**Classical Physics**
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**Lecture No. # 11**

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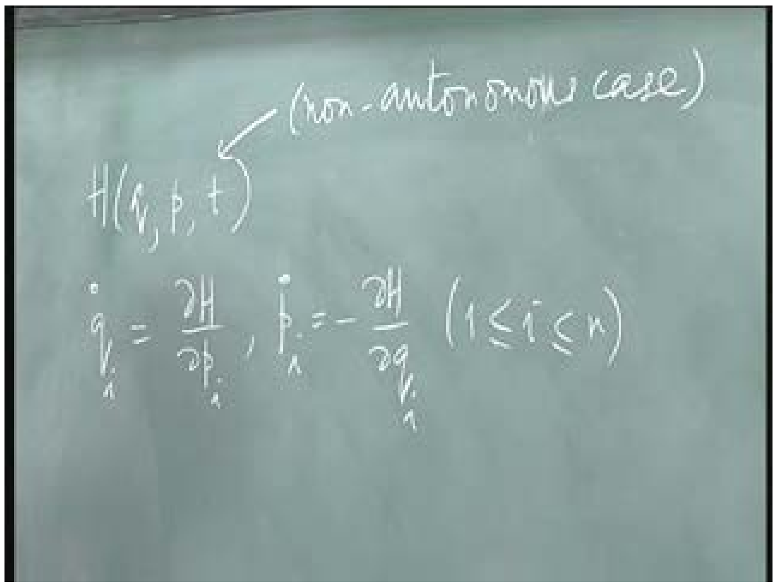


So we have been looking at Hamilton insistence and just to put things in reference, we have H of q p. Sometimes if it is non autonomous, you also have a t. So this will imply non autonomous case. But most of the time, we are going to look at autonomous situations. And we saw that a Hamilton's equation of motion were q i dot is delta H over delta p i and p i dot equal to minus delta H over delta q i 1 less than equal to i less than or equal to n.

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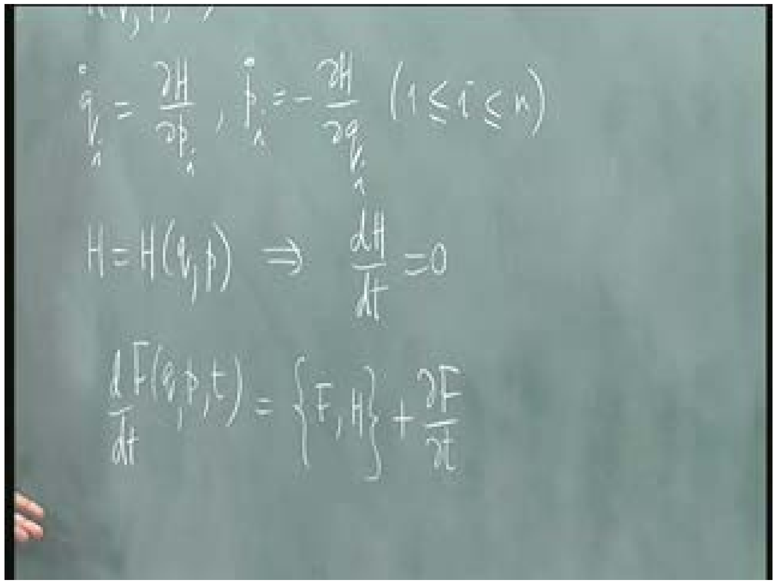


We also saw just to recapitulate that in the case when H was the function of q and p alone the autonomous case this imply that d H over d t is equal to 0. In other words, the Hamiltonian is a constant of the motion on the solution set of Hamilton's equations. We further saw that the general equation of motion for any function of the dynamical variables and possibly time.

The rate of change of this quantity was equal to the Poisson bracket, this side of F with H plus delta F over delta t. And of course, if this is a constant of a motion, then this quantity on the right hand side should vanish. In particular, if F is function of just the q's and p's and not t, then it is a constant of the motion if and only if it is Poisson bracket to the Hamiltonian vanishes identically. So this was our point, which we got.

