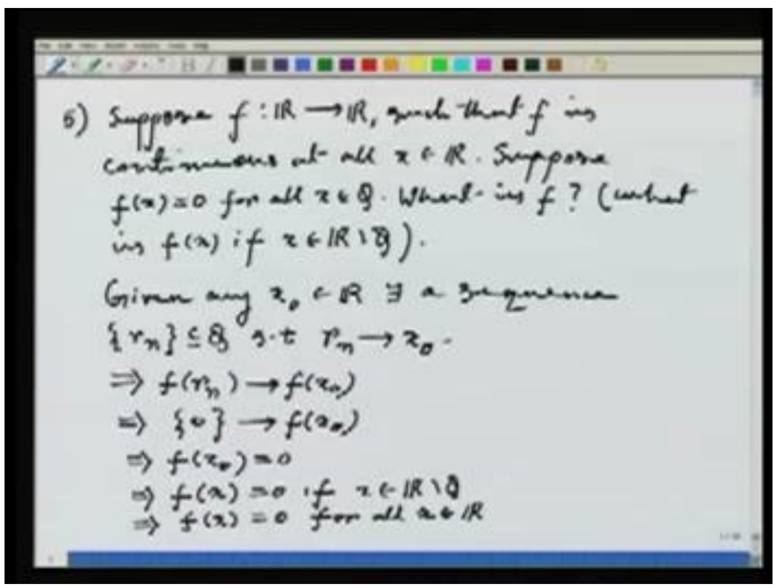
Mathematics-I
Prof. S.K. Ray
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture No. 6
Properties of Continuous Functions

Now let us see some more applications or problems which will help us in understanding the definition of continuity in better way. So let us take this following problem.

(Refer Slide Time: 00: 31)

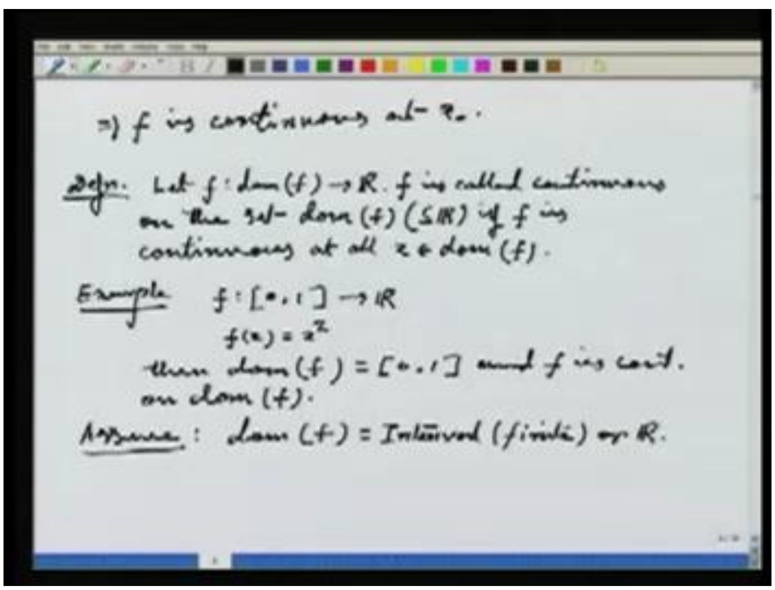


Suppose, f is a function from R to R. Also assume that f is continuous at all real numbers x. So this is the assumption of the function. It is continuous everywhere. Suppose f x is equal to 0 for all x in the set of rational numbers, then the question is what is f? The meaning of the questions is, what is the value of the function at all points which are irrational. That is, what is f x if x belongs to R minus Q? That is the question we are asking. Let us see how to solve this.

See, there is something involved here about the rational numbers. What do you know about the rational numbers in real line which is going to help us here? Well, rational numbers has the following property that they are dense, means what? It means you take any real number, call it x naught then there exist a sequence of rational numbers which converges to that real number. That is, we know that given any x naught in R, there exist a sequence r n which is contained in Q such that r n converges to x naught. Now the rest is really simple.

I know that f is continuous at every point on the real line. It means, in particular, f is continuous at x naught and since r n converges to x naught, I know that this implies that f of r n converges to f of x naught. But what do I know about f of r n? It is given to us that f of r n is 0 because f is 0 on the set of rational numbers. So this implies the constant sequence 0 converges to f of x naught but the constant sequence always converges to the same constant. This implies f of x naught is 0. It means in particular, f x is 0 if x belongs to R minus Q. That is, f is identically 0 function. Now before proceeding any further, let me first make some definitions.

(Refer Slide Time: 04:42)

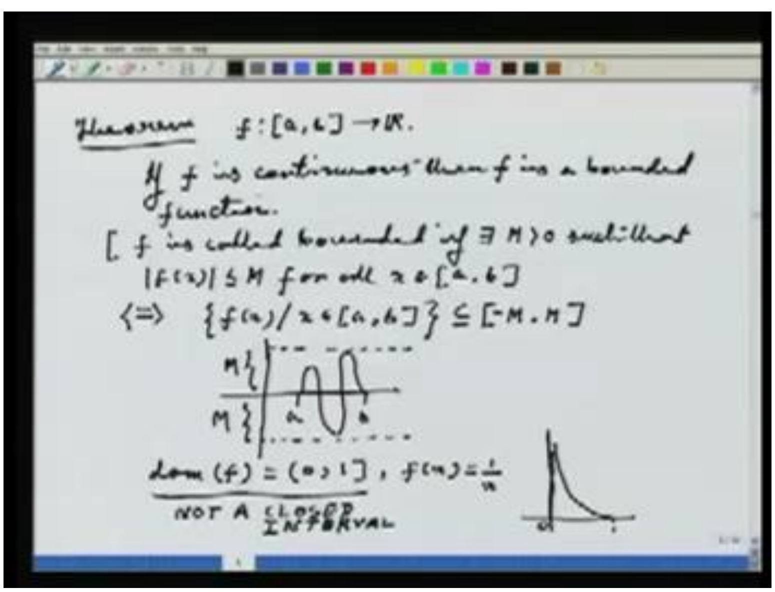


The point is, so far we have discussed only about continuity of function at points. Now I want to talk about continuity of functions on sets. So the definition we will see, what I am going to make, is a very natural one. Let f be a function from domain of f to R. f is called continuous on the set domain of f which is certainly a subset of real line. If f is continuous at each and every point of the set domain f, if f is continuous at all x in the set domain of f, that makes sense.

