Mathematics-I

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Lecture - 9
Mean Value Theorem

It is a famous theorem called Rolle's Theorem; Rolle's Theorem essentially says the following things, I just draw the picture from this, the statement will be very clear. I take a point a and I take a point b. Suppose, I have given a function defined on closed interval [a, b] to R, assume that f is differentiable on the open interval (a, b). Also assume that f of a is equal to f of b. If I try to draw the graph of the function at the point the value of the function f(a), which has the same height as f(b). Then the curve of the function looks like this, may be, if I draw the graph of the function. From the picture, you can see that at this point, at the top most point, the tangent parallel to the x-axis, at this point also, the tangent is parallel to x-axis.

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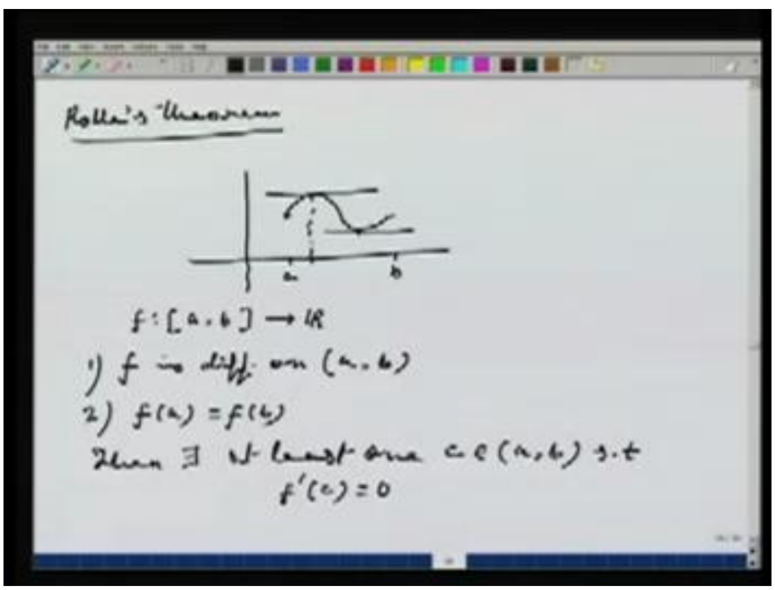

I know what it means, the tangent being parallel to the x- axis can be expressed by saying that the derivative of the function at that point is 0. So based on this picture, I will like to conclude, then there exist at least 1 c in the open interval (a, b) such that, f prime at c is 0, because you see, why I said at least, it could have been at most also. For example, the graph of the function could have been look like this. I take a here, let us say the graph of the function looks exactly like this, then it is only the top most point where the derivative is 0. Now the beauty of Rolle's Theorem is that to prove this theorem, which we know, has to be true if I can draw the curve of the function, the way I look like. But the graph of the function cannot be always drawn as we have seen. So analytically, one has to prove it. But for our intuition, these pictures are quit alright. It tells us that it has to happen, so let us see why it should happen.

The proof is very simple, it just uses the previous observation which we have done and we will use some property of the function which, let me put in the condition that let us say the function f from [a, b] to R is continuous and it is differentiable in the open interval (a, b) and at the end points the function exists and has values.

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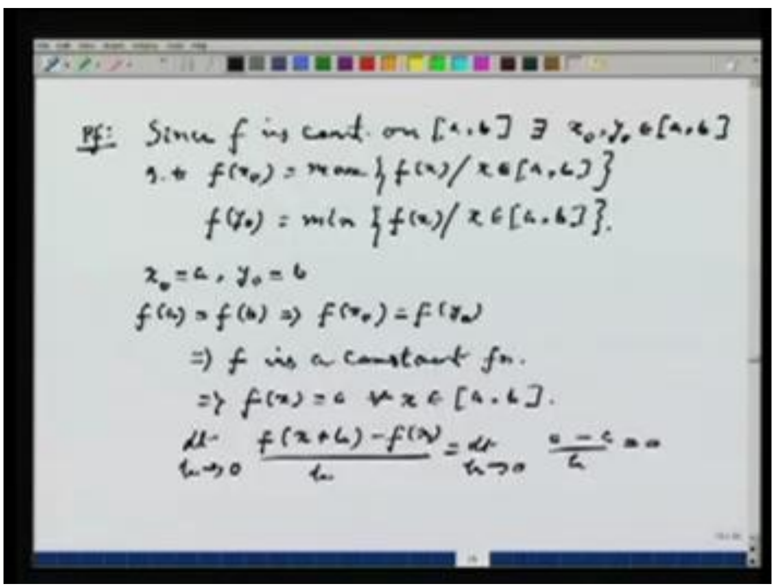

Now since f is continuous on [a,b], on the closed interval [a, b], by a property of the continuous function on the closed intervals, I know there exists 2 points x naught and y naught in the closed interval [a, b], such that f of x naught is the maximum, of all f(x)s x in [a, b] and f of y naught is the minimum of all the f(x)s, where x in [a, b]. Now the possibility is that x naught and y naught are end points. Let us say x naught is equal to a, y naught is equal to b. If that is the case, if f(a) is equal to f(b), this implies that f x naught equals to f of y naught, that is, the function has same maximum as well as the minimum. What does this mean? This implies that f is constant function. That is, f(x) must be equal to c, for all x in [a, b]. Then it is very easy to check, what is the derivative of this function. Well, we just write down the definition.

Limit h going to 0, f of x plus h, minus f(x) divided by h, that is, limit h going to 0, c minus c by h, which is 0. So Rolle's Theorem is proved in this case. Now another case is, x naught is equal to b and y naught is equal to a, but then this is same again. Maximum is same as minimum, implies the function is constant.