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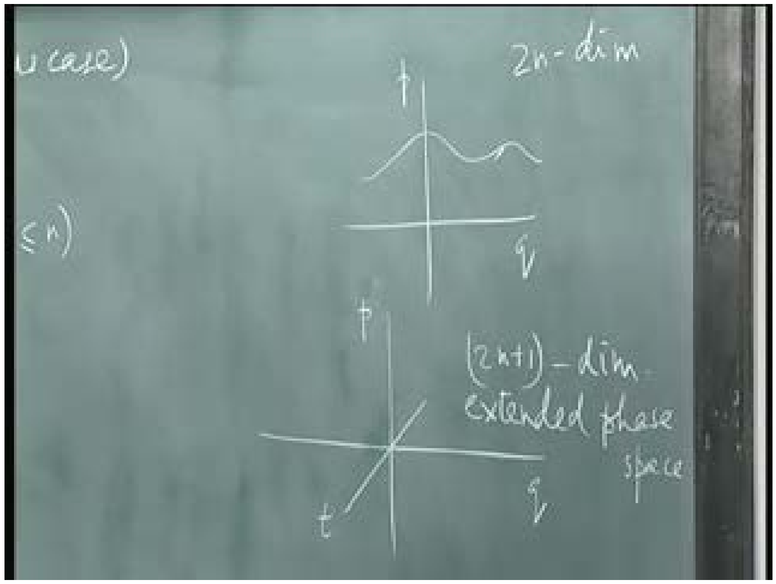


I would like to mention right here that, if you look at, in general there are concepts of the motion which are time dependent, explicitly time dependent. And you have to appreciate the fact that, if you have motion for instance in the q free space. There is some kind of space trajectory. But you could also look at it, in an extended phase space with all the q's and all the p's and time itself as one of the coordinates.

Then in that space, if you have a t here as well; and a q and a p here; in this space too, the phase trajectory is a one dimensional curve and now you could ask how this one dimensional curve formed. It is obviously formed by taking a set of constants of the motion and the mutual intersection of all these constants of the motions, the surfaces would give you the trajectory.

Now this phase space is 2n-dimensional, n of this and n of that. And the phase trajectory is a one dimensional object in a curve. And therefore you know that, you must have two n minus one constants of the motion whose mutual intersection would give you the phase trajectory in principle. In this case the phase space is extended phase space and this is two n plus one dimensional extended phase space and how many constants of the motion do you need here in order to specify the trajectory? 2 n of 1. And at least one of them must be time dependent, because if all of them are explicitly time independent they would be so here too.

So, it is clear for any physical system, you must have at least one constant of the motion. That is time dependent, explicitly time dependent could have more. But it is certainly, it must have at least one. And let us look at a very simple example. You already know this, but let us look at an extremely simple example, of a particle moving on a line, a free particle moving on a line.

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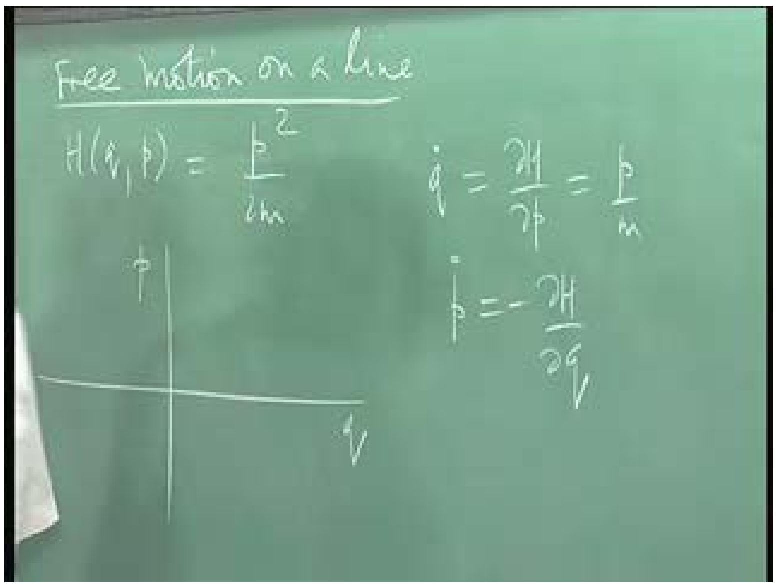


So I have a single q and a p and a Hamiltonian in this case H of q p is just p square over two n. It is motion on a line and it is free motion on a line. A simple example, there is no potential, so it is a particle with no force at all, moves on the x axis or call it the q axis just to keep in touch with this in occasion and then what do the phase trajectories look like. Here is q. Here is p. The reason of course is in Hamilton's equations.

