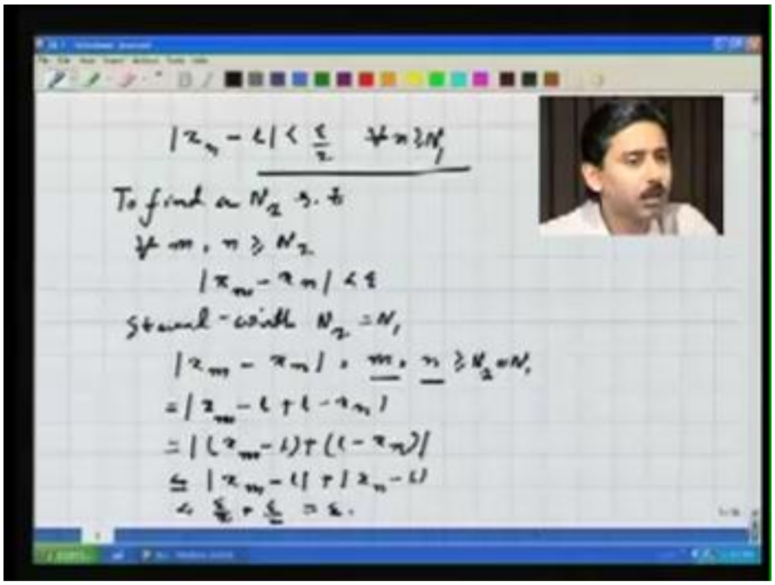

Given epsilon bigger than 0, there exist N 1 such that for all n bigger than or equals to N 1, I have modulus of x n minus l less than epsilon by 2. This is true for all n bigger than or equals to N. So here instead of epsilon, I am starting with epsilon by 2 but epsilon is arbitrary. For any arbitrary epsilon, there exists a stage. So in particular for epsilon by 2 also, there exists a stage after which the difference between x n and l is less than epsilon by 2.

Now, what is my job? My job is to show that the sequence x n is Cauchy. That is to find a N 2 such that for all m n bigger than or equals to N 2, this should be true; that x m minus x n is less epsilon. I say, let me start trying with N 2 equals to N 1 and let us try to estimate the difference x m minus x n, where m and n are bigger than or equals to N 2 which is equals to N 1. Now what I do is, I write it as modulus of x n minus l plus l minus x n. Then I group the terms in the following way, x m minus l plus l minus x n and then I use the triangle inequality that mod x plus y is lesser equals to mod x plus mod y.

If I do that, what I get is.. Notice that mod of l minus x n is same as mod of x m minus l because mod of x is mod of minus x. So there is no loss of generality here but then I already have that m and n, both are bigger than N 1. I appeal to the definition given here. That means, all these terms are less than epsilon by 2 plus epsilon by 2 but which is epsilon? That proves the sequence x n we have is a Cauchy sequence. So this completes the proof.

So one way, it is very clear now that the set of all convergence sequences is actually contained in the set of all Cauchy sequences. The question now is about the converse. Is it true that every Cauchy sequence is convergent? So that is the question we are going to address now.

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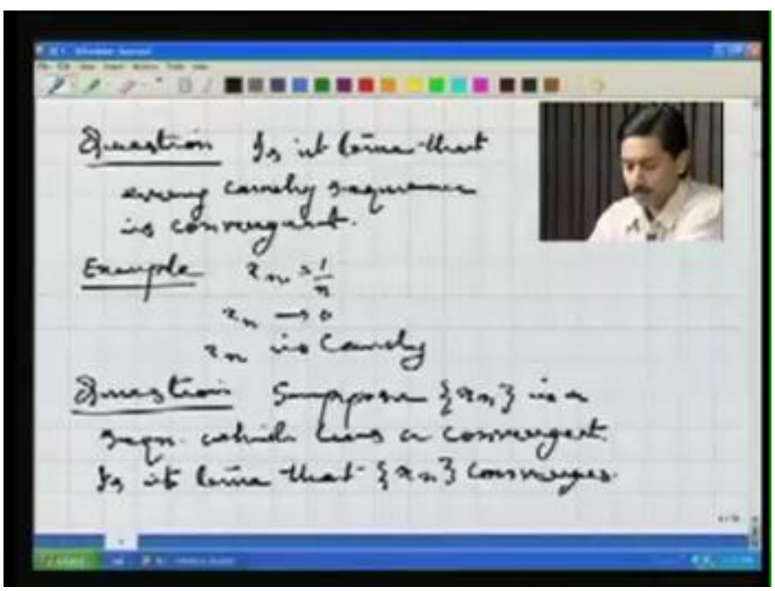

So question is, is it true that every Cauchy sequence is convergent? To start with, I just have one example to tell you, that if you look at this example x n equals to 1 by n, two things I have shown. One is x n converges to 0. That was in the very first lecture. It just uses the Archimedean property and after defining Cauchy sequences if you remember, I have actually proved that x n is Cauchy. This actually illustrates our previous result.

Now we know it has to be very general that every convergent sequence is Cauchy but what about the converse? I have so far not seen any example of a sequence which is Cauchy but it does not converge. So probably it is true that every Cauchy sequence converges. So let us try to prove this. To start with, I have a little lemma to prove. First notice the following question. The question is, suppose x n is a sequence which has a convergent subsequence, does it really imply that the whole sequence converges? Is it true that x n converges?

