Mathematics-I

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Lecture No. 7

Uniform Continuity

Now let us see some more applications of the intermediate value property. It starts with the following problem. The question is as follows

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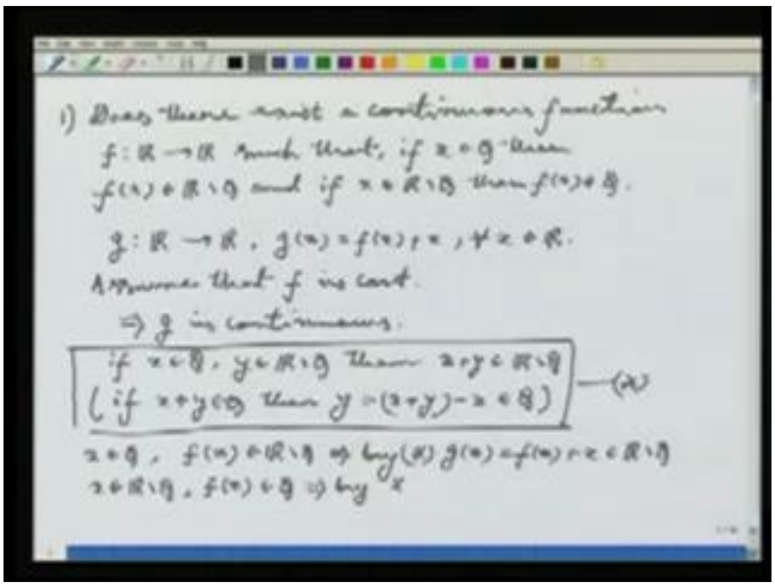

Does there exist a continuous function f from R to R which has the following property, such that if x is a rational number, that is, x is in Q, then f x is an irrational number. That is, it belongs to R minus Q and if x is an irrational number, that is, x belongs to R minus q, then f x is a rational number, that is, f x belongs to Q. So is the question clear? We are asking for, does there exist a continuous function f from R to R, such that, if x is rational then f x is irrational and if x is irrational, then f x is rational. So we proceed with the problem as follows:

Let us define a function g from R to R by the following rule. What is g? I define g x is equal to f x plus x, for all x in R and we are trying to give the contra positive argument. That is, assume that f is continuous. So I want to get a contradiction out of it and then that will show no such a continuous function exists.

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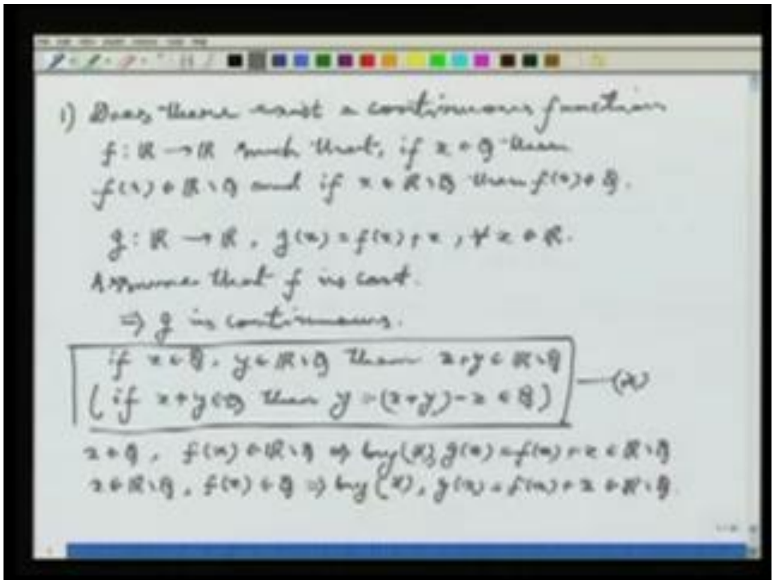

If I assume that f is continuous, then the first thing which follows is that g is continuous. Now in the following arguments, I am going to use one thing time and again. That is this: if x is a rational number and y is an irrational number, then x plus y is an irrational number. This is the observation which I am going to use time and again which is very easy to see because if x plus y is in Q, that means it is a rational number. Then x plus y minus x, that is also in Q. Let us write it down. If x plus y belonging to Q, then y equals to x plus y minus x belonging to Q, which is a contradiction simply because that sum and product of two rational numbers is a rational number.

This is the observation, if I use then I get that sum of a rational number and an irrational number is always an irrational number. This I am going to in my arguments. Now if I have g is continuous, let us look at the property of g. Suppose x belongs to Q. Then I know the defining property of f that f x belongs to R minus Q but then this implies, if I call this star, by star, g x, that is equal to f x plus x, that is an irrational number. That means, it is in R minus Q. If x is rational, g x is irrational. Now let us take x to be an irrational number.

Suppose x is in R minus Q, then I know again that f x belongs to Q but then this implies again by star that g x, that is equal to f x plus x, again belongs to R minus Q. So it follows from these two calculations that g has the following property that whatever x you choose in the set of real number, g x is always an irrational number. That means what I have is g is the map from R to R minus Q and g is continuous. That I am getting because I have assumed that f is continuous.

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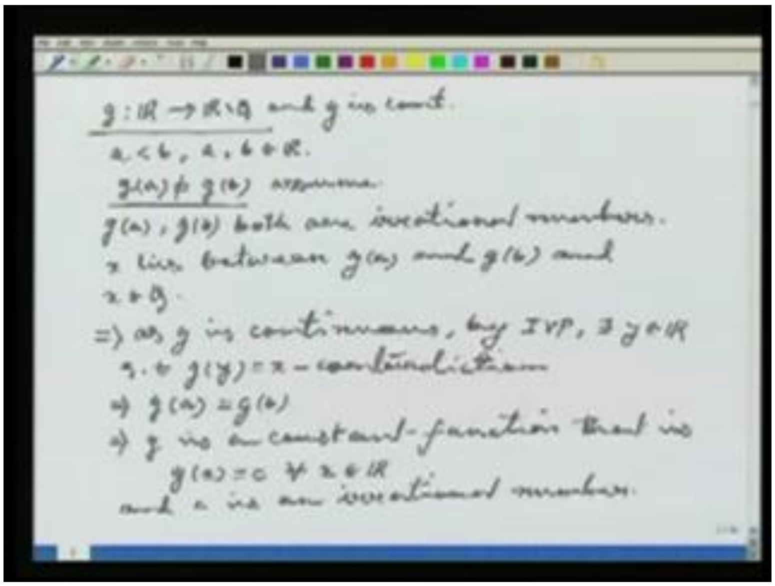

Now let us see what more can I say, more about it. Let us try to use the intermediate value property on the function g. Let us take some, a strictly less than b, where a, b, both are real numbers. Then let us look at g a and g b. It can happen that they are same and it can also happen that they are not same. That means, let us assume this. Now I know that g a and g b, both are irrational numbers. Then what does intermediate value property say? Suppose I choose an x which lies between g a and g b. So x lies between g a and g b and let us choose x to be a rational number. That I can do because if g a and g b are unequal real numbers, then always there exist a rational number in between those two.