You have q dot is equal delta H over delta p equal to p over m and p dot is minus delta H over delta q. And q is now a cyclic coordinate, because there is no q dependence in a Hamilton. And therefore this is zero by definition. This would imply p equal to a constant. What determines the initial value of p? What determines the value of p? The initial condition, what about be the initial condition and what do the trajectories look like? Just straight lines parallel to the q axis either this way or that way depending on the initial p is zero, is positive or negative. So the whole plane is (( )) by this lines, these are the phase trajectories.

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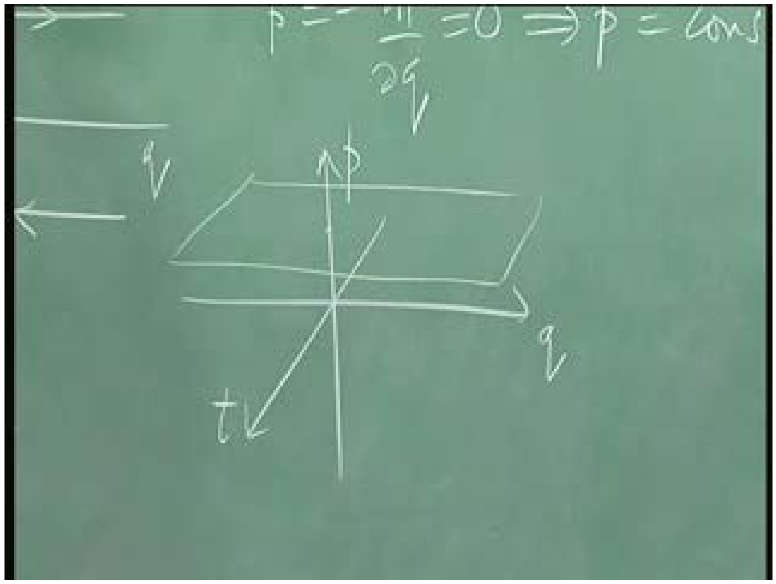


However if I look at it in a extended phase space, so here is q, here is p, and here is t in this direction coming out. Then the phase trajectory is still a line in this extended phase space. But now you need two constants of the motion. One of them is a Hamiltonian or p does not matter. p is a constant. Now what sought of object is p equal to constant in this three dimensional space?

They are actually planes. So whatever was this line becomes a plane. p equal to constant passing through that point. That is not enough to specify the trajectory. You need one more constant of the motion. And it is clear, in order for you to come out of the t of the board. It must be t dependent. It must be some other plane which cuts this plane and produces the trajectory for you. It must be some other surface. What quantity is that? What is that quantity? What is the other constant of the motion? Well, you can do that by solving this equation.

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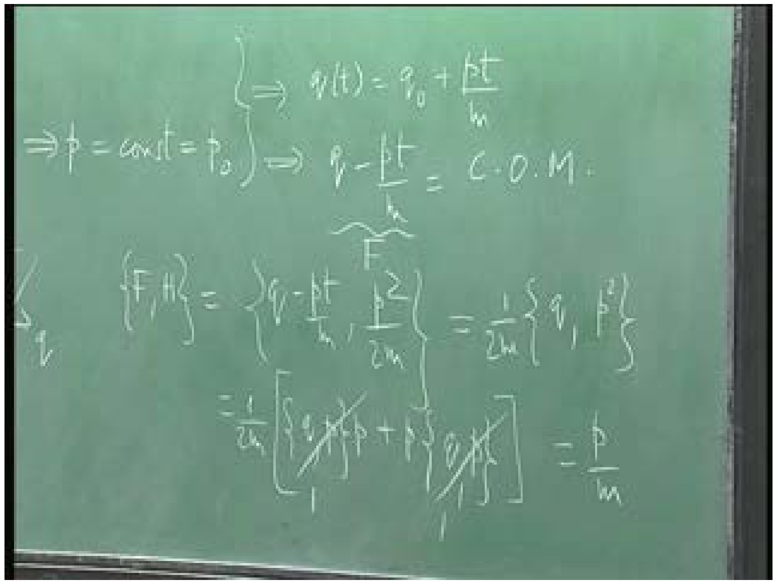