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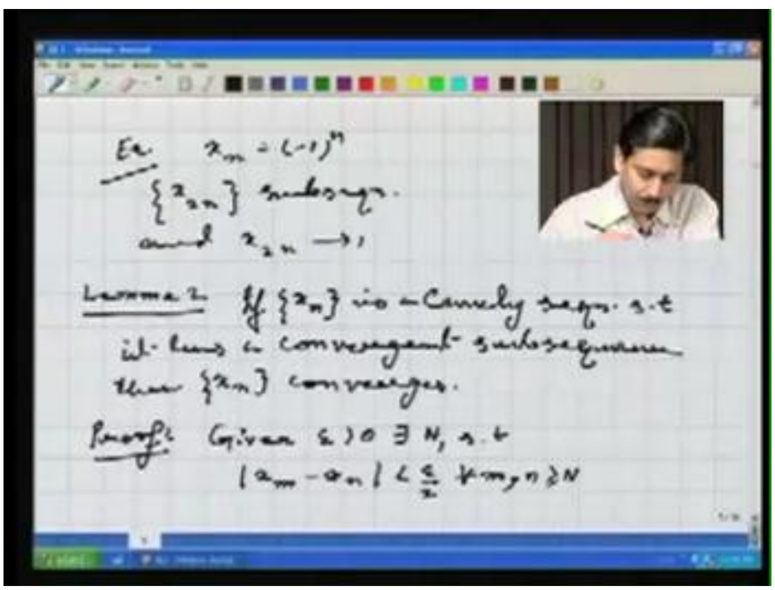

Let us see examples again. Suppose I look at this sequence x n equals to minus 1 to the power n. Then I know x 2 n is a sub sequence and x 2 n converges to 1 and I also know that the whole sequence does not converge. That means, in general it is not true that if a sequence has a convergent sub sequence, then the whole sequence converges but surprisingly it turns out that the same is true for Cauchy sequences. To prove that, let us proceed this way, I call it the second lemma.

If x n is a Cauchy sequence such that it has a convergent sub sequence then x n itself converges. Notice that once I proved this lemma, then if I want to prove that every Cauchy sequence converges, it is enough to extract the convergent subsequence out of the whole sequence. Can you think about the Bolzano Weierstrass theorem which I proved in the last lecture? There it was said that every bounded sequence has a convergent sub sequence. It has not been said every Cauchy sequence has a convergent sub sequence. If it was said, then with this lemma I already got the proof but to have the boundedness, we have to work little bit more but let us first try to prove this lemma.

So first I write down the definition of Cauchyness again, that given epsilon bigger than 0 there exist a stage N such that this is less than epsilon by 2 for all m n bigger than or equals to N 1. This is just the definition of Cauchyness of the sequence. Now let us go to a convergence of sequence.

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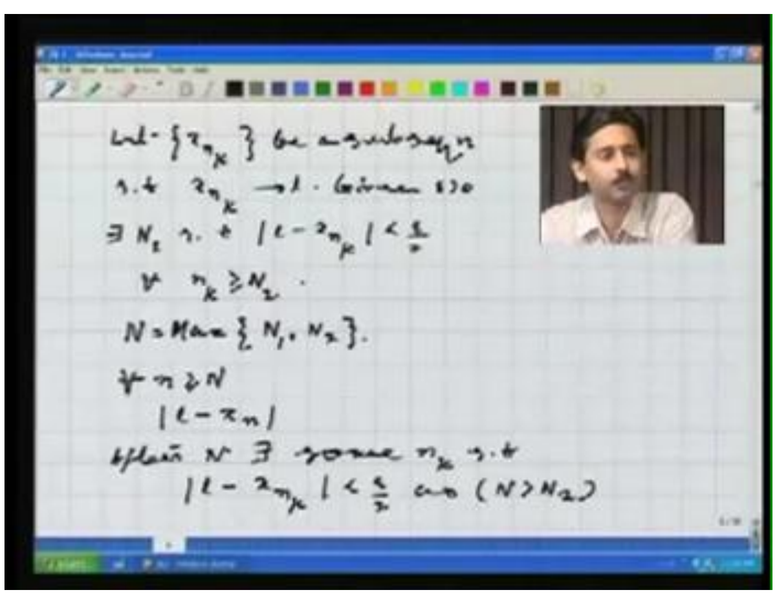

Let x n k be a subsequence such that x n k converges to some real number l and then by definition of convergence I have that given epsilon bigger than 0 there exist a stage N 2 such that mod of l minus x n k less than epsilon by 2 for all n k bigger than or equals to N 2. Now I have to show that the full sequence converges. That means I have to get a stage after which the distance between l and any damn term of the sequence is less than epsilon. What should be that?

I have Cauchyness in my hand. Each individual term is also smaller after certain stage. That is, I am going to exploit now. I define capital N to be equal to maximum of N 1 and N 2 and then I look at, for all n bigger than or equals to capital N modulus of l minus x n. Now notice that after capital N there exist some n k such that this is anyway true as n is strictly bigger than N 2.

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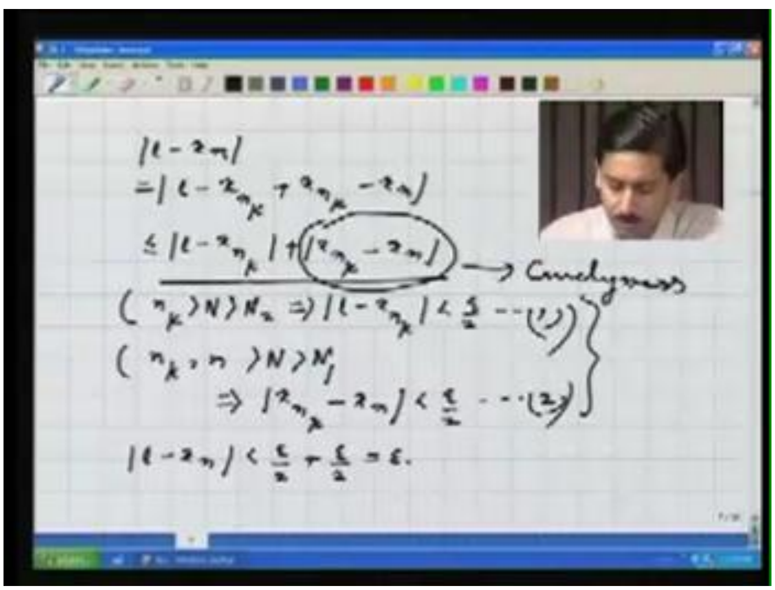

Then I write this l minus x n again as l minus x n k plus x n k minus x n and I again use triangle inequality to get that this is less or equals to modulus of l minus x n k plus mod of x n k minus x n. Now let us try to see what do I know about these quantities. For example, n k is anyway bigger than capital N and by definition of capital N, it is certainly bigger than N 2. This implies that modulus of l minus x n k is less than epsilon by 2. What happens to the second term modulus of x n k minus x n?