By the intermediate value property as g is continuous, by the intermediate value property, I write it as I V P. There exist y in R, such that g of y is equal to x. But then you see, I got a contradiction because g y is standing out to be a rational because I have chosen my x to be a rational number but look at number the definition of g. It is map from R to the set of irrational number R minus Q. That means, for no x in R, g x can be a rational number. That has happened. It has happened simply because I have assumed g a is not equal to be g b which is not true. Then this is a contradiction because g cannot take rational values, implies g a is equal to g b.

What does this mean because I have a b to be arbitrary real numbers? This implies then that g is constant function. That means, that is g x is equal to a number c for all x in R and surely c has to be an irrational and c is an irrational number. Now, since I have got that g x is equal to c, I got some inference about f also.

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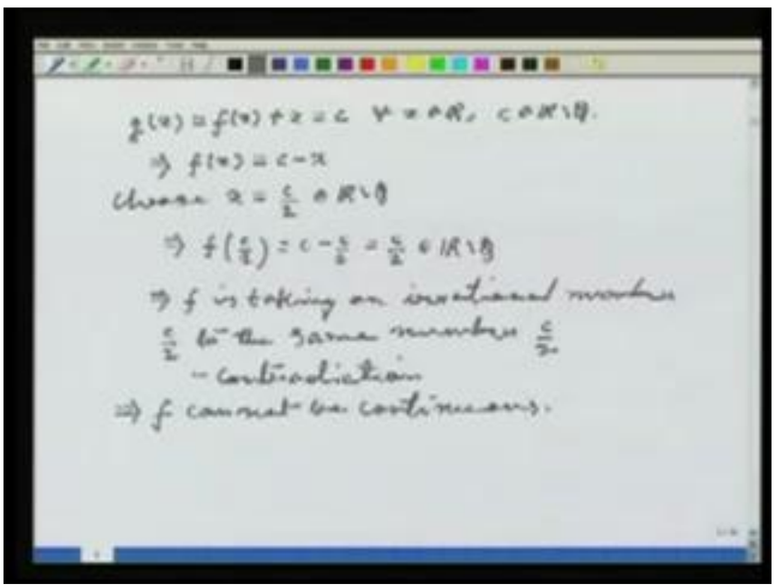

Just writing down the definition of g I get that g x is equal to f x plus x, that is equal to c for all x in R and always remember that c is in R minus Q, that is, it is in the irrational number. Then what do I know about f? That means f x is just a line. That is, it is c minus x. Now in particular let me choose x equals to c by 2. Then I know that this is also an irrational number. This cannot be rational because if c by 2 is rational, call it r. Then, c is equal to 2 r but if r is rational, then 2 r is rational. That means, c will be rational which is not the case. That means c by 2 must be an irrational number but what happens to f x?

What is f of c by 2, that is c minus c by 2? That is equal to c by 2. Notice that c by 2 is an irrational number and f of c by 2 is c by 2, which is also an irrational number then. But you see that cannot happen. f takes rationals to irrationals and irrationals to rationals, but in this case what is happening? f is taking an irrational number c by 2 to the same number c by 2, which is an irrational number which is a contradiction.

This implies that f is taking an irrational number, c by 2 to the same number c by 2. That is a contradiction to the defining property of f; it says that f takes rationals to irrationals and irrationals to rationals. So this is a contradiction which implies, f cannot be continuous. That solves our problem. Now, let us see another application of the intermediate value property.

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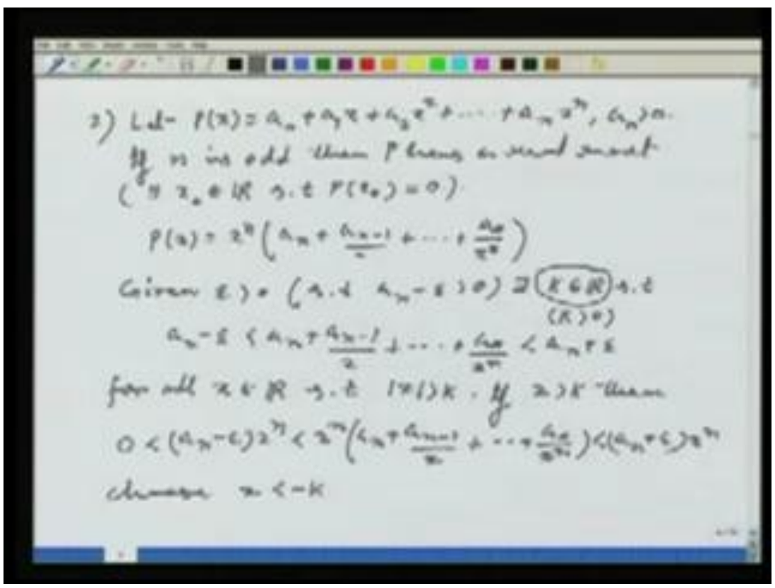

Let P x stand for a polynomial. We know how do polynomial looks like. It is a 0, plus a 1 x, plus a 2 x squared, plus a n x to the power n and I say a n is not equal to 0 and assume a n is positive. Now I say that if n is odd, that means it is of the form 2 k plus 1. Then P has a real root. What does that mean? That means there exist some x naught in R such that P of x naught is equal to 0. What I want to do is, I want to apply intermediate value theorem to the function P x because I already know that any polynomial function is actually a continuous function.