So let us put this p equal to constant equal to p naught, the initial value, back in this equation. And then together these two things will imply that q of t equal to q naught plus p t over m, because p is constant. You could have written p naught, there it does not matter p t over m.

The constant of motion is a function of the dynamical variables and may be time. q minus p t over m. This implies q minus p t over m equal to constant of the motion. And it is directly determined by q naught. So this example illustrates to you what I mean by constant of the motion. Not the initial condition. The initial condition determines the numerical values of constants of the motion. But constants of the motion are functions of the dynamical variables.

And of course time possibly, but you see here q p and t are respectively dynamical variables and the time. And it is explicitly time dependent. This object is explicitly time dependent; so the intersection of this surface in the q p t space. With this surface p equal to constant in the q p t space gives you the phase trajectory. So is this clear? You see why there must be at least one constant of the motion that depends on time explicitly. So do not be surprised, if constants of the motion depends on time explicitly. Incidentally we could apply this, to check, if that guy is really a constant of the motion.

We could apply this formula. So what would this give you in this case? I cling that, this is F and therefore what is F with H. Poisson's bracket with p square over two m and what is that equal to? Well, remember our basic rules; the canonical, the standard commutation relations the Poisson bracket relations.

We had q i with q j equal was 0, p i with p j was 0 and q i with p j was equal to the canonical delta i j. In our problem, there is only one q and only one p. So, there is no need for i and j. This q with q is 0. That is obvious. So with p. q with p is one, on the right hand side. So what is that get us?

This thing here clearly t does not play a role. It is not a dynamical variable. p with p square is zero. Because p commutes with it is Poisson's commutes with itself. And therefore, this reduces to q with p square over two m. But one over two m is a constant. That does not do anything. It just comes out of the bracket. Therefore you have q with p square and one over two m.

How do I simplify that q with p I know, but what about p with q square. I use this chain rule that I wrote down the last time. So this becomes equal to one over two m q with p. The p outside plus the p times the q with the p, I write that p square as p times p. And you have to do that each of them and q with p is one. So this guy is one and this is one. This is bracket round the whole thing.

And therefore I get a two p inside and a two m outside. So that is equal to p over m. That is not zero. That is because this f has explicit t dependence. So we discovered that this quantity here is equal to p equal to p over m, but what is delta F over delta t. minus p over m. The partial derivative with respect to t minus p over m, they cancel and then you get a constant of the motion. So we see how this works.

You need this extra term. Then the total derivative of this is zero. So what has happened in this problem is utterly trivial. The explicit t dependence gave you a minus p over m and this guy here gave you a minus p over m. And they cancel it. The another way of saying it is in this combination q minus p t over m, the t dependence of q cancels the explicit t dependence of the term, minus p t over m. And therefore you have a constant. This is going to be like a general rule. In n dimensions when you have phase space of n dimensions and an extended phase space of two n plus one dimensions, there must be at least one constant of a motion. That is time dependent. That is how motion occurs and the mutual intersection of all these surfaces will give you the trajectory. Now of course immediately you realize that this can lead to incredibly complicated behavior. The reason is as you can visualize even if they are simple surfaces, when they start intersecting with each other, there would be all sorts of little complications in the intersection. And that is how in generality, it turns out, that in a higher dimensional phase space, phase trajectories have exceedingly complicated behaviour, even if the constants of the motion look relatively simple. The intersection can be very complex. And very often it will turn out that, the constants of the motion themselves are not isolating (( )). I will explain what I mean by isolating (( )).