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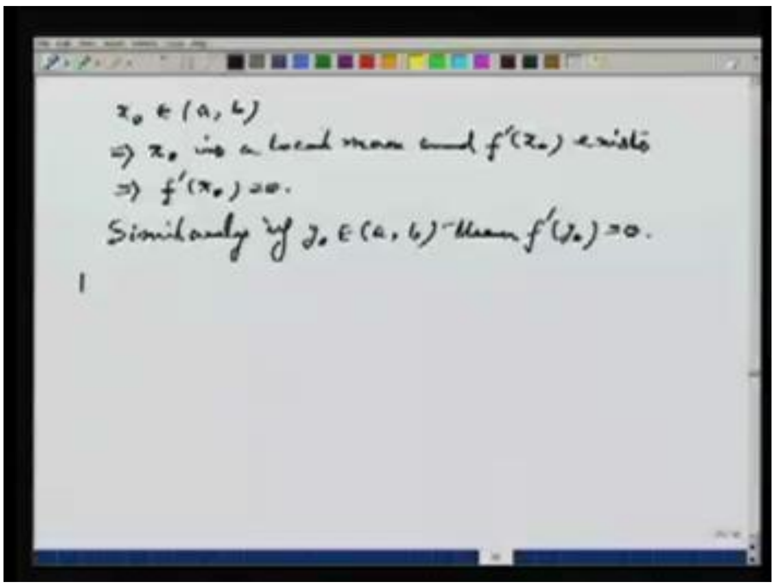

Now the next thing is, suppose that is not the case, that is, either x naught or y naught is not an end point. So let us assume, x naught belongs to the open interval (a, b). Notice that x naught was the maximum of the function. This certainly implies that x naught is a local maximum and I also know that f is differentiable at x naught, that is, f prime at x naught exists. This implies, by the previous result that f prime at x naught is 0.

Similarly, if y naught belongs to (a, b), again by the previous theorem, f prime at y naught is also 0. So that is the proof of the result. But notice here, the most fundamental fact what I am using here is, that since f is a continuous function on the closed interval, it must attain its bounds. That is, there exist a maximum of the function and a minimum of the function in the closed interval. If it happens that these two maximums and minimums are end points, then using the condition that f is same on the end points, that it exists on values on the end points, we get that the function is constant and hence the derivative is 0. Then the next case is, that this maximum or the minimum lies in the open interval (a, b) and in that case, we use the previous theorem which tells us that the derivative is 0. That is the proof of Rolle's Theorem.

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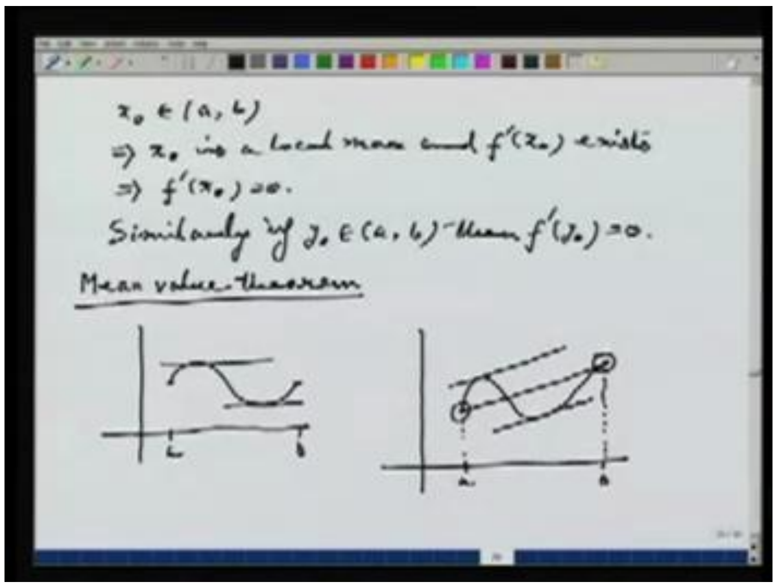

Now let us look at some more generalizations of Rolle's Theorem. The first one is the well known Mean Value Theorem. Look at the situation of Rolle's Theorem again. I draw the picture again here. This is a, this is b. I know the heights are same at a and b. The graph of the function looks something like this. Then there are points where the tangent is parallel to the x-axis. Now it could have happened this way also: I draw another picture that suppose this f(a) is equal to f(b) condition is dropped, that is, the height of the function here is this, height at b is something like this, height at b is certainly more that height at a, and the graph of the function certainly look like this. Then what I do is, I look at this line joining these two points. From the picture, I can see what is happening is, there are points, for example, look at the tangent here or I look at the tangent here, they are parallel to the line joining these two points. Now, how do I explain this geometrically in terms of the derivative? That is what Mean Value Theorem is all about.

So let us notice certain things I want to see. The tangent is parallel to the line joining these 2 points. That means these two lines must have same slope. So let us draw the picture again and try to calculate things here that I have the axis here.