Fine. What I do first is, I write P x as x to the power n a n plus a n minus 1 divided by x plus a 0 by x to the power n. The naive idea goes as follows that if I look at the quantity, if you take x large, it does not matter in which direction, positive large x or negative large x, you feel, that this quantity should converge to a n because all the individual quantity which are involved, they are having x in the denominator and that is getting large. So those quantities will be small. So you expect that the underlined portion converges to a n, which is positive.

Now since n is odd, if I take x as positive, then anyway x n is positive. If I take x as negative then x to the power n is negative. That means, you expect, there will be a change of sign for different values of x and then 0 comes in between those different values. That means by the intermediate value property there should exist x naught such that P of x naught is 0. That is the idea we want to use. But we have to make it rigorous. Let us first write down. What does it mean to say the portion in the parenthesis converges to a naught? I say that given epsilon bigger than 0, I choose it in such a way that a n minus epsilon is positive.

I say there exist a real number k in R, such that, a n minus epsilon is less than a n plus a n minus 1 divided by x a 0 by x to the power n, which is less than a n plus epsilon for all x in R such that, mod x is bigger than k. That means the k I have chosen, it is a positive k. This is just meaning of the fact that as x goes to infinity, the portion in the bracket converges to a n. I have just written it in terms of epsilon following the definition of limit. Now if I choose x bigger than k, what happens then?

Then, a n minus epsilon times x to the power n is anyway less than x to the power a n plus a n minus 1 by x plus a 0 by x to the power n, which is less than a n plus epsilon times x to the power n. Notice then that by my choice this also happens because a n minus epsilon is positive. I have taken my x to be positive. That is also bigger than 0.

Now what I can do is, suppose I choose x less than minus k. That means, now I am choosing x to be negative. Notice that since I have taken my n to be odd, x to the power n is also negative if x is negative. So what does this imply?

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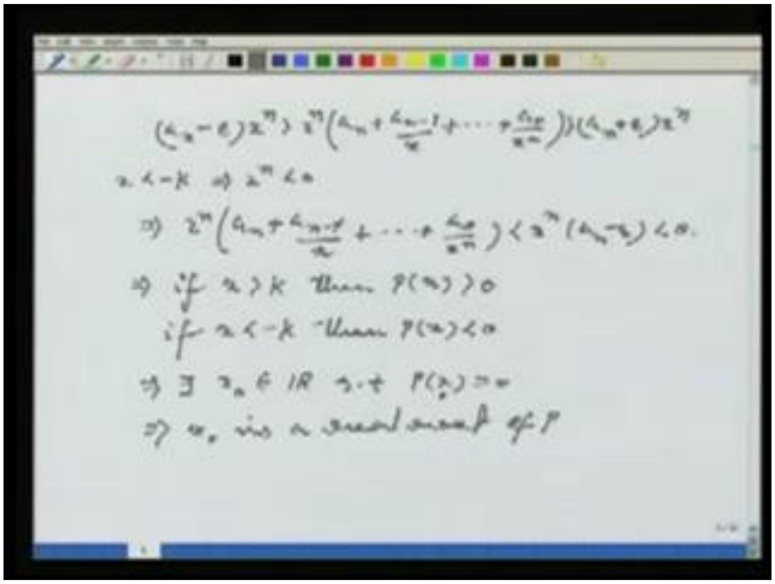

This implies that a n minus epsilon times x to the power n is bigger than x to the power n times a n plus a n minus 1 by x plus a 0 by x to the power n, which is bigger than a n plus epsilon times x to the power n. Now notice that, as x is less than minus k, that is, x is negative, this implies that x to the power n is strictly less than 0. This implies, now since x is less than minus k, it implies x to the power n is also strictly less than 0.

That implies what? It just implies that x to the power n a n plus a n minus 1 by x plus a 0 by x to the power n. This is less than x to the power times a n minus epsilon, which then is strictly less than 0. So this implies what? This implies that if x is bigger than k, then P x is bigger than 0. If x less than minus k, then P x is less than 0. Now this implies then, there exist x naught in R, such that, P x is equal to 0. Why so because there is one value of x, for which P x is positive. There is one value of x P x is negative. That means 0 lies between those two values and hence by intermediate value property, there exist an x naught such that P of x naught is equal to 0 and this implies x naught is a real root, which is exactly what we wanted to prove.

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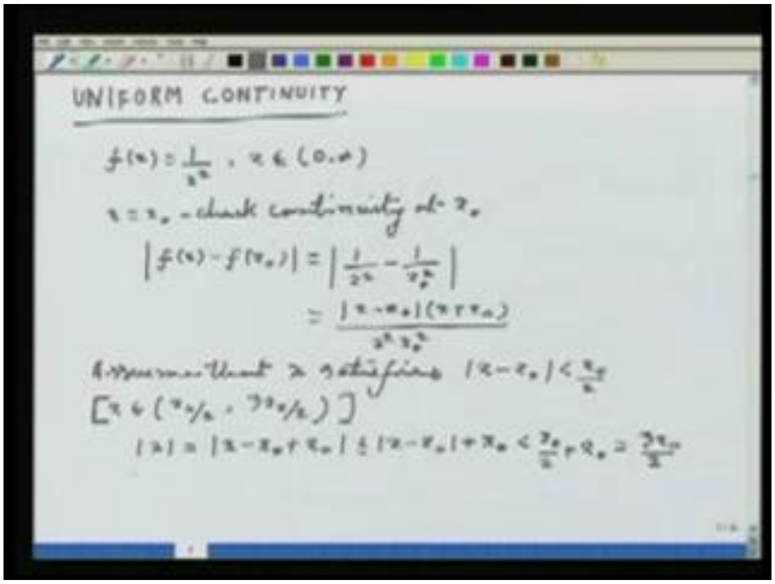

Now we are going to start with concept called uniform continuity. This is the concept which we will interact while talking about Riemann integration of continuous functions. That is the whole purpose of introducing this concept now. So first what we will do is, start with an example of function and check the epsilon delta definition of continuity. First, what we do is we start with the function example of function epsilon delta continuity. So let us look this function, f x is equal to 1 by x squared, where x belongs to open 0 infinity. I certainly cannot include the 0 in the definition because 1 by x squared will not be defined then.