Notice that n k, n, both are actually bigger than capital N which is again bigger than capital N 1. This implies by definition of Cauchyness, modulus of x n k minus x n is less than epsilon by 2. I call this 1, I call this 2. If I apply this 1 and 2 in the previous inequality, that is here, what do I get? I get that modulus of l minus x n is less than epsilon by 2 plus epsilon by 2. That is equal to epsilon. That means, the sequence now converges.

So what we have proved is something fantastic: that if I have a Cauchy sequence which has at least one convergence of sequence, then the whole sequence actually converges, which is not true in general. So what is that trick played by Cauchyness? If you do not assume that the sequence x n is Cauchy, this is the term which you cannot really make small. This is already getting small by Cauchyness. This is where Cauchyness has been used. That is the main advantage. Now, the next result I am going to prove is that every Cauchy sequence is bounded. That will be my third lemma.

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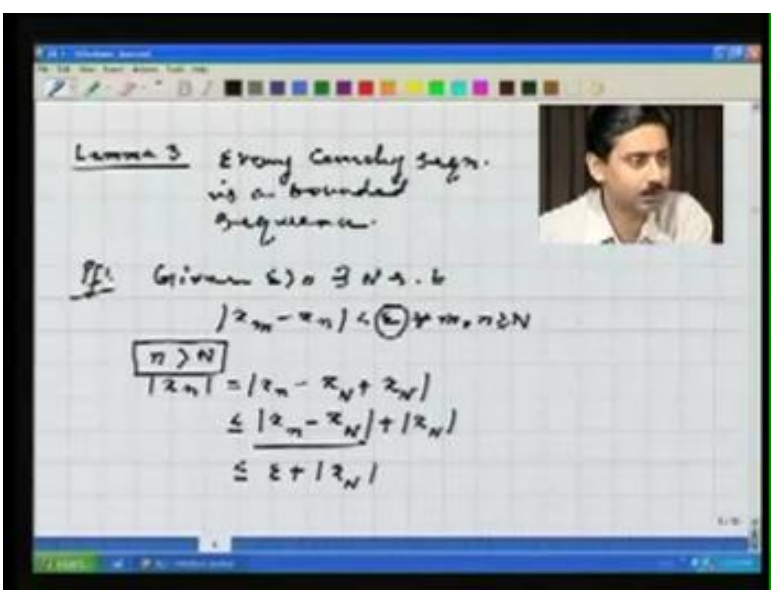

Every Cauchy sequence is a bounded sequence. See, this kind of a lemma is highly expected just because of the following fact. We have noticed that if a sequence converges then it has to be a bounded sequence. Unbounded sequences cannot converge and I am trying to prove that every Cauchy sequence converges but if it has to converge, it has to be bounded. That is why I am trying to prove this first, that every Cauchy sequence is bounded. If it is not then there is no chance of getting convergence. So let us try to prove this first.

Since x n is Cauchy again given epsilon bigger than 0, there exist capital N such that modulus of x m minus x n is less than epsilon for all m n bigger than or equals to capital N. Now if I have to prove that the whole sequence is bounded, that means at least after certain stage the sequence has to be bounded anyway. Let us try to see whether that is true. I take some n which is bigger than capital N. I have to prove that x n is less or equal to some constant for all n. I just simply write it as x n minus x capital N plus x capital N and then I use triangle inequality in a judicious way. That is, x n minus x capital N plus mod x capital N.

Now this quantity x n minus x capital N, that, anyway is less than epsilon, that I have proved. So this is less or equal to epsilon plus mod x capital N. What has happened because of that? Notice that on the right hand side there is no little n. That means, the inequality is valid for all x n s such that little n is bigger than N. It just means after certain stage, the sequence is bounded but what happens to the previous fellows? That means all the x n s such that x n is less than n. That means less than capital N.

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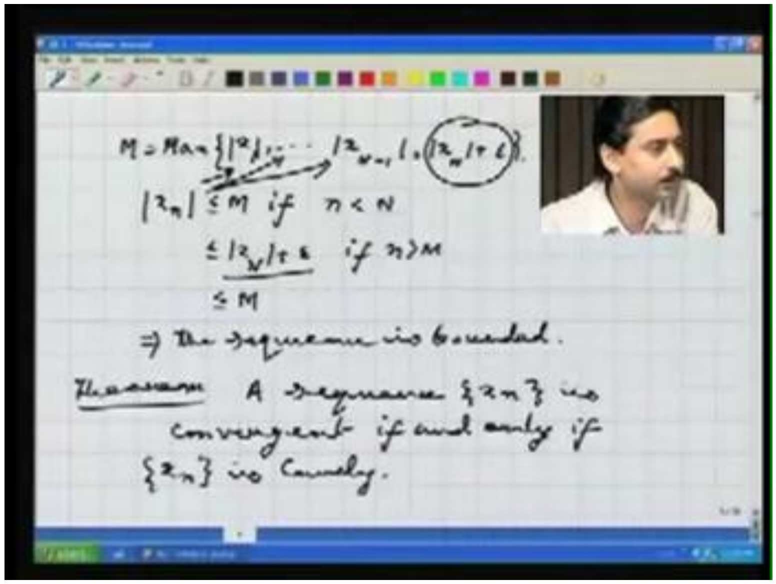

How many terms there are anyway? There are only finitely many terms. So what I do is, now I manufacture the required constant. Let M is the maximum of x 1 up to x N minus 1 and then I add with it mod of x n plus epsilon. Now I take any x n epsilon and look at the modulus, I say this is less than or equal to M if n is less than capital N, which is obvious because if n is less than capital M, then this modulus x n is one of these guys, either this or this.