For example, suppose, I define the function f from 0, 1 to R given by f x is equal to x squared. Then what is domain of f? The way I have written the domain of f is 0 and 1 and f is continuous on domain of f because we know that f x equals to x squared f is continuous each and every points of 0 1. From now onwards I will be only illustrating the cases when domain of f is an interval. It might be an open interval, might be closed interval, might be semi open interval but always, it is an interval. Sometimes it might be the whole real line also.

So domain of f, I will assume so far that the domain of f is equal to interval, a finite interval or the whole real line and with all these assumption, there are essentially two results on continuous functions which come quite handy and have far reaching implications. The first one, which I am going to talk about now, deals with the case when domain of f is a finite closed interval. That is the first theorem I am going to talk about now, so the theorem.

(Refer Slide Time: 08:12)

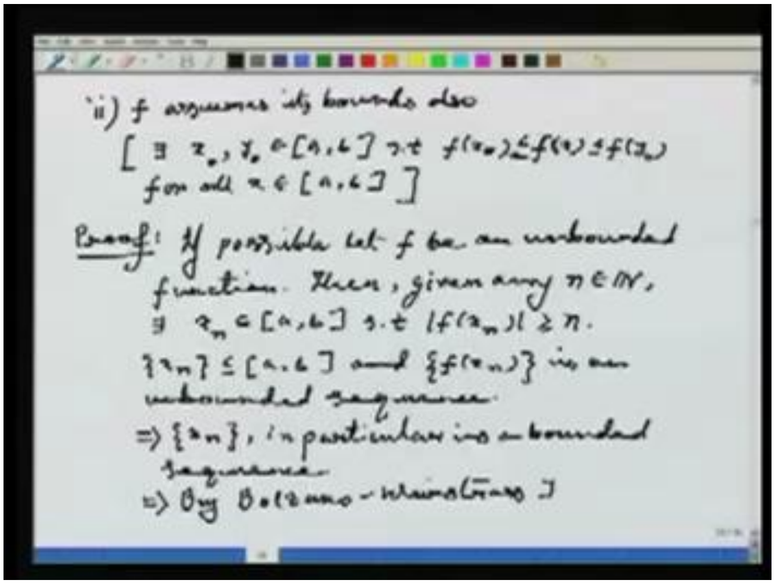


Let f be a function from a closed interval [a, b] to R. If f is continuous, then f is a bounded function. What does a bounded function mean? It means this, that if you look at the collection of all f x s as x vary over the closed interval, then those f x s, they also lie inside a closed interval. That is, f is called bounded if there exist some M bigger than 0, such that, mod of f x is less than or equals to M, for all x in the closed interval [a, b], which is same as saying that the collection of all x s where x is in the closed interval [a, b] is contained inside the set [– M, M]. That means in the picture, it will look something like this that this is a and this b. Suppose this height is M and this height is also M, then the graph of the function will not cross this line. This is what a bounded function means.

So shall we see an example of an unbounded function? Well, if I look at this set that domain of f is open interval 0 closed 1 and I define function f x is equal to 1 by x. Then what happens if you draw the graph of the function it looks something like this. This is 0, this is 1. The graph of the function look likes this, goes on and on and on, you know; it never touches the y-axis. This is certainly an unbounded function because given any real number, I can actually find an x as small as I like so that 1 by x is bigger than the real number. It follows really from the Archimedean property.

So unbounded do exist. But the thought of the theorem, is that if f is continuous and domain of f is a closed interval, then f has to be a bounded function. Notice that in my example, the domain of f, which I have taken, is not a closed interval. That is just one part of the theorem.

(Refer Slide Time: 12:44)



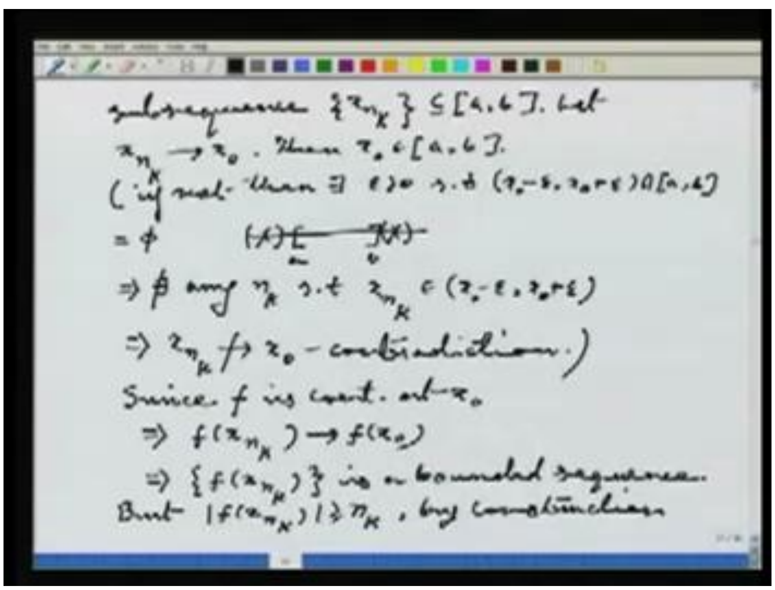
Now the second part says, f assumes its bounds also. What does that mean? It means that there exist x naught, y naught in [a, b], such that f of x naught is less than or equal to f x which is less than or equals to f of y naught for all x in closed interval [a, b]. In this case, you can certainly see that f x actually lies between two lines whose heights are f x naught and f y naught. Let us try to prove the first part first.

The idea is, we use the contra positive arguments. That is, you assume that f x is not bounded. So if possible, let f be an unbounded function. What does this mean? f is not bounded? Well, f bounded means there exist a real number M such that mod of f x is less than or equal to M. It is not bounded, that is, unbounded it means what? It means, whatever real number you take, there exist x in closed interval [a, b], such that mod of f x is bigger than or equal to that number in particular. Then it is going to be happen that if you take any natural number, let us say n, then there exist number x in the closed interval [a, b], such that mod of f x is bigger than or equals to M and I can call that x is x n. Let me write it down then.