What I want to check is first, this function is continuous at every point of 0, infinity and to do that what I will do is, I will just try to show that given epsilon, there exist a delta. I want to see how do I find out delta and how does delta depends on epsilon x, whatever. Choose a point x equals to x naught to check continuity at x naught. How are you going to do about this problem? I first look the quantity f x minus f x naught and then just I write down the definition of f. So that is modulus of 1 by x squared minus 1 x naught squared and then if I expand it what I get is, mod of x minus x naught times x plus x naught. I am not putting modulus here. Both x and x naught are positive divided by x squared times x naught squared.

Now let us assume that x satisfies the inequality x minus x naught is less than x naught by 2. That is, x belongs to x naught divided by 2 and then thrice x naught divided by 2. That is a choice of neighborhood but if I have this, then what I have is that mod of x that is equal to mod of x minus x naught plus x naught and then I use triangle inequality to get x minus x naught plus x naught but x minus x naught according to my choice is less than x naught by 2. So this is less than x naught by 2 plus x naught, which is equals to thrice x naught by 2 and then this implies that mod of x plus x naught, by triangle inequality is less or equals to mod x plus x naught mod x is less than 3 x naught. So this is less than thrice x naught by 2 plus x naught. That is, 5 x naught by 2.

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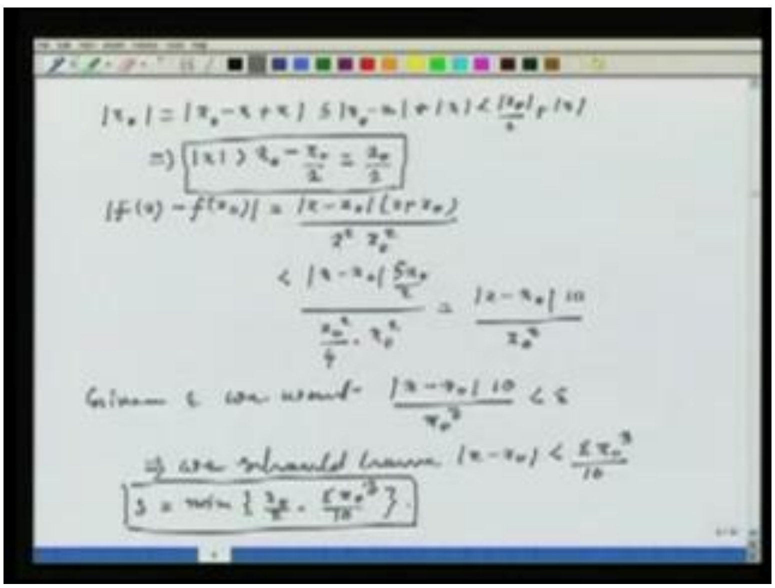

Now, if I start with mod x naught, which I can write as mod of x naught minus x plus x and apply triangle inequality, that is x naught minus x plus x, which then strictly less than, what was the estimate on x naught plus x? Let us see. x naught plus x is 5 x naught by 2. What I do is, I look at x naught minus x plus mod x. But mod of x naught minus x is less than x naught by 2. So what I have is here, I have mod x naught by 2 plus mod x. Actually many of those mods are unnecessary because x and x not are positive.

Anyway, this implies then that mod x is bigger than x naught minus x naught by 2, which is equal to x naught by 2. Now I want to use all these information in my estimate. That is, mod of f x minus f x naught, which I know is mod x minus x naught into x plus x naught divided by x squared times x naught squared. Now I am going to use the above estimate. What I get first is, this is less than mod x minus x naught. I am going to do anything with it. We just keep it in the rough form and I know x plus x naught by previous work that is less than 5 x naught by 2 and in the denominator, what I am going to use is that mod x is bigger than x naught by 2. That means, 1 by x squared is less than x naught squared by 4.

What I am going to use here is, x naught squared by 4 and there is another x naught squared remaining here. What I get then is, this is mod x minus x naught, which is intact times 10 divided by x naught cube. All now I want is, given epsilon we want mod of x minus x naught times n by x naught cube to be less than epsilon. If I want to have this, this implies we should have mod of x minus x naught is less than epsilon into x naught cube divided by 10. Now what is the delta I am going to choose? Certainly, the choice is obvious. I choose delta equals to the minimum of x naught by 2 and the quantity epsilon x naught cube divided by 10. Now the point to note here is that the delta which I got, whose precise expression I have now here, it involves x naught. At the same time, it involves delta.

That means, given epsilon, when we want to choose delta, it depends on epsilon and the point x naught where you are checking the continuity of the function. Now notice one more thing that when I get my x naught to be small, that implies this minimum, which is appearing that will also be small.

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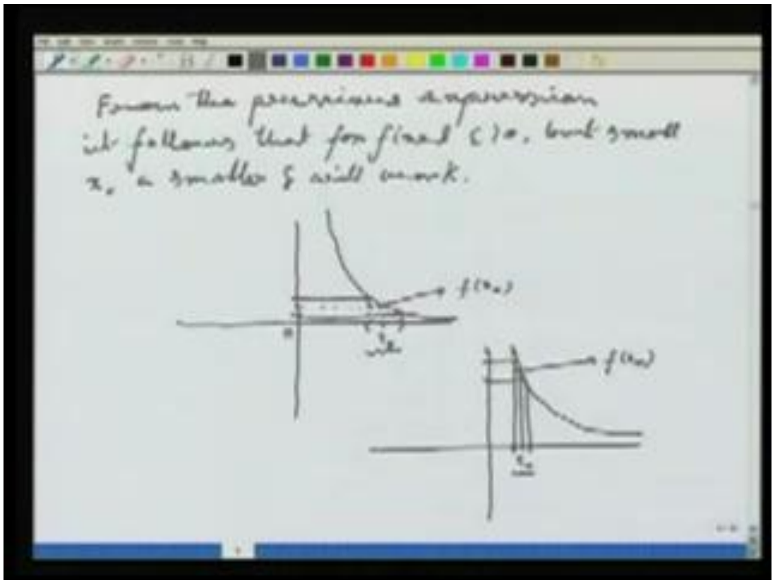

Let us note it down. From the previous expression, it follows that for fixed epsilon but small x naught, a smaller delta will work. This is what I got analytically, but I want to see it from the picture if that is possible. Let us try to draw the picture. This is the y axis. This is x axis. I have 0 here. Let us try to draw the curve for f x is equal to 1 by x squared. It looks something like this. Now first, let us choose the point x naught here. This is my x naught and I take fixed epsilon. Here, I have f x naught. Here is where is my f x naught.