These any way less than capital M because M is the maximum of all those fellows and if it is bigger than M, I anyway know that it is dominated by this chap. That means, it is less or equal to. But this fellow is anyway less than or equal to capital M because capital M is the maximum of all the quantities. Then I have found is M which is dominating all the x n s and that is true for all n. So this implies the sequence is bounded and now I am in a state to prove the final result. That is, the celebrated theorem of analysis.

The theorem says a sequence x n is convergent if and only if x n is a Cauchy sequence. Let us first look at the statement. Suppose x n is a convergent sequence, I have already proved x n that it is a Cauchy sequence. So one way the theorem has already been proved. Now it is the other way. If x n is a Cauchy sequence, I have to prove x n is convergent. So that is the part I am going to prove.

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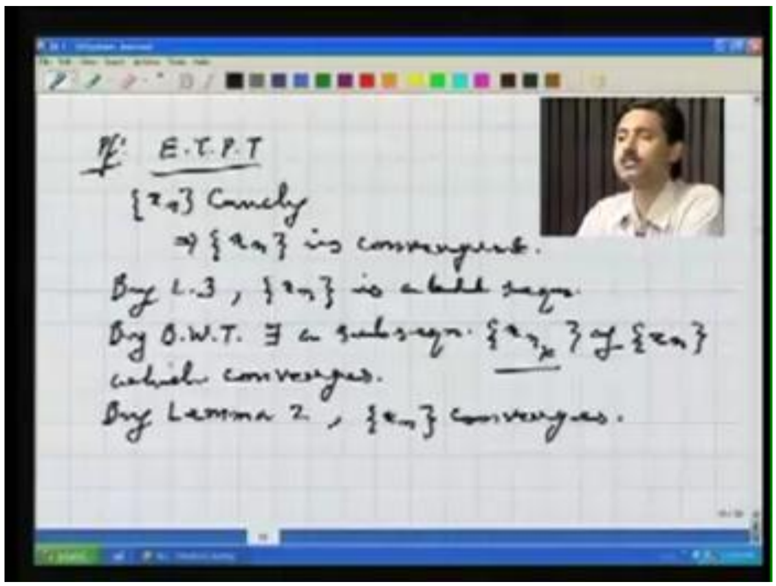

So I will write, it is enough to prove that x n Cauchy implies x n is convergent. First I apply lemma 3. By lemma 3, I will write l 3, x n is a bounded sequence because I have proved that every Cauchy sequence is bounded. I have assigned x n to be Cauchy. So it must be a bounded sequence. Then I use Bolzano Weierstrass theorem. By Bolzano Weierstrass theorem there exists a sub sequence x n k of x n which converges. See, I am in a position to apply Bolzano Weierstrass theorem because it says that every bounded sequence has a convergent sub sequence. I have proved that every Cauchy sequence is a bounded sequence and hence by Bolzano Weierstrass theorem, it must have a convergent sub sequence. Now you can guess the rest of the proof.

Now I apply lemma 2. By lemma 2, what happens? x n is a Cauchy sequence, it has a convergence of sequence and my lemma 2 says that if a Cauchy sequence has a convergence sub sequence, then the whole sequence converges. x n is a Cauchy sequence, it has one convergent sub sequence x n k and hence by lemma 2 the whole sequence converges. This completes the proof.

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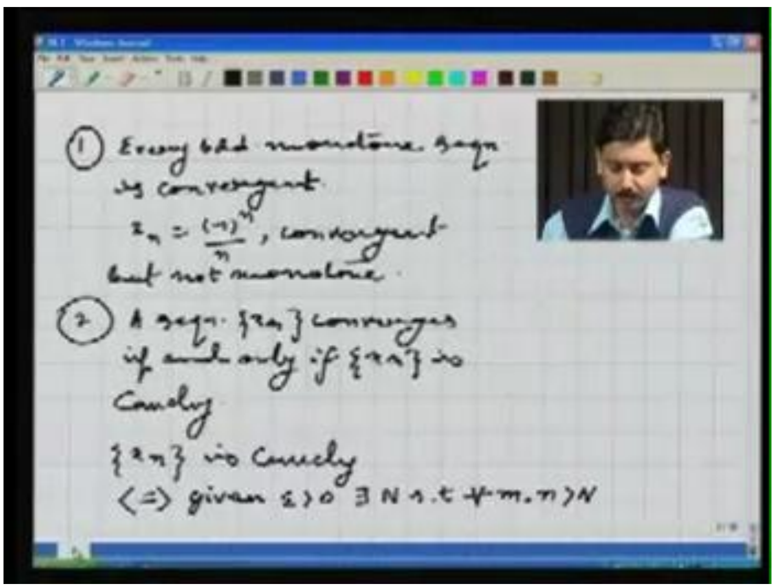

Let us again look back at the notion of convergent sequences. What we have exactly found out? The number one information we have is that every bounded monotone sequence is convergent but at the same time it is not true that every convergence sequence which obviously has to be bounded, is necessarily monotone. The example is, if I define minus 1 to the power m by n, one can easily show by sandwich theorem that this sequence converges but it is not monotone. So this is convergent but not monotone.

The second is the most fundamental one, which we have proved already, that a sequence, if it converges then the necessary and sufficient criteria for that, that it has to be a Cauchy sequence. That is a sequence x n converges if and only if x n is Cauchy and we know what exactly is the definition of Cauchy sequence. So let me write it once again for yourself. That x n is Cauchy implies and implied by given epsilon bigger than 0, there exist a stage N such that for all m n bigger than capital N, the following inequality is true.