Given any natural number n in N there exist x n a number in the closed interval [a, b] which depends on M, that is why I am writing x n, there exist x n in the closed interval [a,b] such that mod of f(x n) is bigger than or equal to n . Notice that in one stroke I got the two sequences. One is the sequences of {x n} which is lying in the closed interval [a, b] and I got another sequences f({x n }) which is an unbounded sequence. So first consider this sequence {x n}. This is contained in [a, b] and the sequence f({x n }) is an unbounded sequence. Fine, what do I know about {x n}?

{x n}is a sequence which lies in the closed interval [a, b]. That means, in particular, is a bounded sequence the {x n}. Now let us recall Bolzano Weierstrass theorem. What does it say? It says that every bounded sequence has a convergent sub-sequence. If I concentrate on the sequence{x n} which I have constructed, it is a bounded sequence. So by Bolzano Weierstrass theorem, it will have a convergent sub-sequence Call that sub-sequence, {x n k}.

(Refer Slide Time: 18:12)



This implies, by Bolzano Weierstrass theorem, there exist a sub-sequence {x n k} which certainly is contained in [a, b]. Let, since this is convergent, assume that it converges to some number x naught. Then I claim x naught is also in the closed interval [a, b]. Why so? Because if not, then there exist epsilon bigger than 0, such that, x naught minus epsilon x naught plus epsilon intersection closed interval [a, b] is null set. This is one property of closed interval I am using.

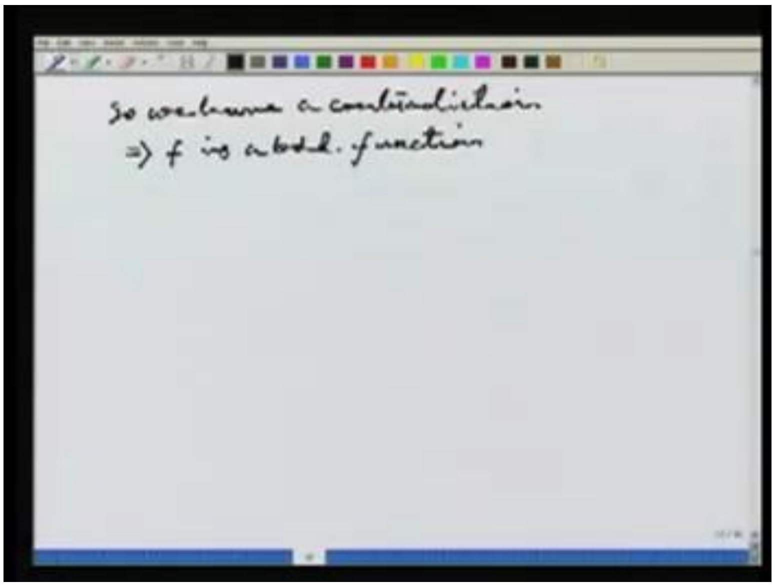
Pictorially, what I mean is suppose this is a and this is b and x naught in the interval means, it might be somewhere or it might somewhere here. But whatever it is, it will have some distance with a or b. So exploit that distance. So you can manage an epsilon, such that, you get this interval or you will have this interval around the point which does not intersect the point but then this implies there does not exist any n k, such that, {x n k} belongs to x naught minus epsilon x naught plus epsilon because all element of the sub-sequence {x n k} are actually contained in the closed interval [a, b] and I have got an interval which is outside the closed interval [a, b].

Those {x n k} which I have looking at, they cannot be in that interval. As a result, this then implies {x n k} does not converge to x naught which is a contradiction because I know the sub-sequence {x n k} converges to x naught. So I have established that {x n}, x naught, belongs to the closed interval. Also, we will see why it is important, because somehow I have exploited the continuity of the function which is given only on the closed interval. Once I guarantee that x naught belongs to the closed interval [a, b], I can use the fact that f is continuous at x naught. This is why we are trying to show that x naught belongs to closed interval [a, b].

Well, since f is continuous at x naught, that I know because x naught belongs to the interval [a, b], this implies, by the definition of continuity that f of {x n k}, look at this sub-sequence, this converges to f of x naught because x n k converges to x naught. f is continuous at x naught f of {x n k}, certainly converges to f of x naught. But now, we anyway know that every convergent sequence is a bounded sequence and f of {x n k} is a sequence which is convergent to f x naught. So it is a convergent sequence. So in particular, it is a bounded sequence.

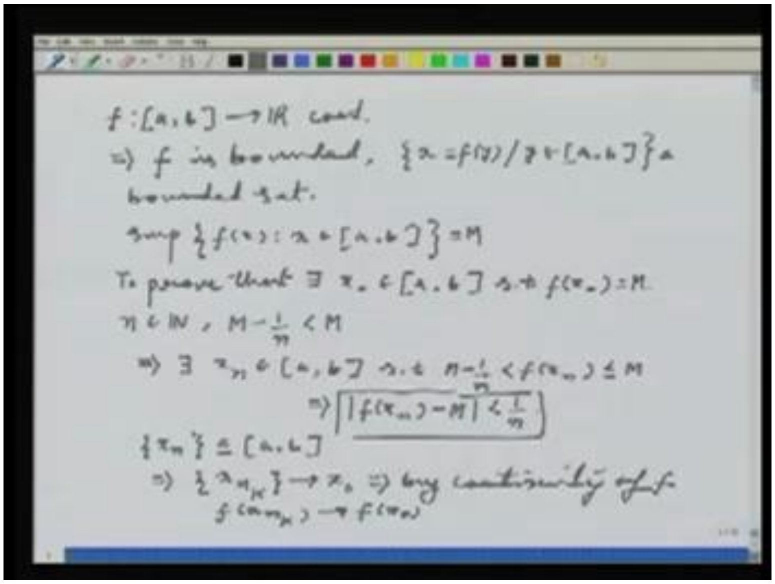
This implies, the sequence f of {x n k} is a bounded sequence but by construction of the sequence {x n}, I know that mod of f(x n k) is bigger than or equal to n k. This follows from the construction. So, on one hand, I have the sub-sequence f of {x n k}, which I have constructed is a unbounded sequence anyway. On other hand, using continuity of the function, I get that the sequence f of {x n k} is a bounded sequence. Now a sequence cannot be a bounded sequence and in the same time unbounded. So that it is a contradiction. Why this contradiction happens?