Now notice, if I choose epsilon band around f x naught, this is f x naught. An epsilon band, how does that look like? Let us say it is here, it is here. That means, I want a choice of delta such that f x lies in this band. Then the obvious choice is, you put this down. It seems this delta, this kind of neighborhood of x naught will work, for which I will delta get. Now let us draw another copy of this picture. This is y-axis and this is x-axis. The graph of the function looks like this. Let me choose x naught here. That means I have chosen a smaller x naught. Then the corresponding, this is my f x naught. It is much higher. That is obvious because if x naught is small, f x naught has to be big because it is 1 by x naught squared.

Now I choose an epsilon around, which is the same epsilon which I have worked. So in the picture, it will look something like this and this, the same epsilon band, then what should work as delta? That means you just have to project it, down below, that is somewhere here and here, you will get somewhere here. That means, this is now your neighborhood. See the length of neighborhood? This much here, the length of the neighborhood was this much. It follows from the picture, if my drawing is perfect, that for this smaller x naught, the delta is getting small. That means delta does depend on x naught.

That might be because of I got delta because of my sloppy arguments. Analytically I need to prove that there is not a single delta which will work for all x not bigger than 0 for this function f x. So in the previous example it seems that if I give you a epsilon, then the delta you are going to get, it depends on epsilon as well as the point x naught. Uniform continuity means this does not happen. That means, delta can be chosen independent of x. The precise definition is as follows.

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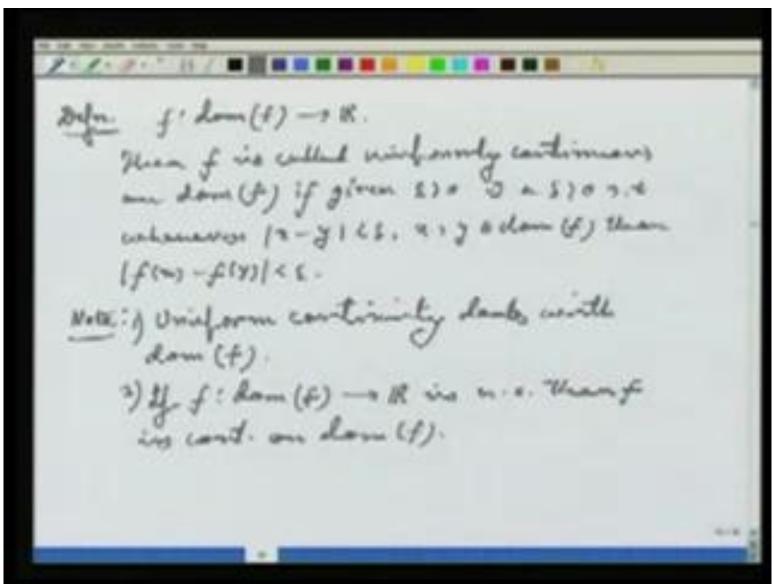

Suppose if I have a f from the domain of f. That is a sub set of the real line to R. Then f is called uniformly continuous on the domain of f. If given epsilon bigger than 0, there exist a delta bigger than 0, such that whenever mod of x minus y is less delta for x, y in domain f then mod of f x minus f y is less than epsilon. That is, you take any two points x and y such that mod of x minus y is less than delta where x and y are in the domain of f then mod of f x minus f y is always less than epsilon.

So the thing to notice, first of all, that uniformly continuity deals with domain of f. It is not about points; it is about the set. This is number 1. Number 2 is an easy exercise which you can easily check. If f from domain of f to R is uniformly continuous, then f is continuous on domain of f. Now what I want to show you is that the function which I started with, f x is equal to 1 by x squared defined on 0 infinity is not really a uniform continuity function. That means there exist an epsilon for which no delta works. Let us try to see that next, so the example.

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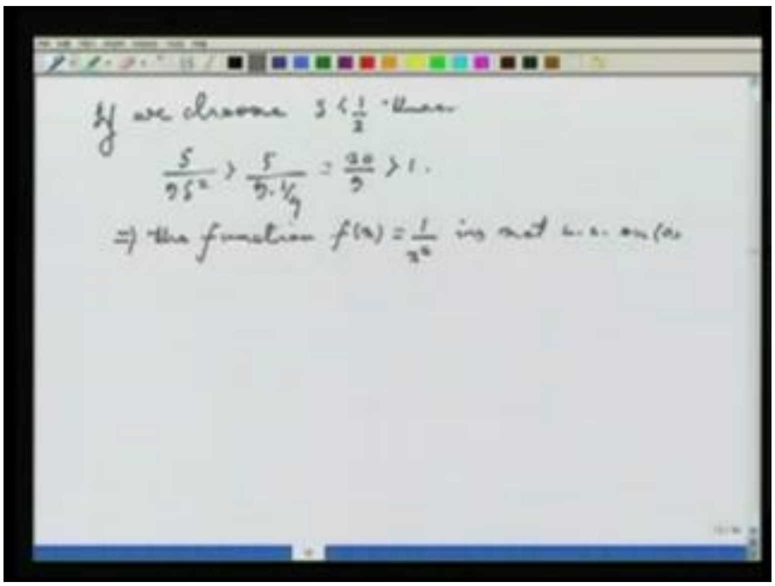

Later we will see an easy proof of this. I look at the function f x equals to 1 by x squared, where x belongs to 0 infinity. I say given epsilon is equal to 1, let us choose a delta naught, which works for some x naught. Now I am going to choose an x 1 and I would show this same delta naught does not work for x 1. So what I do is, I choose x 1 is equal to delta naught by 2 and I choose x is equal to delta naught, then modulus of x 1 minus x, that is, delta naught by 2 which is strictly less than delta naught and then modulus of f x 1 minus f x. That is, modulus of 4 by delta naught squared minus 1 by delta naught squared, which is 3 by delta naught squared. That is bigger than 3, as delta naught is bigger than 1. This implies, mod of f x minus f x 1 is not less than epsilon. This implies the function f x equals to 1 by x squared is not uniformly continuous on 0 infinity but let us do some more experiment with this.