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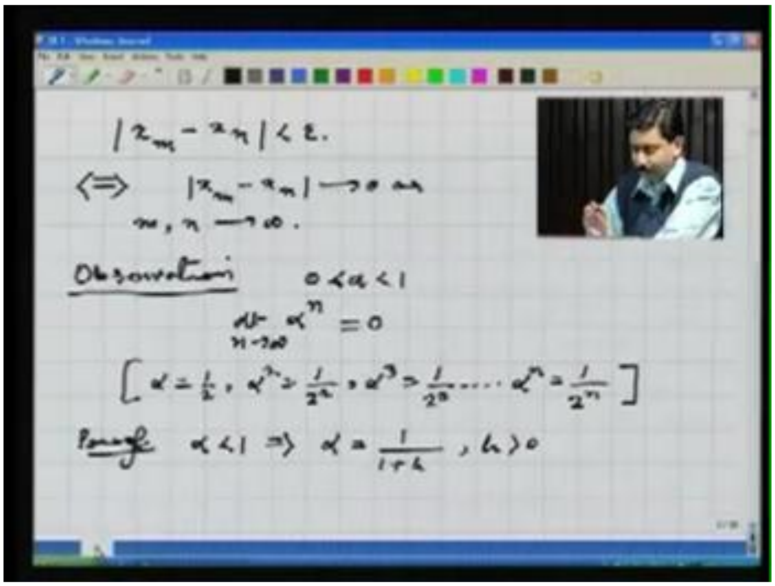

Actually we can simplify this definition a little bit. I say it is equivalent to say that mod of x m minus x n goes to 0 as m and n both go to infinity and using this, now we are trying to determine which sequences are Cauchy. If we look back at the definition of Cauchy sequence, at first sight it looks pretty complicated, that given a sequence if I want to know whether it converges or not, how do I exactly find out whether the sequence is Cauchy or not. That means given any arbitrary epsilon, that is the first weakness because epsilon can vary, I have to find a stage which depends on epsilon such that after that the difference between any 2 terms is less than epsilon.

If you want to do it just by hand, it will turn out to be extremely complicated. So you want to develop certain sorts of mechanism, you know, which will enable us to see that certain sequences are Cauchy. Of course, the defect is this mechanism will not apply to all possible sequences which are Cauchy but certainly it will show that there exist a large number of sequences which are Cauchy. So to start that, first let us make an observation which is just calculation of a very fundamental limit which keeps appearing time and again.

I take some number alpha which satisfies.. So it is a positive number and it is strictly less than 1. Then I can look at alpha to the power n and then I get the limit. What is this limit? Alpha is less than 1. Now if you look at the powers of something less than 1, for example, let us take alpha to be equals to half, if alpha equals to half, then alpha square is 1 by 2 square. Alpha cube is 1 by 2 cube and then alpha to the power n is 1 by 2 to the power n.

Now this limit is very easy to calculate because I know that 2, 2 square, 2 cube 2 to the power n; it actually increases. It crosses any real number however large it is. That means, the sequence 2 to the power n actually converges to infinity. That means, 1 by 2 to the power n has to converge to 0 because of our quotient rule. From that we expect that the answer of this is also 0 but now this has to be proved. The proof is very easy.

So this is just an observation which will help me in guessing what should be the limit. This is a fundamental lesson in sequences once you can guess what exactly the limit of a sequence going to be, then it becomes possible to show that that actually is the limit but the most difficult job is to guess the limit first. In this case we could with some example and now I just try to justify, that my guess is correct.

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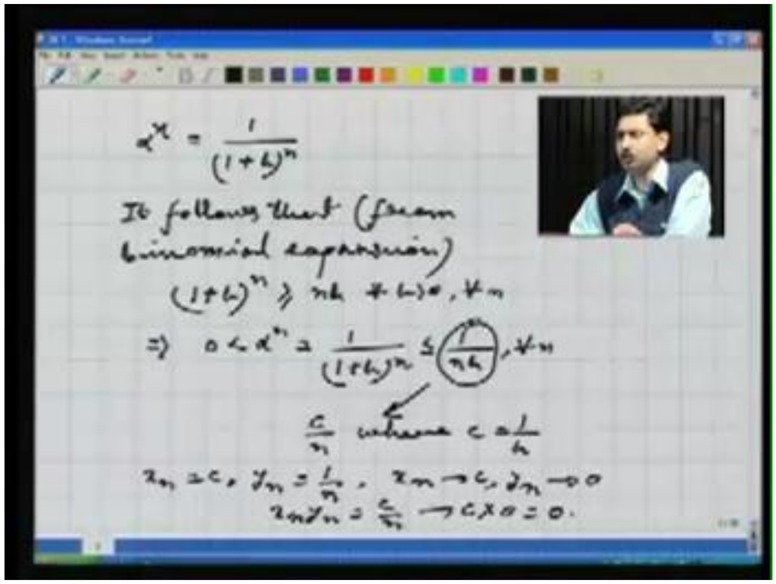

Since alpha is less than 1 this implies alpha must be looking like some 1 by 1 plus h, where h is bigger than 0. In that case what is alpha to the power n? It is 1 by 1 plus h whole to the power n. Now anyway this 1 plus h whole to the power n can be expanded binomially and from the binomial expansion, it will follow. I will write down that it follows from binomial expansion that 1 plus h whole to the power n is bigger than or equals to n h. This is true for all h bigger than 0.

What is the consequence then? This implies that 0 is anyway strictly less than alpha to the power n, which is 1 by 1 plus h whole to the power n, which is now less or equals to 1 by n h. Notice that in the whole discussion h is constant. So this is true for all n. Once alpha is given, h is fixed. So this is of the form some constant c by n where c equals to 1 by h. Then what is the limit?

Look at the constant sequence x n equal to c and y n equals to 1 by n. I know x n converges to c. I also know that y n converges to 0. Then, by my product rule x n, y n, which is c by n converges to c into 0, which is 0 and that is what we wanted to prove. Now I am going to use this result to produce examples of Cauchy sequences in that direction. First I want to prove a theorem.