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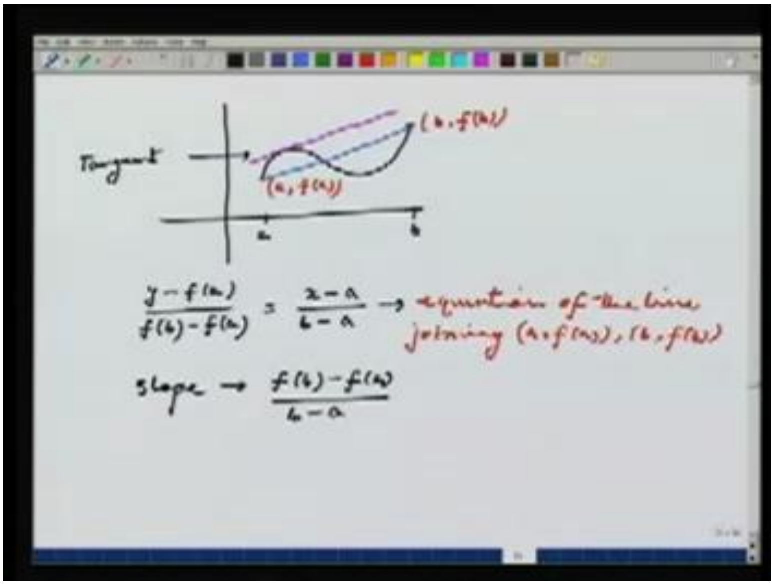

This is the point a, this is the point b. My function has to different heights. One is here, other is, let us say, here. My function looks like this. What are the points? Well, the points are (b, f(b)) and here the points are (a, f(a)). Now I can draw the line joining a, f(a) and b, f (b). This is the line. Then I can draw the tangent line also, let us say, at this point. This is the tangent line. Now what is the slope of the line joining a, f(a) and b, f (b)? Well, I can actually write down the equation of this line. That is what I am going to now.

So the equation of the line is, this is the equation of the line joining the points a, f(a) and b, f (b). So the slope of the line anyway is f(b) minus f(a) by b minus a. I want to say that there exists a point where the tangent is parallel to the line joining a, f (a) and b, f (b). That would mean that the tangent at the same slope at the line joining a, f(a) and b, f (b). That means, there should be a point where f prime of that point is same as the slope of the line. So the precise statement of the Mean Value Theorem goes as follows.

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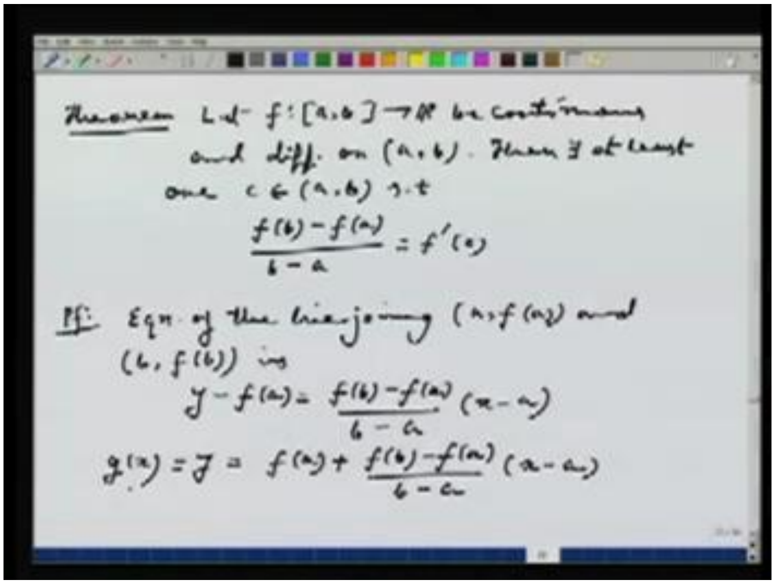

Let f from the closed interval [a, b] to R be continuous and differentiable on the open interval (a, b). Then there exist at least one c in the open interval (a, b) such that f(b) minus f(a) by b minus a is equal to f prime c. How do you go about proving it? If you notice the previous thing carefully, we are actually dealing with 2 functions. One function is certainly the given function f(x). Another function is the line joining a, f(a) and b, f(b). So again, I will write down the equation. Equation of the line joining (a, f(a)) and (b, f(b)) is y minus f(a) is equal to f(b) minus f(a) by b minus a into x minus a. That is, y is equal to f(a) plus f(b) minus f(a) divided by b minus a into x minus a. Now this is another function. I can actually call it g(x).

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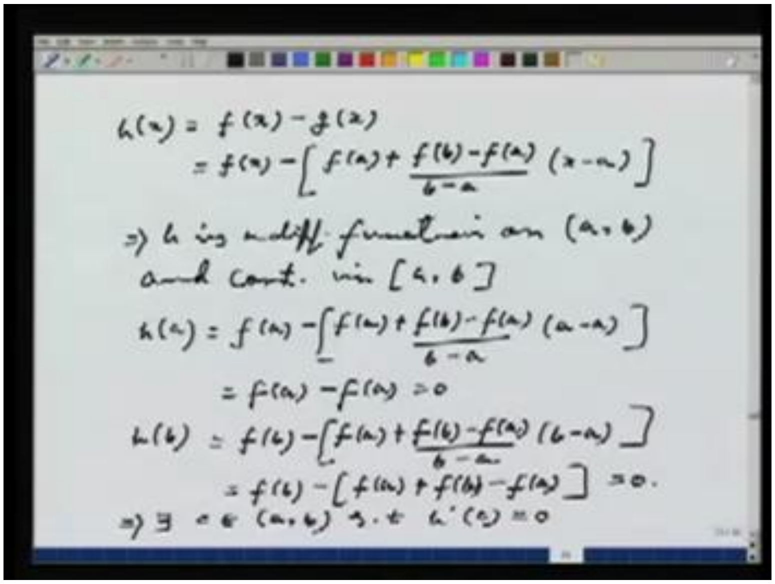

What I want to do is, I just want to compare these two functions with the given function f(x). That is, I define a new function h(x) is equal to f(x) minus. That is, f(x) minus f(a) plus f(b) minus f(a) by b minus a into x minus a. Notice h is a differentiable function on the open interval (a, b) and continuous in the closed interval [a, b]. This is simply because h is given in terms of f and some other constants and the function x also. This function has the property, they are differentiable in the open interval (a, b) and continuous in the closed interval [a, b]. So each also the same property. Now notice what is h(a)?