(Refer Slide Time: 23:26)



This happens because I assume that f is not bounded. So this is a contradiction. So we have a contradiction which implies that f is a bounded function, which is precisely what we wanted to prove. So what we have proved in the first part of the theorem, let us see again. We have proved that suppose f is a continuous function from closed interval [a, b] to R is continuous, then f is bounded.

(Refer Slide Time: 23:51)



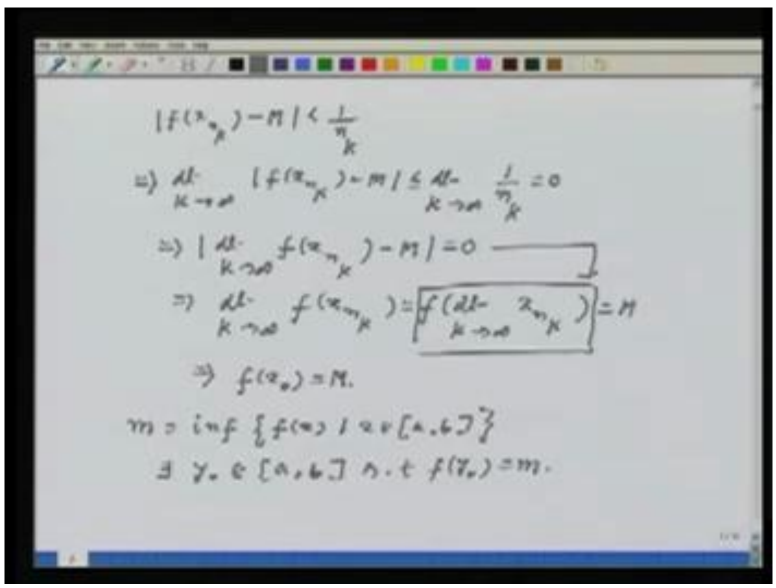
That is the set x, which is equal to f(y), where y belongs to [a, b], this is a bounded set. Now the second part of the theorem was to show that f attains its bounds. What does that mean to show? That is, let us say, that the supremum of the set f x where x belongs to [a, b]. Let us say, this is equal to capital M. Then I know the supremum exists. That is the first part of the theorem because the set of the values of f, that is the bounded set. So does have the supremum, call that as the supremum f. We want to prove that there exist x naught in [a, b], such that, f of x naught is equal to M.

Now how propose to show that? The only weapon we are going to use is the definition of the supremum. Fine, choose some natural number n in N and then consider the number M minus 1 by n, which is strictly less than M. That means, since it is less than the least of the upper bounds, that means this number M minus 1 by n is no longer an upper bound. This implies that there exist some x n in [a, b] such that M minus 1 by n is strictly less than f of (x n) which is less than or equal to M. Good, this in turn implies mod of f of (x n) minus M is strictly less than 1 by n.

Finally, I have managed a sequence because this is happening for every n. Now where do the sequence x n lie? I know that this actually contained in the closed interval [a, b]. That means it is a bounded sequence. This implies, by Bolzano Weierstrass theorem, there exist a sub-sequence which I might call {x n k}, now which is convergent, say that it converges to some x naught and notice, since the whole sequence inside the close interval [a, b], the limit of the sequence also has to be in the closed interval [a, b]. It cannot go outside. This kind of a logic, we have used before also.

Now this implies, by continuity of f, now we have that f(x n k) converges to f of x naught. You must have guessed by now that I want to prove that f of x naught is equal to M. How do we propose to show that? Well, this is where I am going to use this inequality. What does this inequality tell me?

(Refer Slide Time: 28:36)

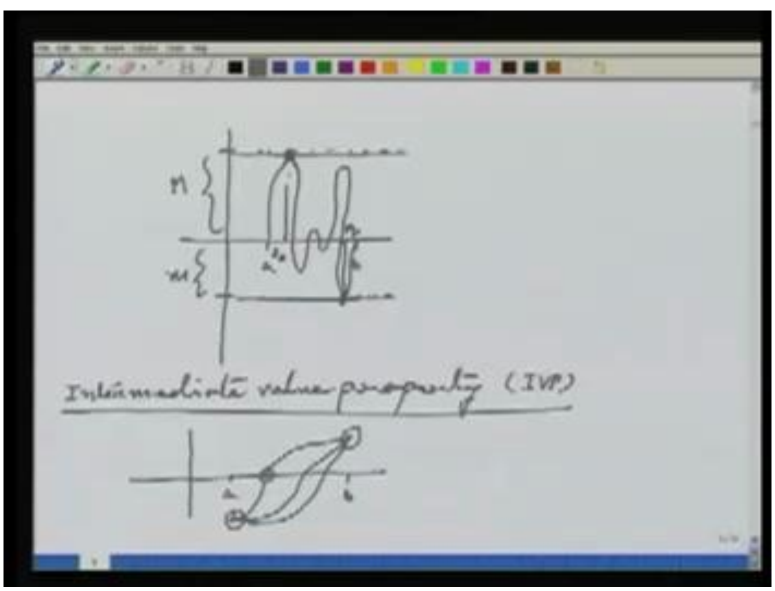


It tells me that mod of f x n k minus M is less than 1 by n k. Now this implies, if I take the limit as k going to infinity, what do I get? I get f of x n k minus M. This is less or equal to limit k going to infinity 1 by n k and I know the right hand side limit 0, so I put 0 here but now, if I use the fact that modulus is a continuous function and f itself is a continuous function. It then implies that mod of limit k going to infinity f of x n k minus M. That is equal to 0.

This certainly implies that limit k going to infinity, f of x n k, which by continuity f of limit k going to infinity x n k, which is equal to M from this inequality. But then implies, as this quantity is actually f of x naught, I have that f of x naught is equal to M, that is, the upper bound has been achieved. Certainly, that is not the end of the proof. The next step is, if you look at, as little m is equal to infremum of f x with x in [a, b] and then exactly assume the previous argument that I can show there exist y naught belongs to [a, b]. The proof is exactly similar, you can try this way, such that, f of y naught, that is also is equal to m.