That is our next task. Now given this, I have to say a few things about how we could go about solving Hamilton's equations of motion. And then we go to some examples, but first a little more formally. How would you solve a set of equations of this kind. It is quite intricate and the reason is accepting the simplest instances, this is some horrible function of all the q's and p's and it is a very non linear function.

So is this and therefore you discover that the problem is quite intricate, solving this set of equations is not possible, except numerically in most cases. But here is what happens, in some instances and it turns out that, it is the only way in which, this set of equations is explicitly integrable. And this is what can happen. Like in all these problems, with many variables, it may be possible to change variables and write the problem down in terms of simpler set of equations.

So you might want to, for instance use this fact that, if there is a cyclic coordinate. If there is any coordinate q on which the Hamiltonian does not depend the corresponding p the conjugate p is a constant of the motion. So it would be very nice if you could change coordinates, if you could change variables to new set of variables, such that the Hamiltonian does not depend on some of them. In other words the idea is try to produce cyclic coordinates.

But you cannot in general do this for an arbitrary Hamiltonian you might want to do this by changing variables. But you do not want to change variables in an arbitrary fashion. You would like to preserve some of the properties of these set of equations. What the primary property of these set of equations. Well if it is at all first order, you like to preserve that. I would like to preserve the structure of these equations, because I know that for autonomous systems, h is a constant of the motion. I do not want to lose that I would like to preserve that. So let me for a while talk about what happens when you have autonomous Hamiltonian and then we come back to non autonomous situations like this. I would like to use this fact.

I would like to use this fact, even independent of this fact the flow in this phase space. Is it conservative or dissipative? We can easily check this out. Because remember we had a general criterion for checking out whether flows were conservative or dissipative. What you think is going to happen.

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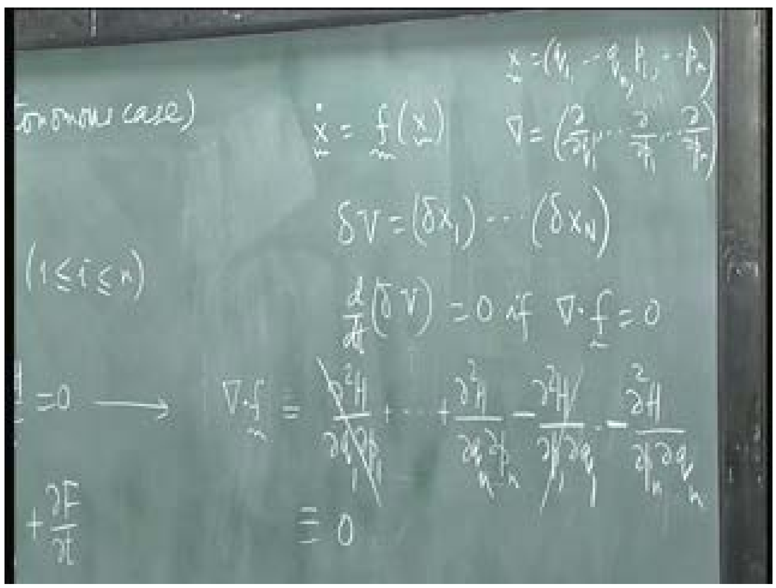

Well here is what is going to happen. I took a flow like this, of this kind. It was an autonomous flow. And then I said ok. This flow is conservative in the sense that any volume element delta V, which was defined as delta X one product up to delta X N. This volume element remained unchanged, as time went on provided the divergence of f was zero. If del dot f was zero in phase space, in the capital N variables. Let us ask that is true, let us ask is, if that is really true here. Well let us look at conservative. Let us look at autonomous Hamiltonians first. And ask is this true. What should you do? So, I have del dot f in the present case. In this case, is equal to. Remember that the coordinates in my problem the coordinates x equal to q one up to q n p one up to p n. And what is the del operator; the derivatives with respect to these guys so delta over delta q one, delta over delta p one up to delta over delta p n. Now what happens if I did, del dot f.