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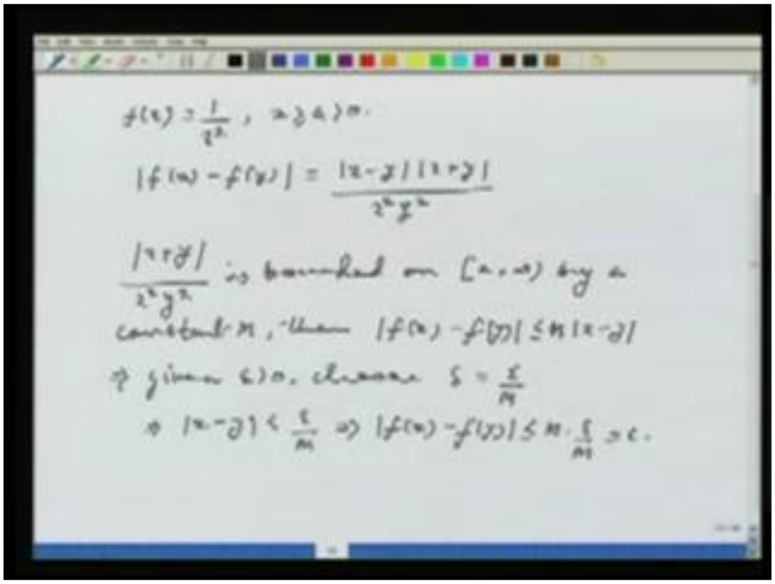

Again, I look at the function f x is equal to1 by x squared. Now let me assume that x is bigger than or equal to a, which is bigger than 0. Now I can show this function is uniformly continuous. What does it show then? It actually shows that the concept of uniform continuity depends on which domain of f you are working with.

Let us see why is this uniformly continuous. So I look at f x minus f y. That is, mod x minus y mod x plus y divided by x squared into y squared, fine. That means, I have to find a delta which works for all x and y and a given epsilon. Since x and y both are bigger than a, what I get is, this is less than equal to mod of x minus y into mod of x plus y divided by a to the power 4. If I can, somehow show that mod of x plus y divided by x squared y squared is bounded on a infinity by a constant M, then what I get is that mod of f x minus f y is less or equal to M times mod x minus y.

If I can show that, then this implies that given epsilon bigger than 0 choose delta to be equal to epsilon by M. Now notice that this delta is not depending x and y because if I choose my delta to be less than epsilon by M, this would imply mod x minus y less than epsilon by M implies mod of f x minus f y is less than or equal to M times epsilon by M, which is epsilon. So all I need to show that this quantity is bounded by some constant M which is independent of x and y. That is pretty easy.

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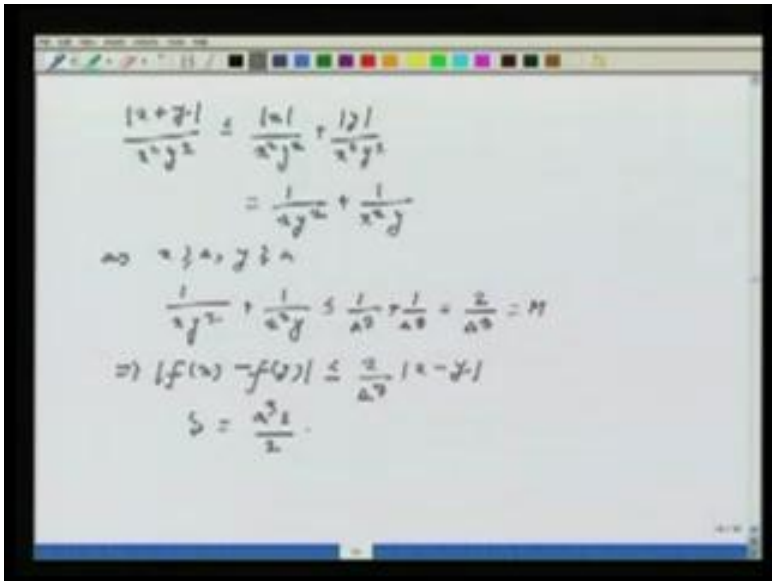

I just write mod x plus y divided by x squared y squared as the mods are unnecessary because I am working with positive x and y. Anyway, then this is 1 by x y squared plus 1 x squared y, but as x bigger than or equal to a and y is bigger than or equal to a. What I have is, 1 by x y squared plus 1 x squared y is lesser equal to 1 by a cube plus 1 by a cube which is equals to 2 by a cube which I can take to be equal to my M. So this implies mod of f x minus f y is less or equals to 2 by a cube times mod of x minus y. That means my choice of delta actually is, a cube epsilon divided by M.

See, delta depends on epsilon, not on the points, but certainly depends on the set. Notice that if I had taken a to be equal to 0, that was 0 infinity I wanted to work with. Then my delta will be 0, so close to 0. I have got a problem, but away from 0. If the function is defined for x bigger than or equals to a, then it quickly become uniformly continuous.

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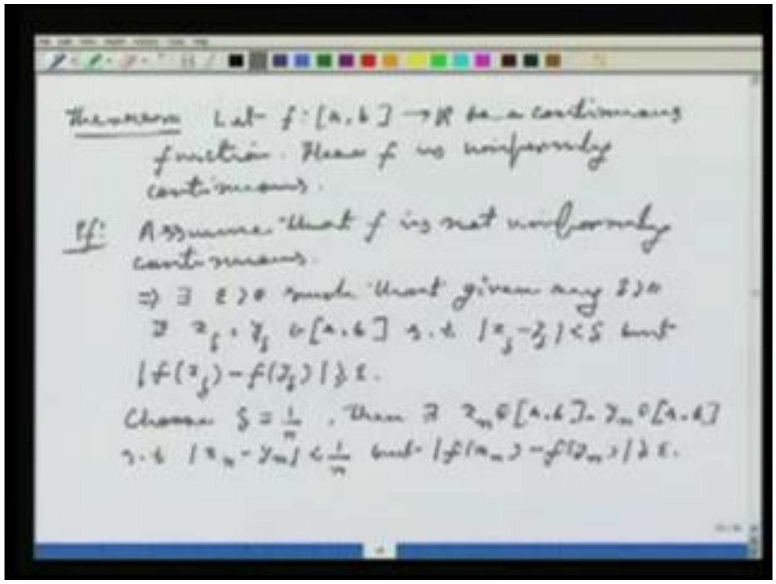

Now I come to most important theorem of uniform continuity and why we wanted do it. We have seen that if I have uniformly continuous function, then function is always continuous. We have also seen, there exist continuous functions which are not uniformly continuous. For example, f x is equal to 1 by x squared on 0 infinity. That is a continuous function certainly I have checked but it is not a uniformly continuous. But there are certain domains on which continuity always gives me uniform continuity. That is what I am going to prove now. That is the theorem.