Now since h is a differentiable function, what we will try to do is, we try do see whether it satisfies the conditions of the Rolle's Theorem. That means, we will try to evaluate first what is h at the end points. So let us first calculate h of a that by definition f of a minus f of a plus f (b) minus f(a) by b minus a into a minus a, which anyway is 0. So what I get is f(a) minus f(a). That is 0. Now I look at what is h of b. That again by definition is f(b) minus f(a) plus f(b) minus f(a) by b minus a into b minus a, which is f(b) minus f(a) plus f(b) minus f(a) which is 0. That is, h is the function which differentiable in (a, b) and continuous in the closed interval [a, b] and at the end points it takes the value 0. That means, by Rolle's Theorem that there must exist a point c where the derivative of the function h vanishes. So this implies there exist a point c in the open interval (a, b) such that h prime at c is 0.

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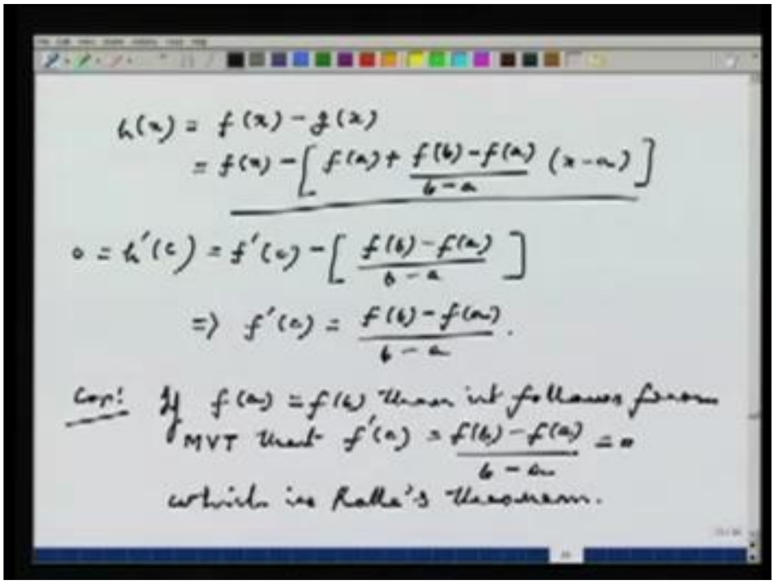

But now I try to calculate what is h prime at c because I know the precise description of h. So let us try to calculate that so our definition h is given as this h(x) is equal to f(x) minus g(x) which is f(x) minus this quantity. So if I look at what is h prime at c which I know is equal to 0 that it would give this is equal to f prime c minus, f(a) is a constant. So derivative is 0. Then what I get f(b) minus f(a) by b minus a. This is equal to 0. This, then in turn implies that f prime c is equal to f(b) minus f(a) by b minus a. Now I said that this is generalization of Rolle's Theorem. It is actually in the following sense:

If f(a) is equal to f(b), then it follows from Mean Value Theorem, which I will denote by MVT that f prime c is equal to f(b) minus f(a) by b minus a is equal to 0, which is Rolle's Theorem. So this way, Mean Value Theorem actually generalizes Rolle's Theorem. If you put the condition f(a) is equal f(b) in the Mean Value Theorem, then Rolle's Theorem follows. At the same time, we should not forget while proving the Mean Value Theorem, the most fundamental things is actually Rolle's Theorem. What we have done is, we have constructed certain functions out of the given f and I have applied Rolle's Theorem to conclude Mean Value Theorem.

Next we go to some more general form of Mean Value Theorem called Cauchy's Theorem, Cauchy's Mean Value Theorem. Now we come to the Mean Value Theorem due to Cauchy, Cauchy's Mean Value Theorem. In short I will mention it as CMVT which says the following.

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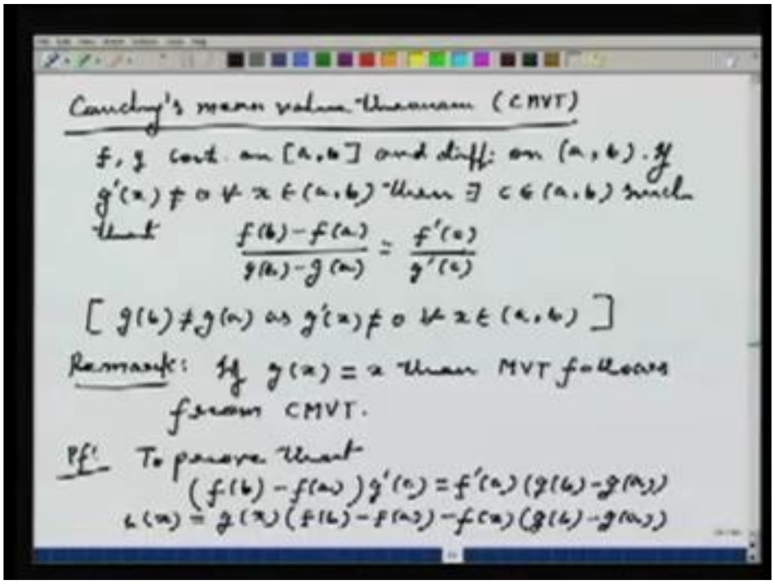

Suppose I have a function f and a function g which are continuous on closed interval [a, b] and differentiable on the open interval (a, b). If g prime x is non-0 for all x in the open interval (a, b), then Cauchy's Mean Value Theorem says that there exist a c in the open interval (a, b) such that f(b) minus f(a) by g(b) minus g(a) is equal to f prime c divided by g prime c. Notice here, the c in the numerator and the denominator are same. If I could have allowed different cs then it straight away follows the usual Mean Value Theorem.