I must do delta f one over delta X one. So the first term is delta f one over delta X one but X one is q one and f one is delta H over delta p one. So this gives me p two H over delta q one delta p one plus all the way up to d two H over delta q n delta p n and then, I go on to the next one which is delta f n little n plus one over delta X little n plus one, but what is that equal to. That is equal to a minus delta H over delta q 1, differentiated with respect to p one. So the next term is minus d two H over delta p 1 delta q 1 up to the minus sign d 2 H delta over delta p n delta q n.

What is the answer? 0. This guy cancels that, because you can take partial derivatives in either order. So this is identically zero. Hamiltonian flow, therefore is volume preserving in phase space.

Very crucial observation and now you begin to see the importance of these minus signs which followed by the Euler Langrangian equations. These things cancel out in pairs automatically. So it gives us this very crucial piece of information that the Hamiltonian flow preserves volume and phase space. And I already mention, that the volume preserving flow in phase space is the analog of, in fluid dynamics and incompressible fluid.

It is, as if the phase space the volume elements that the flow of an incompressible fluid. They could change in shape. You know a fluid element can change in shape as it goes along but its total volume must remain the same. Any initial volume you take must flow along such that the volume does not change. Could change very badly in shape as we will see but it must remain constant. That is our first important lesson.

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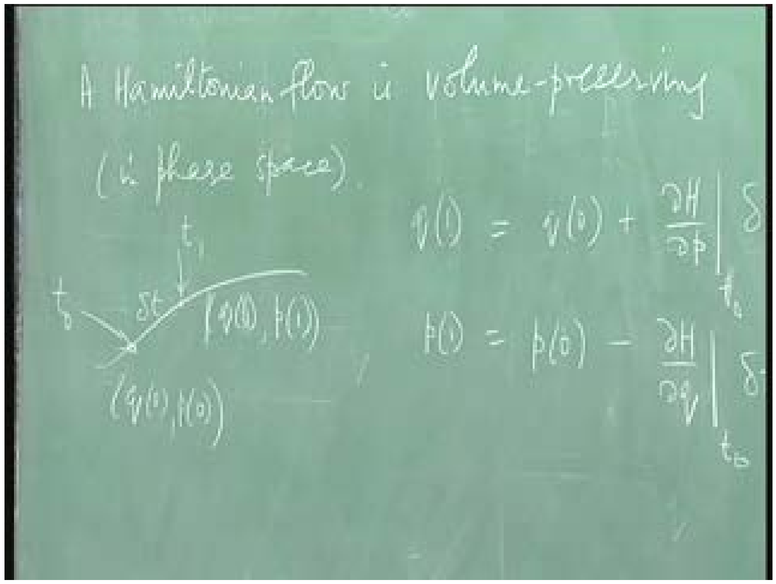


Hamiltonian flow is volume preserving in phase space. Of course you could say look you prove this for an autonomous Hamiltonian system. I used the fact that the Hamiltonian system was autonomous somewhere along the line. I used this here. This theorem was true for autonomous system. What if the system is non autonomous?. What would you then say for the magic is the Hamiltonian flow continues to be volume preserving even if the flow is non autonomous?

And that is not hard to see, because suppose you took some point here. This is your initial point q naught, p naught and you went to some point q one p one. I should not use this symbol. Let us call it q at time zero. p at time zero and it went to a point which is q at time one. Say here time one, a little time, delta t later. So this was at t zero. This is at time t zero.

And that was at time t one. And let us say t one is separated from t zero by a delta t, an infinite decimal interval of time. So this is an infinite decimal line element in phase space, part of the trajectory. Then this point is determined by this point by Hamilton's equations. And let me write down without the i j, for a moment to see exactly what is happening.