That means, the supremum of the values of f x, which is the best possible upper bounds you can have, is actually f of x naught for some x naught. Similarly, the infremum of the values of f x, which is the best possible lower bound which you can have, is actually achieved by some value y in [a, b], such that, f of y naught is M. Now, this finishes the proof of the result. What we have proved, we have proved that if I have continuous functions on a closed interval [a, b], then first of all, the function is bounded. That means there is a height. You can give in this way that this is the plane R 2, a is here and b is here.

(Refer Slide Time: 31:38)



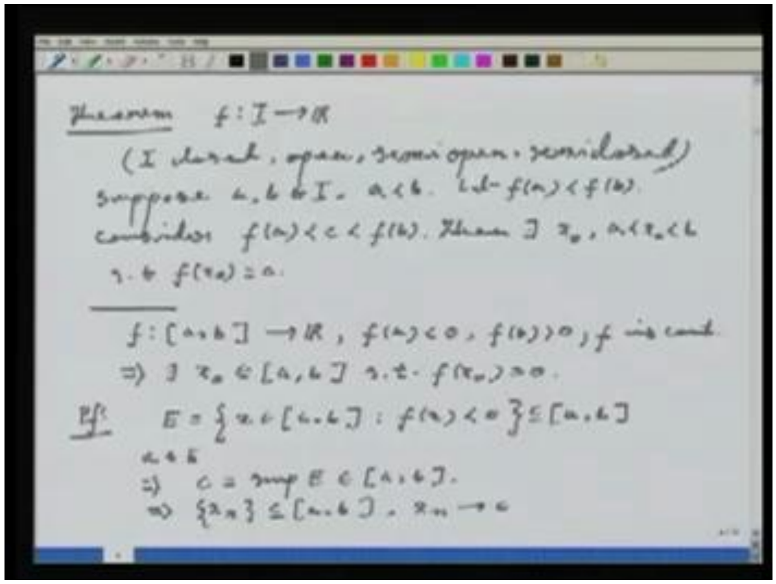
If f is a continuous function on the closed interval [a, b], then all that I am saying is there exist height here and some height here. I might call this height, M and this, little m. Then I draw these lines. Then f is bounded means the graph of the function over this closed interval [a, b] lies within these two lines. That is, the graph of the function has to be something like, you know. It cannot cross these two lines which I have drawn. That is, what boundedness of a function means and it achieves the bounds here may be. I will put height a bit m naught here. The corresponding value is x naught. Here, it is x naught, this y naught. That is what the theorem means.

Now let us go to next important property of continuous functions on intervals. This is called the intermediate value property. In short we call it IVP. What does the statement says? Suppose, just look at this kind of an example. This is again the axis. Suppose I have the point a here and the point b here. Suppose at the point a, f of a is negative, that is the value somewhere here and suppose at the point b, the value of f is positive.

Now f is a continuous function. It means, if I can draw the graph of the function, then I can draw it without any break. That means, in the graph, there is no break. Now if we have to do in that way, that means, we have to join this point and this point through a graph which has no break. It is obvious to see that the graph has to look like this. It might be something like this also. I do not care. Without any break, I am drawing the graph. That means, there is some point on the x- axis which is the value f of x.

For example, this point, this point and this point. That means there exist some x in that interval where the value of f x is 0. The intermediate value property actually generalizes this. What does that say then? Let us come to the precise statement of the theorem.

(Refer Slide Time: 34:41)



Suppose f is a continuous map on an interval. Now I, it can be closed, it can be open, semi open or semi closed. While doing the proof, we will notice it is only the closed interval which matters but the result is true for any interval. What does it say? Suppose a and b are two points in I with a less than b. Let f(a) also satisfy the condition. It is less than f(b). Now just consider any number between f(a) and f(b). Consider some c which satisfies.

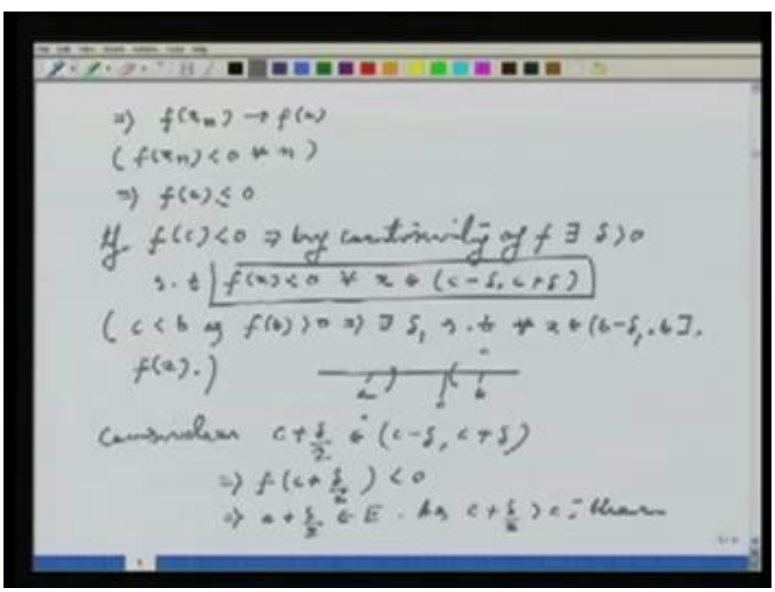
Then I want to say, there exist some x naught which satisfies a less x naught less b, such that, f of x naught is equal to c. It just means that between two functional values, if you take any real number, that real number itself is a functional value. That is what intermediate value property says. Now, to prove this we will essentially do a special case and we will see from that how does the usual proof follow.

Let me assume first that I have a map f from closed interval [a, b] to R. It satisfies two conditions that f(a) is strictly less 0 and f(b) is strictly bigger than 0 and f is continuous, of course. Fine, the claim is, then this implies there exist x in [a, b], such that, f of x naught is equal to 0. The thing to notice here is that, if you look at f(a) and f(b), f(a) is negative number and f(b) is a positive number. So 0 is the number which lies between these two numbers and then what I am trying to show is, there exist a point x naught, so that f of x naught is equal to 0. That is almost like intermediate value property, right? There is a value which intermediate between f(a) and f(b). Then that value is actually a functional value. That is what we are planning to prove. Let us try to prove this first.