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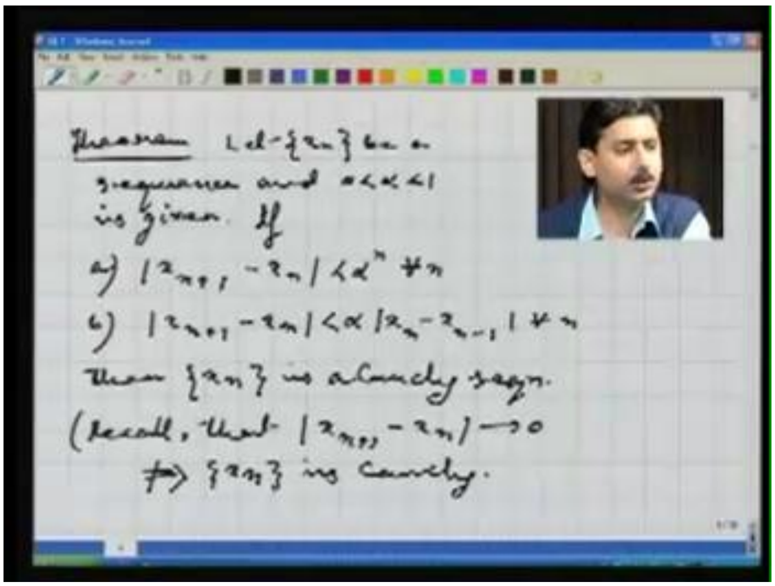

Let x n be a sequence and 0 strictly less alpha strictly less 1 is given. If a mod of x n plus 1 minus x n strictly less than alpha n for all n or mod x n plus 1 minus x n strictly less than alpha times. Then x n is a Cauchy sequence. Notice that earlier I have given example of a sequence x n which has the property that mod of x n plus 1 minus x n goes to 0 but the sequence is not Cauchy. So recall that modulus of x n plus 1 minus x n going to 0 does not imply x n is Cauchy. Why so? Which example I had given?

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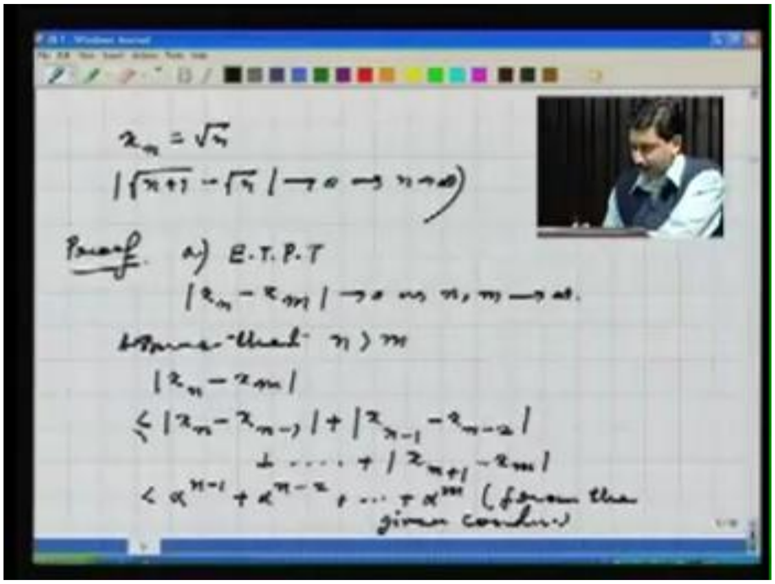

The example was x n equals to root n. Why x n is not Cauchy? Because if it is Cauchy then it has to converge but this sequence x n which is certainly unbounded and hence cannot converge because if a sequence is convergent it has to be bounded. This sequence is not bounded. That means, it has to diverge but we have checked that this is true. So what is happening?

Actually the theorem states that, if the difference of the consecutive terms goes to 0 fast enough, that is, it is rate of going to 0 is faster than alpha to the power n, then it turns out that the sequence actually is Cauchy but just an arbitrary sequence like mod of x n plus 1 minus x n going to 0, that surely does not ensure that the sequence is Cauchy. That is what this example shows. Now I will give you a proof of part a of the theorem. Part b is exactly analogous you have to reiterate the arguments which are involved. So let us first try to prove a.

I say it is enough to prove that modulus of x m minus x n goes to 0 as n m goes to infinity. That is what was our observation. Assume that n is strictly bigger than m and let us start trying estimating the term x m minus x n. I can surely write it as, it is lesser equals to modulus of x n minus x n minus 1 plus mod x n minus 1 minus x n minus 2 plus modulus of x m plus 1 minus x m.

How do I get this? I just insert the terms, extra terms which I am writing down and then subtract and just use triangle inequality. From that you get this. Now you use the given condition in a. I can say that this is less than alpha to the power n minus 1 plus alpha to the power n minus 2 plus alpha to the power m. This follows from the given condition. Once this is given, I know alpha to the power m is the smallest power which is appearing here. So I take that fellow common.

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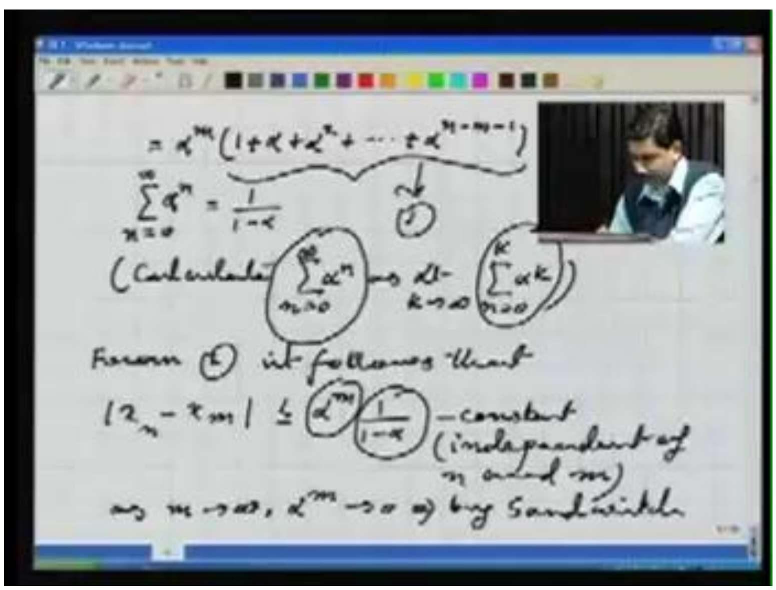

I get that this is equals to alpha to the power m into 1 plus alpha plus alpha square plus. Now notice that summation alpha to the power n, n starting from 0 to infinity is nothing but 1 by 1 minus alpha. This is just a geometric series. You calculate the partial sums, I will say calculate summation n equals to 0 to infinity alpha to the power n as limit k going to infinity. Summation n equals to 0 to k alpha to the power k. What is involved here? What is involved is essentially this, this sum anyway I can find out just by using G P series. Then whatever sum you get you take the limit as k goes to infinity.