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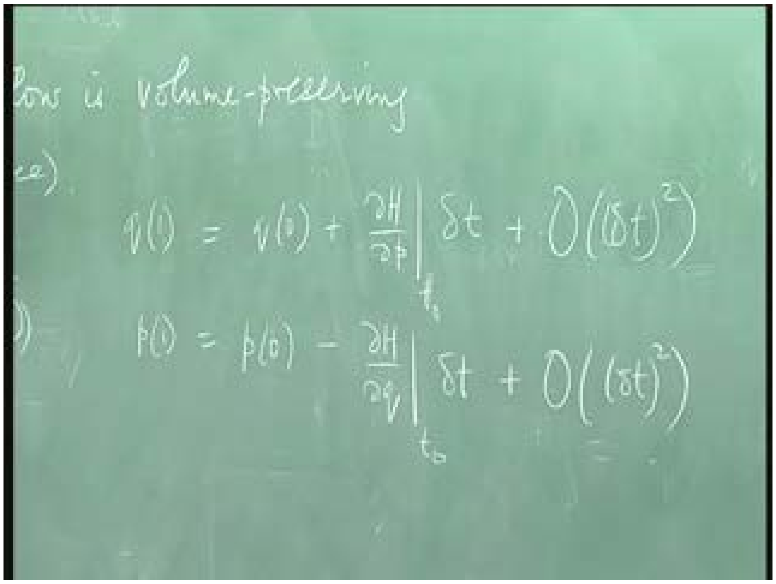


So what is q at 1. This is equal to q at time zero plus directly q over q dot multiplied by delta t. But what is q dot, delta H over delta p. And this is at 0. By 0 I mean at time 0 and t 0. And the point is p 0 and q 0. Time is delta t plus order delta t whole squared. And similarly p at time one is p at time zero plus delta H over delta p. Am I making sense here? Delta d p over p dot, which is minus; and this is delta H over delta q at t zero delta t plus order delta t whole squared.

Now I can regard the point q one p one p one, as having then obtained from q zero p zero, by a certain change of variables induced by the Hamiltonian. Imagine that, this is a new set of variables, which has come out from the old set of variables by a certain transformation. And Therefore, I can find the Jacobian matrix of this transformation.

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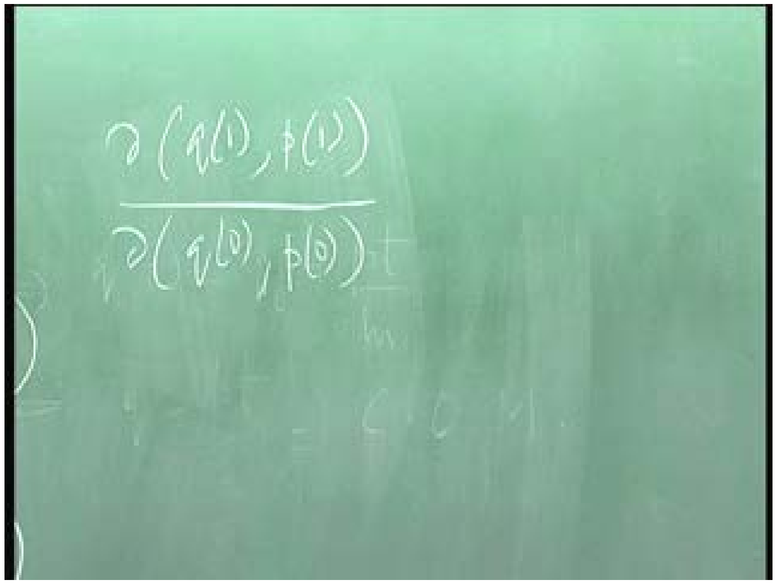


So, I can actually find delta q one p one over delta q zero p zero. I can find this transformation. This is the transformation matrix. It is going to be a two n by two n matrix; and all these partial derivatives; and that is what the Jacobian matrix is. Now the initial volume element is related to the final volume element by a simple relation, which says the final volume element is equal to the initial volume element multiplied by the determinant of this Jacobian matrix. That is how volume change. And now, I leave it to you as a very simple exercise, to show that the Jacobian of this matrix is one plus order delta t whole squared.