The beauty of Cauchy's Mean Value Theorem is you get the same c in the numerator and the denominator. Now, the first question is why g(b) is not equal to g(a)? Why the left hand side makes sense? So I will just note it here g(b) is not equal to g(a) as g prime x is not equal to 0 for all x in the open interval (a, b). How does this follow, because if g(b) is equal to g(a) then by Rolle's Theorem, there will exist x in the open interval (a, b) such that g prime x is 0, but g prime is always non 0. That is given. That means g(b) cannot be equal g(a). Now I can remark that if I take g to be the particular function, if g(x) is equal to x then usual Mean Value Theorem follows from Cauchy's Mean Value Theorem, which is obvious because just in the expression, if you assume g(x) is equal to x then g(b) minus g(a) turns out to be b minus a and in that case g prime c is anyway 1 because g(x) is equal to x has the derivative 1 everywhere.

Now, to prove this, I just note what I have to prove is to prove that f(b) minus f(a) into g prime c is equal to f prime c into g(b) minus g(a). This is what I need to prove. So for that, I just look at the function h(x) is equal to g(x) into f(b) minus f(x) into g(b) minus g(a) and then just hope that perhaps h satisfies the conditions of Rolle's Theorem. Let us see. g and f both are differentiable functions on the closed interval [a, b]. Both are continuous functions on the closed interval [a, b]. Then certainly h is a continuous function on the closed interval [a, b]. Moreover, f and g, both are differentiable functions on the open interval (a, b). So h is also a differentiable function on the open interval (a, b).

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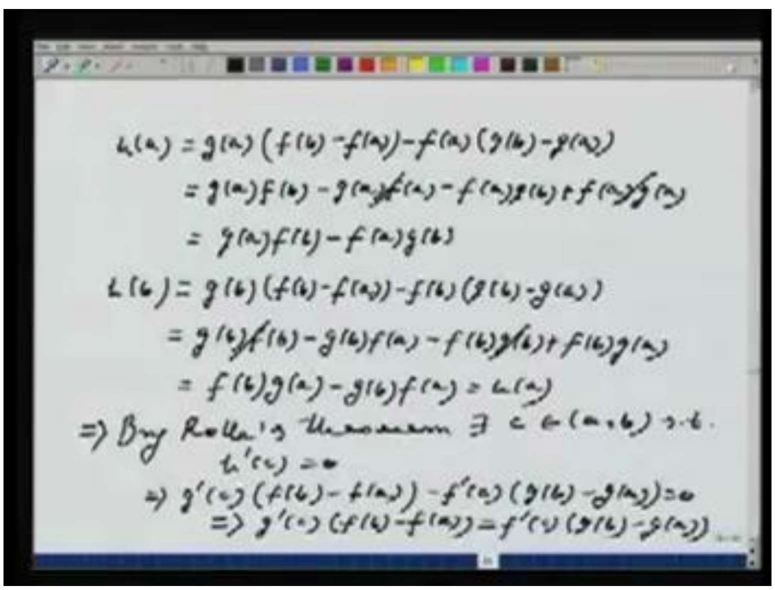

Now, I just want to check what is h at the end points. Well, what is h(a)? So h of a is g(a) into f(b) minus f(a) minus f(a) into g(b) minus g(a). If I expand this, what I see is that f(a) and g(a) cancel each other. We just open it up and check what happens. It is g(a) f(b) minus g(a) f(a) minus f(a) g(b) plus f(a) g(a). Then f(a) g(a) cancels each other. What I get is, g(a) f(b) minus f(a) g(b). Now let us look at what is h of b. That, by definition, is g(b) into f(b) minus f(a) minus f(b) into g(b) minus g(a). I again open it up, it is g(b) f(b) minus g(b) f(a) minus f(b) g(b) plus f (b) g(a). Here I see what cancels is, it is f(b) g(b) cancels each others. What I get is, f(b) g(a) minus g(b) f(a) which is same as h(a). This implies, by Rolle's theorem, there exist c in the open interval (a, b) such that h prime c is equal to 0 but from the definition of h again I can calculate the derivative.

This implies, g prime c into f(b) minus f(a) minus f prime c into g(b) minus g(a) is equal to 0. This then implies that g prime c into f(b) minus f(a) is equal to f prime c into g(b) minus g(a). This is precisely what we wanted to prove. Next, let us go to the application of these tools which we have developed so far. So we start with one obvious thing.

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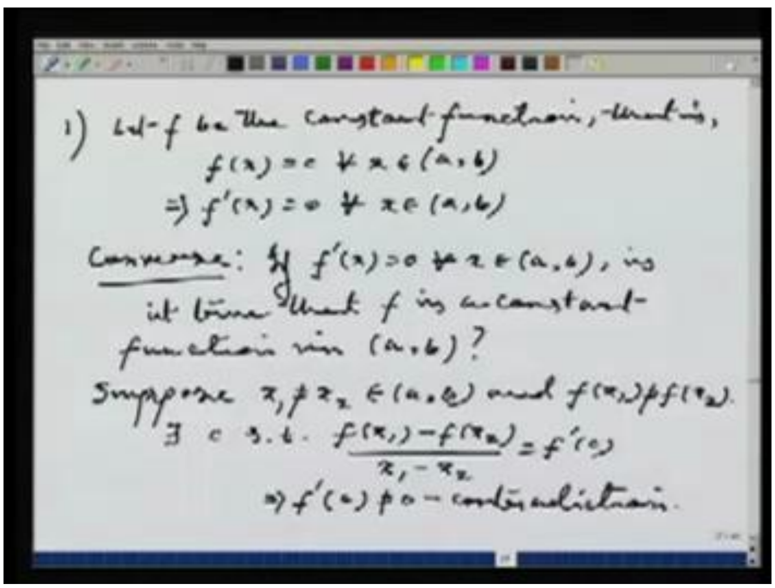