How do I prove? First let me consider a set E, which is collection of all x in [a, b], such that, f of is strictly less than 0. Notice that this is a sub-set of [a, b] because all the x s, which I am taking, they are from [a, b]. Now, notice that this set E is certainly non empty, because a belongs to E because f(a) is given to be strictly less than 0. That means this set E which I constructed, is a non-empty set and it is bounded above, means this set E must have a supremum in R.

So this implies, call that supremum c. Now also notice that since E contained in the closed interval [a, b], the supremum of E is anyway less than or equal to b. At the same time, it is also bigger than a because a epsilon E. This implies, this c actually belongs to [a, b]. I wish to show that f(c) is actually 0. Why is that so? Now since c is supremum, this implies, there exist a sequence{x n}, which is contained in [a, b] and also {x n} converges to c, that we have observed before.

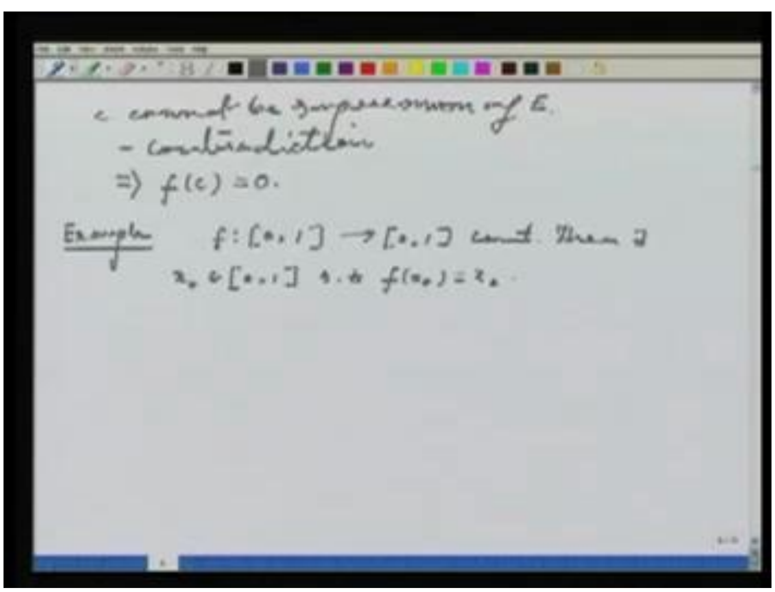
(Refer Slide Time: 39:57)



Now this implies by continuity that f of {x n} converges to f of c. Now I also have that f of {x n} is strictly less than 0, for all n. What does this imply about the limit? If I have a sequence, all whose terms are negative then the limit of the sequence cannot be positive. At the most, it can be 0. So this implies then that f of c is less than or equal to 0. What I need to show is, f of c is equal to 0. So if not, if f of c is strictly less than 0, I have to get a contradiction. Now the f of c is less than 0. This implies that c is less than b. So pictorially, it means just this, this is a, this is b. The value of the function is negative here.

That means, there exist a neighborhood where it is negative and b, it is positive. That means, there exist a neighborhood where it is positive. So c is somewhere here maybe. That means, if f (c) is less than 0, by continuity of f, as I already said, there exist a delta bigger than 0, such that, f of x is less than 0 in the whole neighborhood of c. What does that mean? So consider then c plus delta by 2. Now c delta by 2 belongs to the interval c minus delta c plus delta. This implies f of c plus delta by 2 is also less than 0 because I said that f of x is less than 0 for all x in the interval c minus delta c plus delta. Then this implies c plus delta by 2, it belongs to E by my definition because E consists of all the points, where f of x is less than 0. It seems that c plus delta belongs to E but then c cannot be a supermum as c plus delta by 2 is strictly bigger than c.

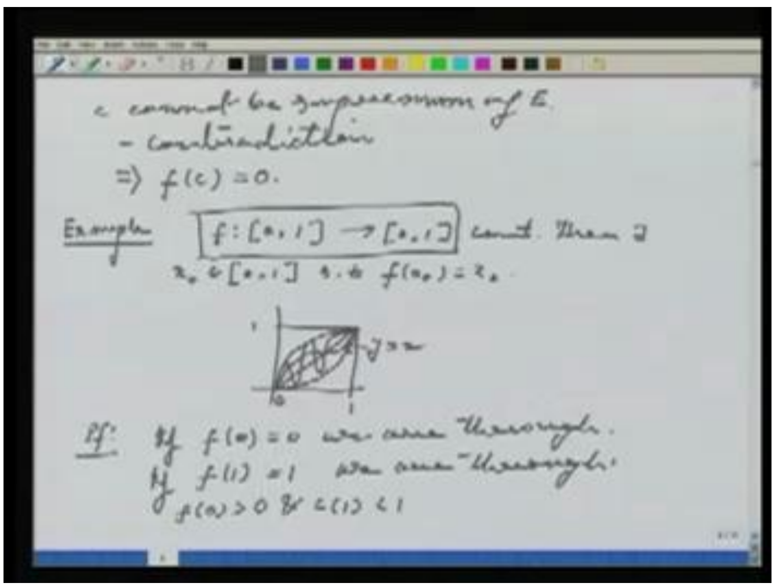
(Refer Slide Time: 42:40)



Then c cannot be a supremum of E but this is contradiction because we know c is the supremum. Why this contradiction happens? Because I have assumed f of c is strictly less than 0. That cannot happen but I already know f of c is less than or equal to 0. So this implies what? It implies f of c is equal to 0. Now, let us see some application of intermediate value property, you know, so the first application.

Consider the function f from 0 1 to 0 1 which is continuous. I want to prove that then there exist x naught in 0 1 such that f of x naught is equal to x naught.