Then by the definition, what I get is this quantity. That is, how we can calculate this sum, which is actually an infinite sum. You see that there is something involved here. It is not really a finite sum and we do not know what is the definition of infinite sum but still this seems to be something reasonable and I am going to use it. So if I call this star, let me call this, let me give it a different name. Let me call this 1. Then I say from 1, it follows that modulus of x m minus x n is less or equals to alpha to the power m. Now what remains is this sum and because of the calculation of the sum and since the terms involved here are non negative, I can say that this is actually less than 1 by 1 minus alpha.

Now notice that this is a constant. What I do mean by a constant? It is independent of m and n. So the only contribution of n and m in this inequality is just this fellow and from our previous calculation I know, as m goes to infinity alpha to the power m goes to 0.

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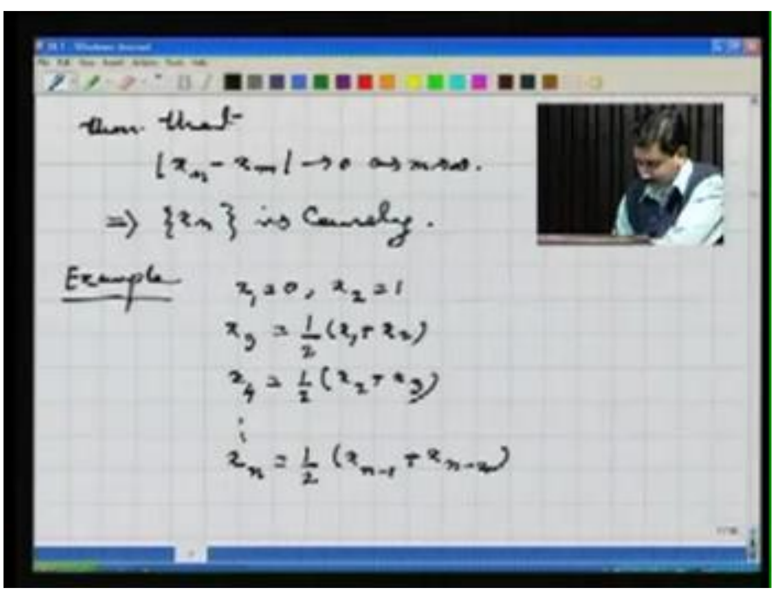

This implies by sandwich theorem that modulus of x n minus x m goes to 0, as m goes to infinity but does it prove it is Cauchy because what happens to n, I do not know. n is anyway l is bigger than m. That is what was my assumption. As m goes to infinity, so does n. That means, as n m goes to infinity modulus of x n minus x m is going to 0. This does imply that x n is Cauchy.

By the similar kind of methods actually but it will require little bit of worth from your part that the condition b also implies that x n is Cauchy. So what we have learnt? We have learnt that if a sequence x n satisfies either condition a or condition b then it is Cauchy sequence. Now I am going to apply this in certain particular cases. So I will study an example.

In this example, I am going to give you an example of a sequence, which if you look at the sequence, it is difficult rather difficult to calculate explicitly its limit and first to guarantee the existence, that the sequence it itself has got a limit, so the idea we are going to use is given that sequence, somehow I will try to prove first that the sequence is a Cauchy sequence and then the existence of the limit is guaranteed. Then the next thing is to calculation of the limit.

The sequence is a very natural one. What I do is, I take x 1 to be equal to 0. I just define it that way. I define x 2 to be equals to 1. Then I define x 3 to be equals to half of x 1 plus x 2. Then I define x 4 to be equals to half of x 2 plus x 3. In general, what I am doing? For anything, for any number n, I am looking at the two preceding terms of the sequence and then looking at their arithmetic mean. That is the general definition, I want to give is x n is half of x n minus 1 plus x n minus 2. So this certainly produces the sequence and I want to know whether this sequence converges or not.

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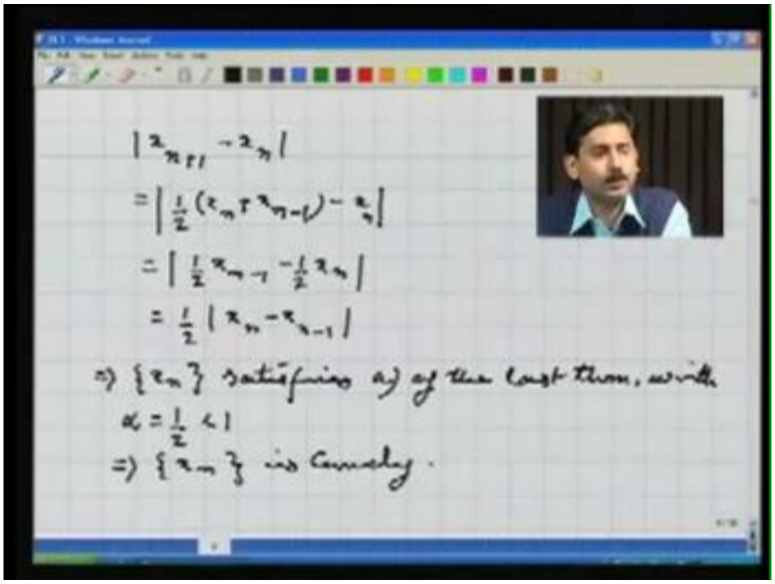

If it converges then I want to find out its limit. So what I do is, I start with modulus of x n plus 1 minus x n. Let us just try to calculate modulus of x n plus 1 minus x n. So I just write it as modulus of half x n plus x n minus 1 minus x n. That is, I am using the definition of x n plus 1. This is modulus of half x n minus 1 minus half x n. Then I take half common and since I am inside modulus, I can write it as x n minus x n minus 1. This implies the sequence x n satisfies a of the last theorem with alpha equals to half. That certainly implies x n is Cauchy. The next thing would be to calculate the limit of the sequence since it is Cauchy. I know it converges. Let us say l is the limit and I want to know explicitly what l is.