Let f from closed interval a b to R be a continuous function. Then f is uniformly continuous. That is, the statement of the theorem says that if I have a continuous function defined on a closed and bounded interval, then the continuous function actually is uniformly continuous. Let us try to prove this. The proof actually uses the contra-positive arguments. Let us assume that f is not uniformly continuous. Assume that f is not uniformly continuous. What does this mean?

Some function is not uniformly continuous. It means there exists an epsilon for which no delta works. The definition of uniformly continuous says that given any epsilon there is a delta satisfying some inequalities. It is not uniformly continuous then it means that there exists an epsilon for which there is no good delta. Let us just try to write it down. It implies there exist epsilon bigger than 0 such that, given any delta bigger than 0, there exist two points which I will call x delta and y delta in the closed interval a b, such that mod of x delta minus y delta is less than delta but modulus f of x delta minus f of y delta is bigger than or equal to epsilon.

That is what not uniformly continuous means. Fine. With respect to this bad epsilon, I am going to choose delta according to my will. So I will start choosing delta is equal to 1 by n. Then corresponding to this delta there are bad two points x delta and y delta which now I will call x n and y n. Then there exist x n in a b and y n in a b, such that, mod of x n minus y n is less than 1 by n but mod of f x n minus f y n is bigger than or equal to epsilon.

So the thing is, I have got hold of two sequences. One is x n, other is y n on the closed interval a b but I do not know anything about convergence of those sequences. But see, I am on the closed interval a b and I know Bolzano Weierstrass theorem. It says that the given any sequence in the closed interval a b, I have got a convergence sub sequence. Let us consider convergent sub-sequence of x n, call it as x n k.

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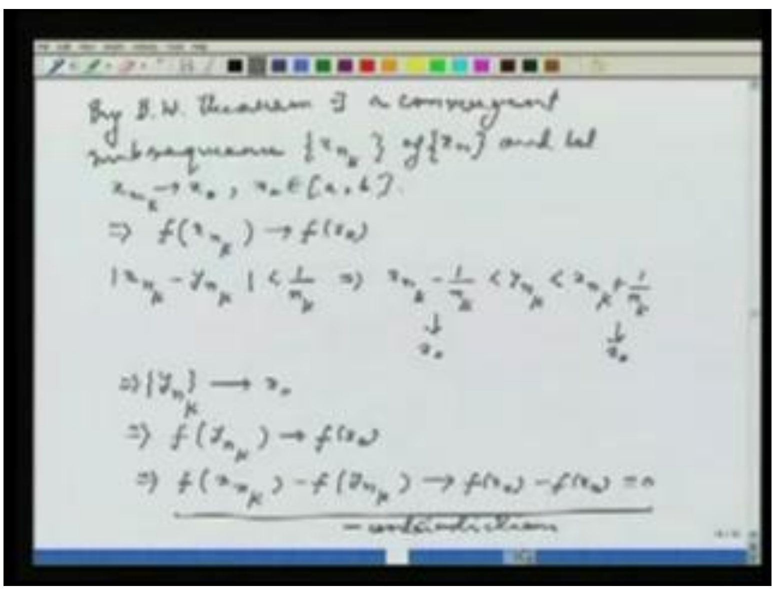

By Bolzano Weierstrass theorem, there exists a convergent sub-sequence x n k of x n and let x n k converges to x naught which is also in the closed interval a b but since f is given to be continuous, I know that this implies f of x n k converges to f of x naught. Also notice that mod of x n k minus y n k, that is less than 1 by n k. What does this imply? It implies, x n k minus 1 by n k is less than y n k which is less than x n k plus 1 by n k. Apply Sandwich theorem now. This converges to x naught. This converges to x naught.

This implies, y n k also converges to x naught but again, by continuity, this implies that f of y n k converges to f of x naught. Then this implies that f of x n k minus f of y n k, that converges to f x naught minus f x naught which is equal to 0. But if you look back the condition on f x n and f y n was this: that is bigger than or equal to epsilon for all n but in that case it cannot happen that it has got a subsequence which converges to 0 but that is what I have got here which is a contradiction. This happens because I assume that f is not uniformly continuous. So it follows that f is uniformly continuous.

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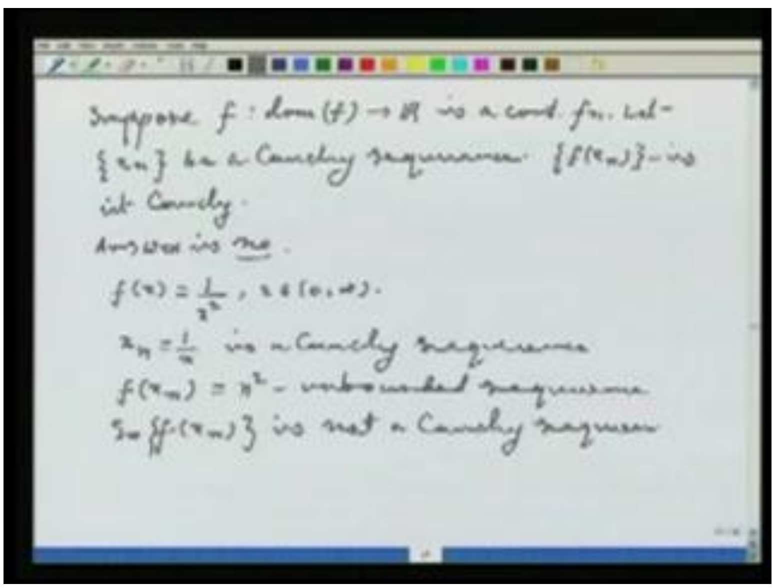

Now let us ask the following question. Suppose f from domain of f to R is a continuous function and let x n be a Cauchy sequence, then the question I want to ask is, look at the new sequence f of x n. The question is, is it Cauchy? That is, the question we are asking is that suppose, I have given a continuous function, then is it true if takes a Cauchy sequence to a Cauchy sequence? The answer is no. Well, I look at this function f x equals to 1 by x squared where x belongs to 0 infinity. I look at this sequence, x n is equal to 1 by n. Then this sequence is contained in the domain of f certainly and also it is a Cauchy sequence.