(Refer Slide Time: 44:09)



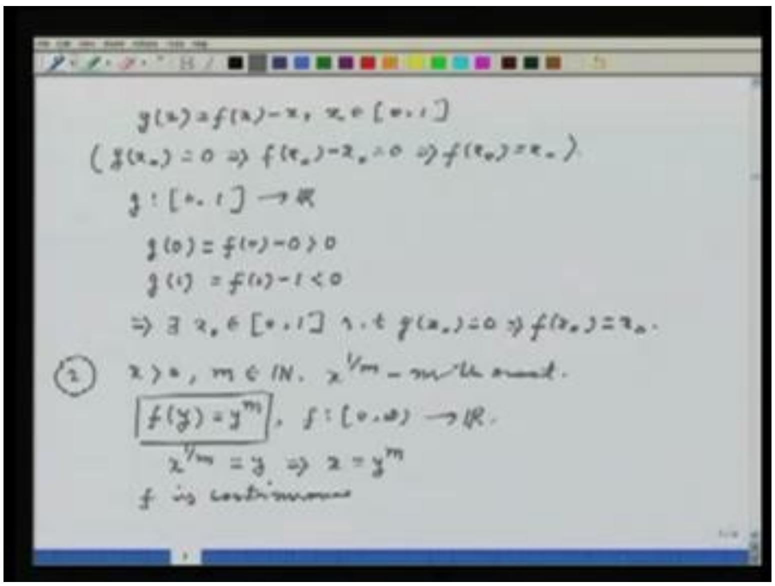
Pictorially, it means just this. First, consider 0 1. This is 0, this is 1. Consider 0 1 here. Look at the square. Now the graph of the function lies between this square. That is what this condition says. Then the assumption says, if you look at the diagonal which is the line, this is the line y is equal to x. Then the graph of the function, at some point, has to intersect this diagonal. It can be this. It can do several times, but it has to.

So how does one prove? See there are two obvious cases. It can happen that f 0 is equal to 0. If f 0 is equal to 0, we are through. That means, I can take 0 to be equal to x naught. If f 1 is equal to 1, again, we are through. Now the third case is when this is not true.

What does it mean to say this is not true? Notice that these two are not true implies that f of 0, since it has to be in the closed interval 0 1. But it is not 0.

That means, it has to be strictly bigger than 0 and at the same time, I am also saying that f of 1 is not 1. But it has to be in the closed interval 0 1. That means, the possibility is then f of 1 is less 1. Both of these has to true, right? If none of the above happens, this is the assumption now I am working with. That f of 0 is strictly bigger than 0 and f of 1 is strictly less than 1, very good.

(Refer Slide Time: 46:51)



Now I define another function, call it g. So I define g(x) as f x minus x. The purpose of considering of g is very easy. It just that I have to show that at some point x naught, g of x naught is 0. Just notice that g of x naught is equal to 0 implies f of x naught minus x naught is equal to 0. This implies f of x naught is equal to x naught. This is the motivation to construct the function, the function g. Now, what do I know about g? g is a function from 0 1 to R. That is obvious because I have defined it for x in 0 1.

Also, the important thing is, since f x is equal to x is a continuous function and the given function f is a continuous function; the difference of two continuous function is always a continuous function. That is why the function g which I have defined, that is also a continuous function. Now notice that whatever g 0, by definition, this is f 0 minus 0 but I know that f 0 is strictly bigger than 0. That means, this is bigger than 0. What about g 1? This f of 1 minus 1, but I know that f of 1 is less than 1. That means this is less than 0.

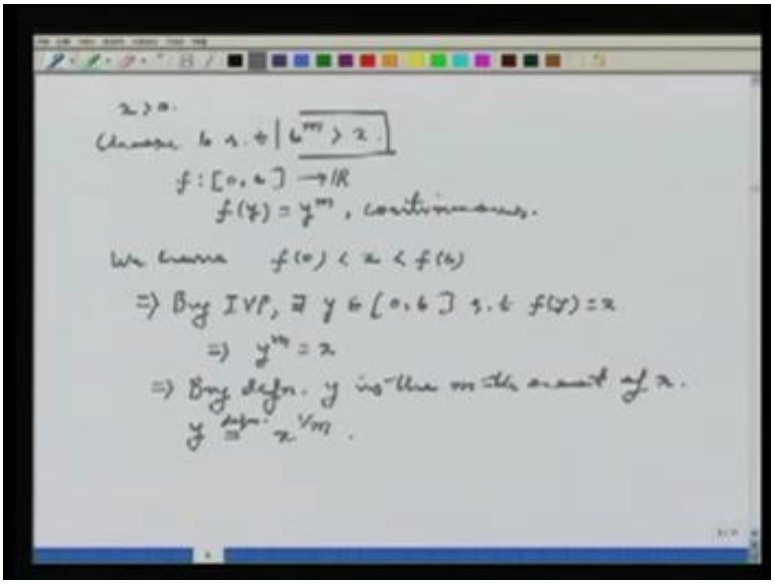
Now I am in the set up of the previous theorem. g is continuous function defined on 0 1. At 0, it is positive; at 1 it is negative. That means, at some point g has to 0, by the previous result. This implies, there exist x naught in 0 1 such that g of x naught is equal to 0 and this implies, as we have observed before, that f x is equal to x naught. This is precisely what we wanted to prove. Now let us see some more important applications of this result. The first is the following.

Suppose I take a number x which is bigger than 0 and let us take a natural number m in N. I want to understand the meaning of x to the power 1 by m called the mth root. See, the point is we want to about the m-th root is, we do not have precise definition of real line with us. We are trying to develop it without going into the details of the definition of the real line.

Once we do that, several questions are arises: things like what do you mean about the square root of a real number, cube root of a real number and so on because the precise definition of the real number is not there. But we want to show that some thing like x to the power 1 by m should exist and this is where you will see the thing like supremum axiom and whatever we have developed will come handy. It will give you the existence of some such things.

Let us see, how to do that. So look at the function defined by f of y is equal to y to the power m where the function is defined for f from 0 infinity to R. What do I want to show? I want to show that given a real number x, I can talk about x to the power 1 by m. So, if x to the power 1 by m is equal to y, this would then imply x is equal to y to the power m. That motivates the definition of the function which I am looking at. Good, now concentrate on the function f(y) is y to the power 1 by m. Since this is a polynomial function, I know that f is continuous.