Suppose, let f be the constant function. That is, f(x) is equal to c, for all x in, let us say, open interval (a, b). This would certainly imply that f prime at x is equal to 0 for all x in the open interval (a, b). This, we have observed from the definition of derivative. Question is the converse. So the question is, if f prime at x is 0 for all x in the open interval (a, b), is it true that f is a constant function in the open interval (a, b)? This is really the converse. That is, if a function is constant, then its derivative is 0. The question is, if the derivative is 0 at all points in the interval, does that mean the function is constant in that interval? Notice that the question involves the whole interval; it is not about the points. A non-constant function might have a derivative which vanishes at a point. That is not the point. The point is, if you have a function whose derivatives vanish in an interval then can you say the function is constant?

It turns out to be very simple by Mean Value Theorem that suppose not, suppose the answer is no, suppose x 1 is not equal to x 2, belongs to the open interval (a, b) and f x 1 is not equal to f x 2. Then I know by the Mean Value Theorem, there exist c such that f x 1 minus f x 2 divided by x 1 minus x 2 is equal to f prime c. Now since f x 1 is not equal to f x 2, the left hand side has to be non-0 because x 1 is not equal to x 2. This would imply that f prime c is not equal to 0, but this is the contradiction because I said f prime vanishes everywhere. This implies that the function must be a constant function. That is, take any x 1 and x 2, f x 1 has to be equal to f x 2. That is what is meant by saying f is a constant function.

But notice here that you need the condition that the derivative of the function vanishes everywhere in the interval. Why so, because if you look at the c which appearing in the Mean Value Theorem, this c actually depends on x 1 and x 2. I can actually say c, it is a function of x 1 and x 2. Once you change x 1 and x 2, this c will also change. Unless you know that the derivative vanishes everywhere, you cannot show that f x 1 is equal to f x 2 for all x 1 and x 2. Anyway using Mean Value Theorem, now we have seen that if we have a function has the 0 derivative everywhere, then the function has to be a constant function.

Now the next application we are going to do is a famous technique applying limits called L’Hospital Rule. The next application is the famous L’Hospital Rule.

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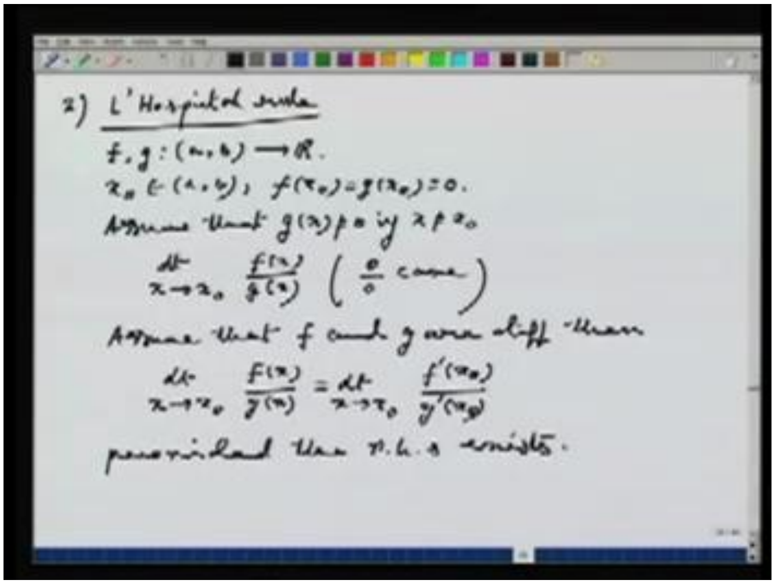

Our situation is something like this. I have a function f and g defined on the open interval (a, b) to R. Let us say, x naught is a point in (a, b) and f of x naught is equal to g of x naught. Let us say it is equal to 0. Assume that g of x is not equal to 0, if x is not equal to x naught. Then I can interested in finding limits x going to x naught f(x) by g(x). You can see there is an obvious problem because I cannot just write f x naught by g x naught has the limit because f x naught is 0, g of x naught is also 0. So it is the famous 0 by 0 case. Now if I assume that f and g both are differentiable functions, then the L’Hospital Rule says that limit x going to x naught f(x) by g(x) is same as limit x going to x naught f prime x naught by g prime x naught, provided that the right hand side exists.