But then what is f of x n? f of x n is standing out to be n squared which is an unbounded sequence. If you remember, we have proved that the first thing a Cauchy sequence does is, it becomes a bounded sequence. This is an unbounded sequence. It cannot be Cauchy. So f of x n is not a Cauchy sequence. So in general, it is not true that a continuous function takes a Cauchy sequence to a Cauchy sequence but I am going to show if my function uniformly continuous, then it does take a Cauchy sequence to a Cauchy sequence. Let us try to prove that. This is the theorem.

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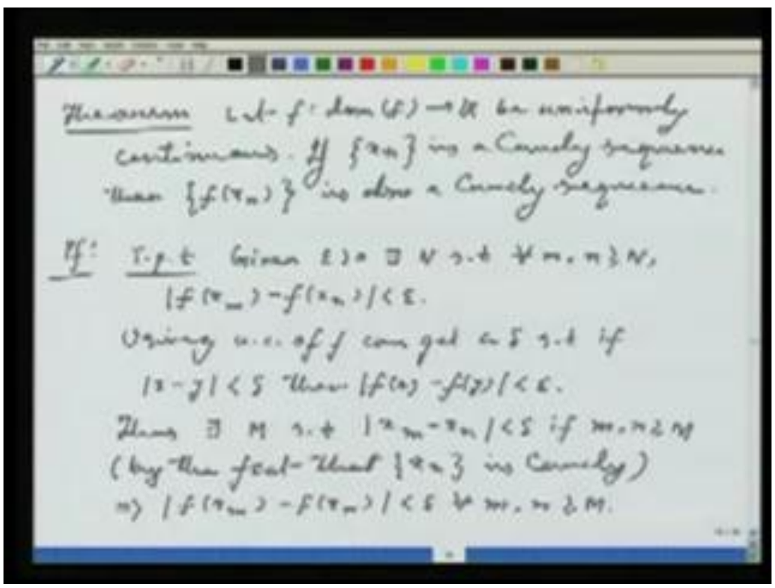

Let f from domain of f to R be uniformly continuous. If x n is a Cauchy sequence then the new sequence f x n is also a Cauchy sequence. The proof is very simple. If I have to show f x n is a Cauchy sequence, what I have to do is, to prove that given epsilon bigger than 0 there exist capital N such that for all m, n bigger than or equal to capital N mod of f x m minus f of x n is less than epsilon. Now using uniform continuity of f, can get a delta corresponding to the given epsilon such that if mod of x minus y is less delta, then mod of f x minus f y is less epsilon. That is the definition of uniform continuity but notice that x n is Cauchy.

So thus there exists capital M such that mod of x m minus x n is less than delta if m, n bigger than or equals to M by the fact that x n is Cauchy. But then by definition of uniform continuity mod of f x m minus f x n is less than epsilon for all m, n bigger than or equal to the same capital M. But this is precisely what I wanted to prove here, that given epsilon bigger than 0, there exist M such that, for all m, n mod of f x m minus f x n is less than epsilon. See what I have found out. I found out that mod of f x m minus f x n is less than epsilon for all m, n bigger than or equal to M. That means, the stage N we called in the previous case, I found it to be M and what is this M? This M works for the delta in the Cauchy sequence x n. This proves the result.

Now using this, it follows quite easily that the function f x is equal to 1 by x squared is not uniformly continuous in 0 infinity. Let us see how so.

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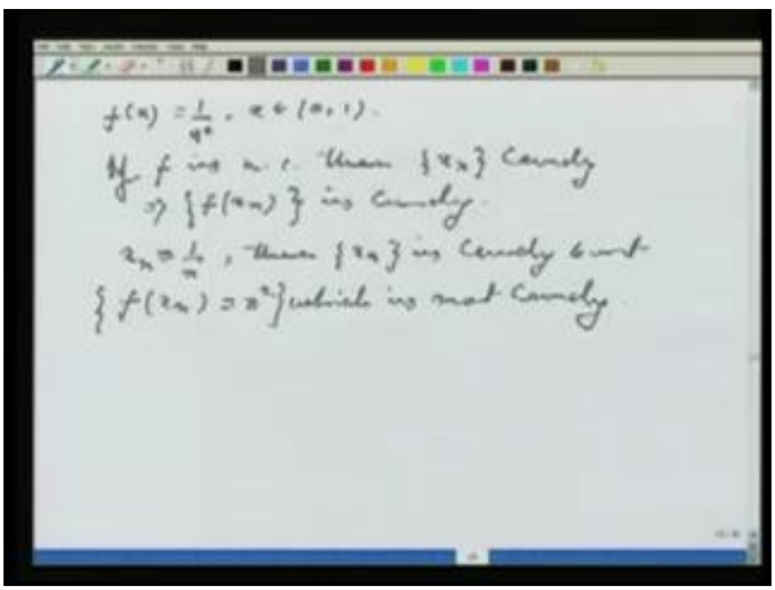

I look at the function f x is equal to 1 by x squared. Let me take x in 0 1. There also it is uniformly continuous. If f is uniformly continuous, then x n Cauchy implies f x n is Cauchy. That is not the case. Let me look at the sequence x n is equal to 1 by n which I have looked at earlier. If x n is equal to 1 by n then x n is a Cauchy but f of x n equals to n squared which is not

Cauchy. But my previous theorem says if f is uniformly continuous then a Cauchy sequence will be mapped to a Cauchy sequence by f.

I have assumed my f to be uniformly continuous on 0 1 and I can see there exist a sequence 1 by n which is not getting map to Cauchy sequence. That means f cannot be uniformly continuous. So now then we have an easier proof of the fact that the function f x is equal to 1 by x squared is not uniformly continuous in 0 infinity. Earlier we have established the same fact using the epsilon delta definition of uniform continuity but here we have an alternative proof using definition of uniform continuity.