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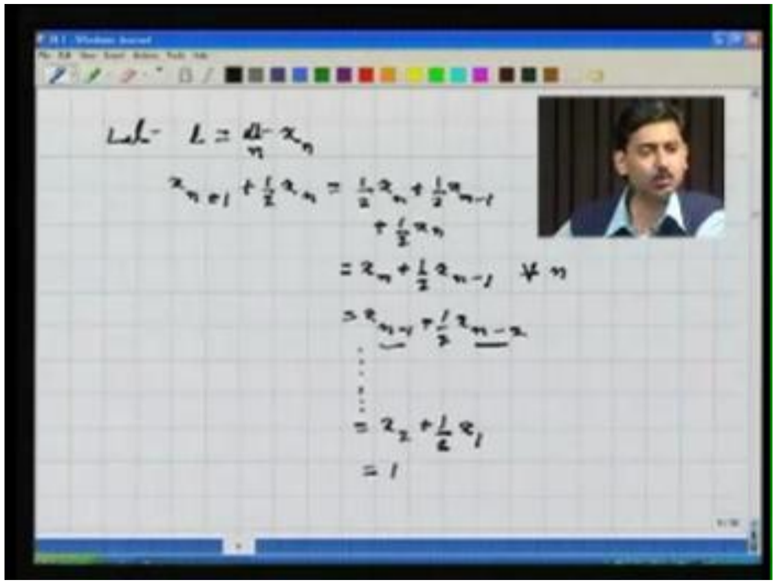

Let l equals to limit n. Now what I do is x n plus 1, with that I add half x n. What do I get? Let me just write down the definition of x n plus 1 again. This is half x n plus half x n minus 1. Now I have added an extra half x n. What I get is, x n plus half x n minus 1 and notice that this is true for all n. Now if I use or if I iterate the argument which I applied on x n plus 1 instead of x n plus 1, if I iterate the argument on x n, what will I get? I will get x n minus 1 plus half x n minus 2.

I can go on doing this but at certain stage, I have to stop. So if I go on doing this the last step, what I get is, now reiterating this definition if I want to come down I have to stop at certain stage, at which stage I will stop? I will stop at the stage after which there is no definition and that stage is x 2 plus half x 1. Notice that we cannot go below this stage because x 2 is, if it was given, it would be given in terms of x 1 and x 0 but there is no such 0. The sequence starts with x 1 but let me now just plug in the values.

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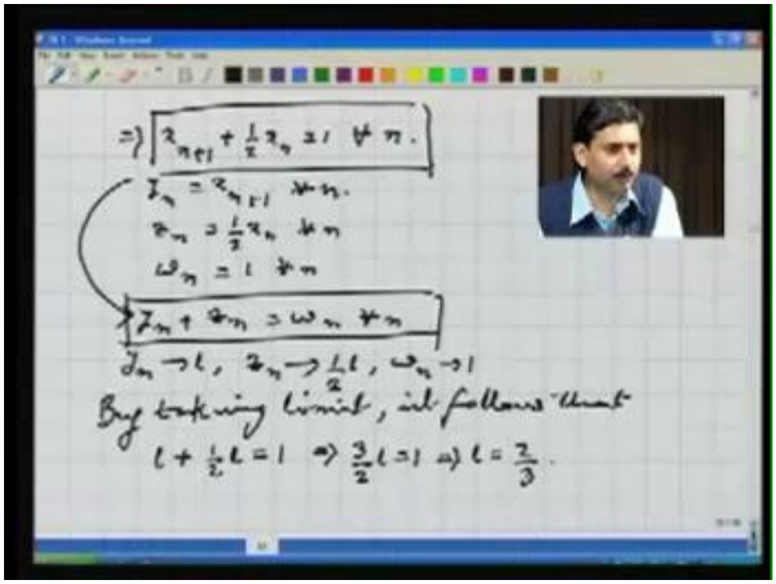

What is x 2? It is 1 and x 1 is again given to be 0. So what I get is that x n plus 1 plus half x n equals to 1 and this is true for all n. Now look at the scenario. I have a sequence y n which is x n plus 1. I have a sequence z n which is half x n and I have another sequence w n which is 1 and this is true for all n and it is given to me that y n plus z n equals to w n. This is what I got here. Now I know that y n anyway has got the limit l because the sequence x n converges. So I know y n converges to l. Where does z n converge?

Again, you use your multiplication formula of two sequences. If x n converges to l, then m times x n if m is a constant converges to m l. By the same argument, I have that z n converges to half l, since w n is a constant sequence, it converges to 1. So if I take limit at this stage, by taking limit it follows that l plus half l equals to 1. This implies 3 by 2 l equals to 1. This implies l equals to 2 by 3. See, we could find out the limit of the sequence.

That is all I wanted to tell you about sequences. We have gone through the fundamental concepts. So let us see what we have done. First, we have defined what does it mean to say convergence of a sequence. Then we did certain algebra with sequences, that if I have two sequences, both of which converge then what happens to the sum of those two sequences, product of those two sequences or quotient of those two sequences.

Then we started investigating the behavior of convergence. That is, when does the sequence converge? The first criteria we got is that every bounded monotone sequence converges with the presumption that every convergence sequence has to be bounded at the least and then we came to the most fundamental concept of sequences, the Cauchy sequence, why so, because we wanted to know without referring to the concept of limit that when does a sequence converge. The answer is it converges if and only if it is Cauchy and then I have given you one theorem, where at least there are two criterias which will help you in determining when a given sequence is Cauchy and then I used it to calculate limits of a sequence.

All these concepts finally will culminate into the behavior of continuous functions. That is the topic of our next lecture. When you want to study the elementary and deeper properties of continuous functions, quite frequently we will refer to the properties of the sequences which we have developed in these lectures.