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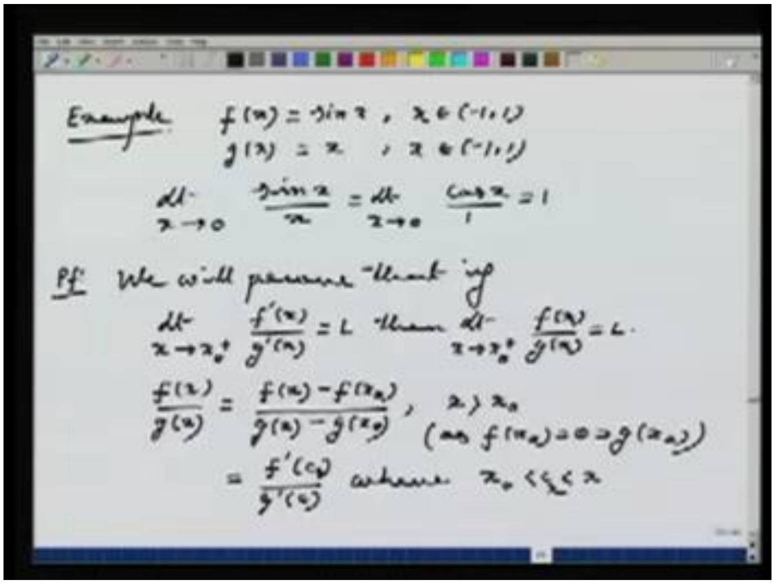

As quick examples, we can look at, let us say f(x) is equal to sin x, x in minus 1 and 1. g(x) is equal to x, x in minus 1 and 1 and I am interested in finding limit h going to 0 sin x by x. Notice that g satisfies the condition of the L'Hospital Rule that g is non-0 at any other point except 0. So if I can L'Hospital Rule, this would say that this limit same as limit h going to 0, cosine of x divided by 1 which turns out to be equal to again, 1. It becomes that easy. Now let us go to the proof of the result. What we show is, we will prove that if limit x going to x naught plus f prime x by g prime x is equal to L, then limit x going to x naught plus f x by g x is also equal to L.

So we start with f(x) by g(x). I write it in the form f(x) minus f x naught divided by g(x) minus g x naught and since I am interested in the limit x going to x naught plus, I will take x to be strictly bigger than x naught. This, I can do as f x naught is equal to 0, which is equal to g x naught. Now I am in a position to apply Cauchy's Mean Value Theorem because in the neighborhood of x naught, f and g both are differentiable functions. So they are continuous. It implies then there exist c, such that this is equal to f prime c divided g prime c, where c lies between x naught and x. Now I want to take the limit as x going to x naught from the right hand side.

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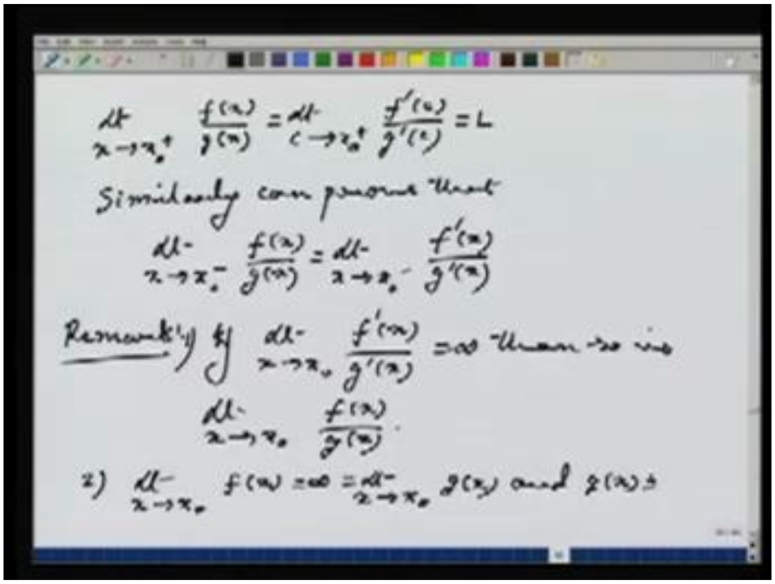

Notice that c is sandwiched between x and x naught. What I can actually write, c suffix x because c depends on x. This would then imply that limit x going to x naught plus f(x) by g(x) is equal to limit c going to x naught plus f prime c divided by g prime c. This actually follows from the Sandwich Theorem that since x converges to x naught and c lies between x naught x, it has to converge to x naught, but now the right hand side is L. That is my assumption. So is the left hand side. Similarly, applying the same technique of Cauchy's Mean Value Theorem, we can prove limit x going to x naught minus f(x) by g(x), that is also equal to limit x going to x naught minus f prime x by g prime x.

I will just make one remark here which become useful that if limit x going to x naught f prime x by g prime x is infinity, then so is limit x going to x naught f(x) by g(x) because our proof does not actually depend on the value of L, which is the limit. As an example, let us again look at the following limit. Now I would just like to mention that, if the situation is something like this that limit x going to x naught, f(x) is equal to infinity, which is same as x going to x naught g(x) and g(x) is not equal to 0 for all x naught equal to x naught and f and g are differentiable, then limit x going to x naught f(x) by g(x) is limit x going to x naught f prime x by g prime x, provided the right hand side exists. This is the famous infinity by infinity form. As an application of this, we can look at the following example.

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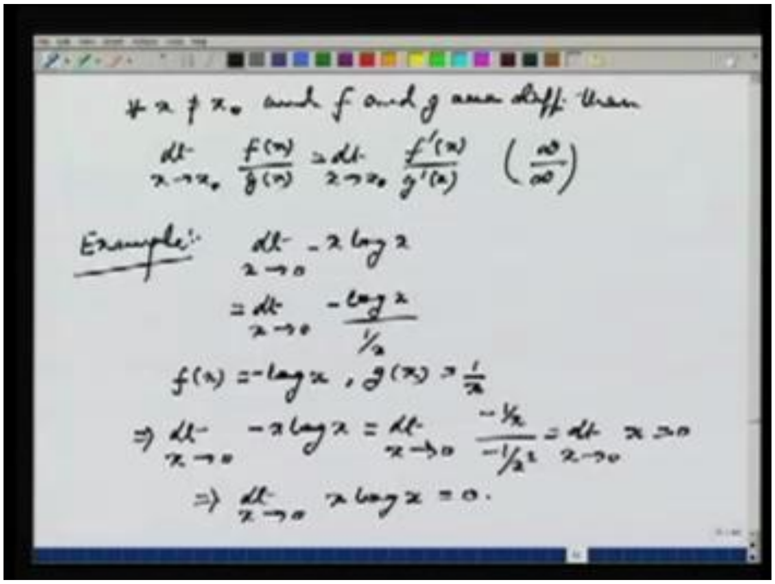