(Refer Slide Time: 51:58)

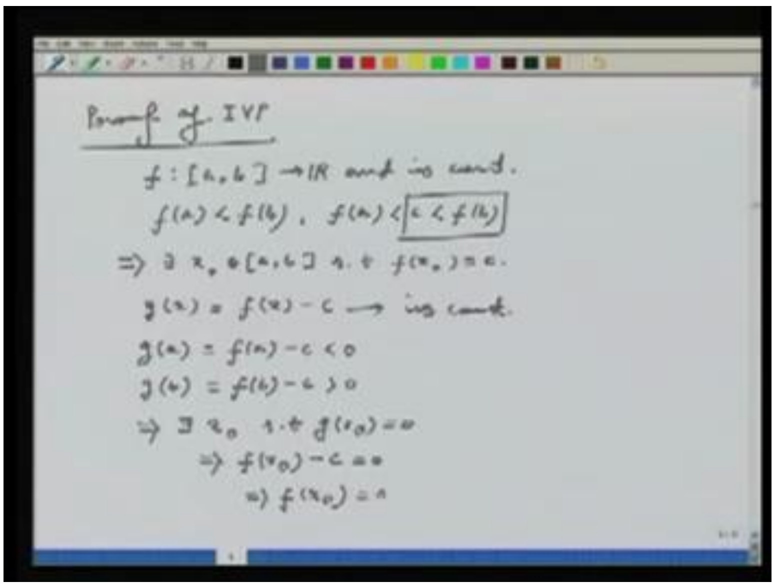


Now x is given to me; x is bigger than 0 and I am just saying that choose b such that b to the power m is bigger than x. That can always be done and then I concentrate on the function f from 0 b to R given by f of y is equal to y to the power m, which is continuous. Now all I want to use is the intermediate value property. Since I have this condition, this implies that we have f of 0 less than x, less than f(b). Then this implies by the IVP, there exist y in the closed interval 0 b such that f of y is equal to x which implies y to the power m is equal to x which implies by definition y is the mth root of x. That is what we wanted to show and this y is actually called, this is the definition, y is called x to the power m.

Right now before the proof of this, there was no definition of x to the power 1 by m. So we define the m-th root of x to be equal to some y, such that, y to the power 1 by m is x, then I what to show the m-th root of real number exist, all I need to show is that there exist y so that y to the power 1 by m is x, which I have shown by the intermediate value property. That means m-th root of a positive real number exists.

But here while doing this I have used the statement of the intermediate value property which I have not actually proved. I have proved the special case of intermediate value property. That is, if I have a continuous function which takes positive value at one end point and negative value at one other end point, then 0 is also a functional value. That is not exactly the statement of the intermediate value property as we stated. Now let us try to prove intermediate value property from the result which I have already used. So now I come to the proof of IVP.

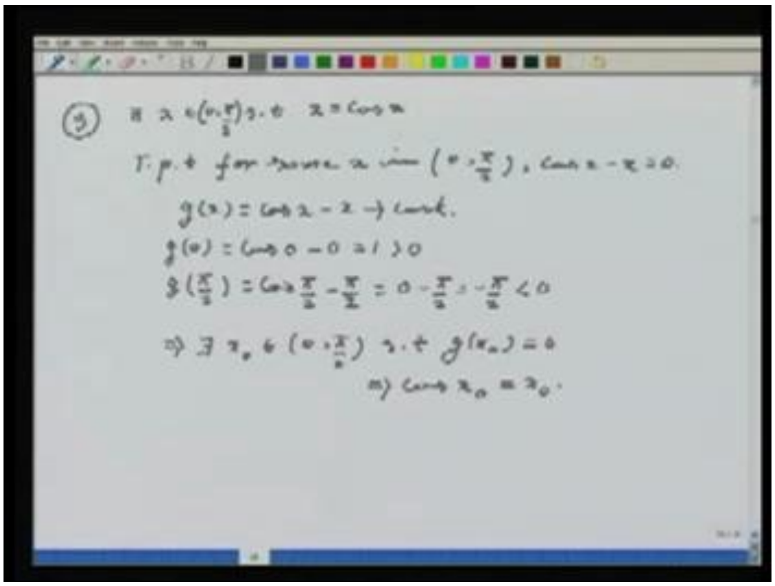
(Refer Slide Time: 55:05)



What does it say? f is from [a, b] to R and is continuous. Suppose f(a) is less than f(b) and also suppose there exists a c which satisfies f(a) less c less f(b). Then I want to prove that there exist x naught in the closed interval [a, b] such that f x naught is c. While doing this, I will use the result I already proved and I define the new function g(x), which is f x minus c. The idea is clear. I want to prove that this function g has a 0. That means, there exist a x naught where g of x naught is 0. Once I proved that, it will follow that f of x naught is c.

That will solve the problem. So first look at what is g of a. This is f(a) minus c but since c is bigger than f(a), this is less than 0. Now if I look at g of b, which is f of b minus c but since I have that f(b) is bigger than c, what I have is, this is bigger than 0. This implies, there exist x naught as g is continuous such that g of x naught is 0. Notice here that I am using the fact g is continuous but g x naught is 0 implies f of x naught minus c is equal to 0 which implies f of x naught is equal to c. This is precisely what we wanted to prove. .

(Refer Slide Time: 57:53)



After finishing intermediate value property, let us look at another example. It says, there exist a x in R such that x is equal to cosine x. In fact, I can represent this by saying, there exist x in 0 pi by 2 such that x is equal to cos x. How to solve this problem? The idea is exactly as in the previous result. I have to show x is equal to cosine x. That means, I have to show that cosine x minus x is 0 to prove that, for some x in 0 pi by 2 cos x minus x is equal to 0.

Then it immediately tells me that now I can look at the function g(x) is equal to cosine of x minus x and then let us see, what is g 0. That is, cosine of 0 minus 0 which is 1 and then which is strictly bigger than 0 and what is g of pi by 2. This by definition of g is cosine of pi by 2 minus pi by 2. But cosine of pi by 2 is 0. So 0 minus pi by 2, which is minus pi by

2, which is less than 0. Notice that g, again, is continuous. Then this implies g must have a 0. This implies, there exist x in 0 pi by 2, such that, g of x naught is equal to 0. This implies cosine of x naught is equal to x naught and this is precisely what we wanted to prove.