Let us try to calculate limit x going to 0 x log x, which I write as limit x going to 0 log x by 1 by x. If you like, I might start with minus here. I will put minus here. Now, I will define f(x) is equal to log x with minus sign g(x) is equal to 1 by x. Then f and g satisfies all the possible criteria here. By L'Hospital Rule, this implies that limit x going to 0 minus x log x is equal to limit x going to 0. Now the derivative comes, that is, minus 1 by x divided by minus 1 by x squared. That is, limit x going to 0 x, which is equal to 0. This implies, limit x going to 0 x log x is also equal to 0.

So notice that here in this set up to apply L'Hospital Rule, we actually needed infinity by infinity form of the L'Hospital Rule. These are the famous indeterminate forms. One is the 0 by 0. The example we have seen is sin x by x. Other is infinity by infinity. Here the example we have is x log x. Now, let us start with the geometric interpretation of the Cauchy Mean Value Theorem.

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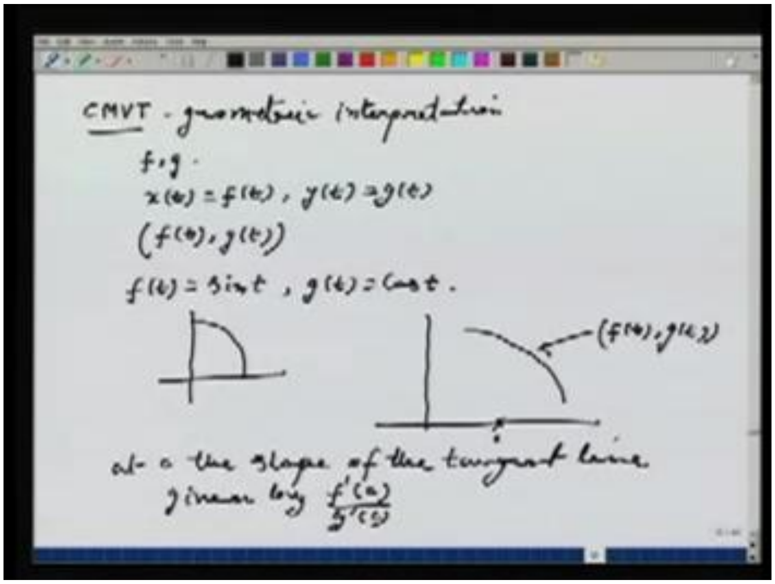

I have given two functions f and g. I look at the curve defined as x(t) is equal to f(t) and y(t) is equal to g(t). So the points on the curves are given by f(t) g(t). As an example, I can look at f(t) is equal to sin t and g(t) is equal to cos t. f(t) varies from 0 to pi by 2. I will get something like this, as my curve. Given this situation, what does Cauchy Mean Value Theorem say? So let us say, this the curve where the points are given as f(t), g(t). Both are differentiable functions. Then it says there exist a point c where at c, the slope of the tangent line which is given by f prime c divided by g prime c must be equal to the slope of the end points. So the end points we know are f(a) g(a) and f(b) g(b).

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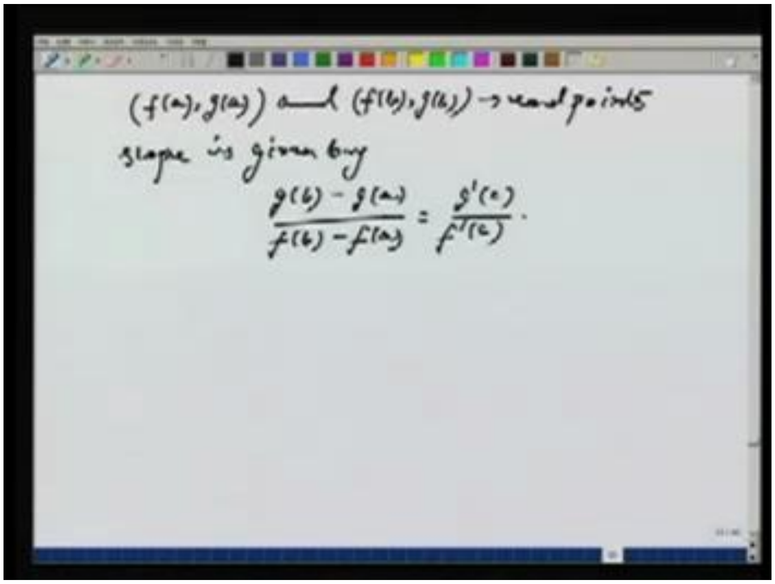

These are the end points of the curve. Then the slope of the tangent line if we look at, that is f(b) minus f(a) by g(b) minus g(a), Cauchy Mean Value Theorem says that this slope is same as the slope f prime c by g prime c. At the point c, the slope of the tangent line is given by g prime c divided by f prime c. On the other hand, if I look at the end points which are given by f(a)g(a) and f(b) g(b), then the slope is given by g(b) minus g(a) by f(b) minus f(a) and then Cauchy's Mean Value Theorem says that this slope is actually same as, pretty similar to usual Mean Value Theorem.