Because of this minus sign, the cross term cancels out the order delta t term cancels out. And it says the volume is preserved and if it is preserved in every delta t then you can see it is preserved along the entire trajectory, because you can do this delta t at a time. Pardon me. It is called calculus, delta t is a first order infinite decimal; and if I choose delta t sufficiently small in the limit going to 0.

Then the Jacobian matrix is one plus higher order infinite decimals, which will go to zero if I divide by delta t, very good point. He says, what if that second derivative glue up. Now, our assumption throughout is that the Hamiltonian is a nice differentiable function. All the second derivatives exist. We assumed right from the beginning we assumed it. There could be isolated points where the matter becomes singular. Then it is no longer true.

Just as even to go from the Langrangian to the Hamiltonian, I needed to assume that the Hessian matrix was not singular. So there is an assumption namely that all these H's and so on are smooth functions. They are differentiable, at least twice differentiable. Later we will see, I am going to require something like analytic behavior, but for the moment twice differentiable. So modular mathematical problems of this kind in general, generically this is going to happen.

So the lesson, I want to mention here at the end of it, is this statement is true whether or not the Hamiltonian is autonomous. Even if it is time dependent explicitly that statement is still true. It is a very powerful statement. We will see, where it is going to lead us. So in that sense, this structure here was extremely helpful and now if I make a change of variables I would like to preserve this, I would like to preserve this property. This is like an incompressible fluid and therefore I am now going to find a set of properties I would like to deduce. I would like to impose on any transformation. It is going to keep my system manageable and this defines what, called a canonical transformation. So let me define these.

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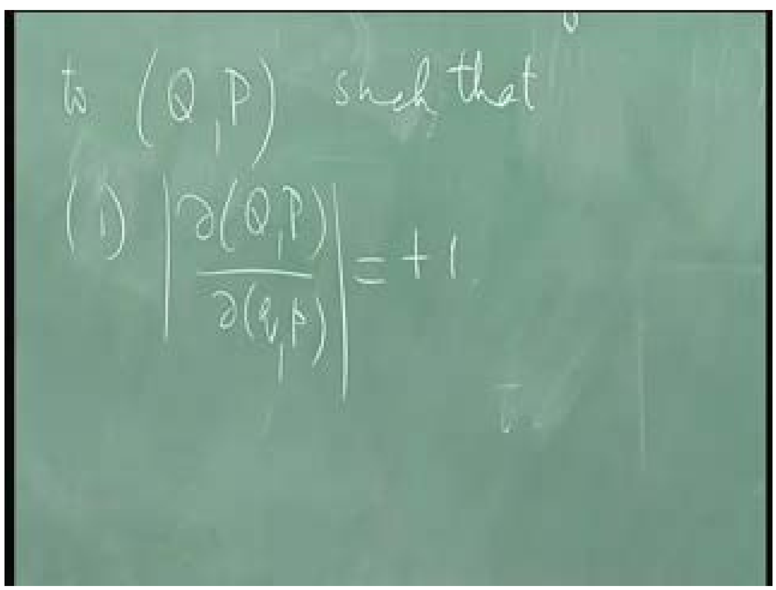


One, we would like to change variables from the set q p all through of them. Two, a new set Q P such that the first thing we would like to have is that Jacobian matrix delta Q P over delta q p. This matrix determinant equal to positive, this ensures that it is volume preserving incidentally it could be a minus one and it could still preserve the volume in magnitude.

When you took this magnitude, this determinant could have value minus one and it would still preserve the volume but what would it change? What could it change to become minus one, what would it need?. It would change what is called orientation. It is like going from a right handed coordinate system to a left handed coordinate system. It is like doing a parity transformation.

Suppose in three dimensions, you start with a set of coordinates x y z and I choose a two set x prime as minus x, y prime as minus y and z prime as minus z. Then a volume element does not change, but the product d x d y d z is minus the product d x prime d y prime d z prime. It changes orientation. I would like to make it plus one for reason which one is not clear right now, but impose. Orientation should also be present.

That is the first thing I would like to tell. I would like to have a little more. I would like to ensure that, the structure of Hamilton's equation in not changed. I would like to ensure that. What would the new Hamiltonian be?

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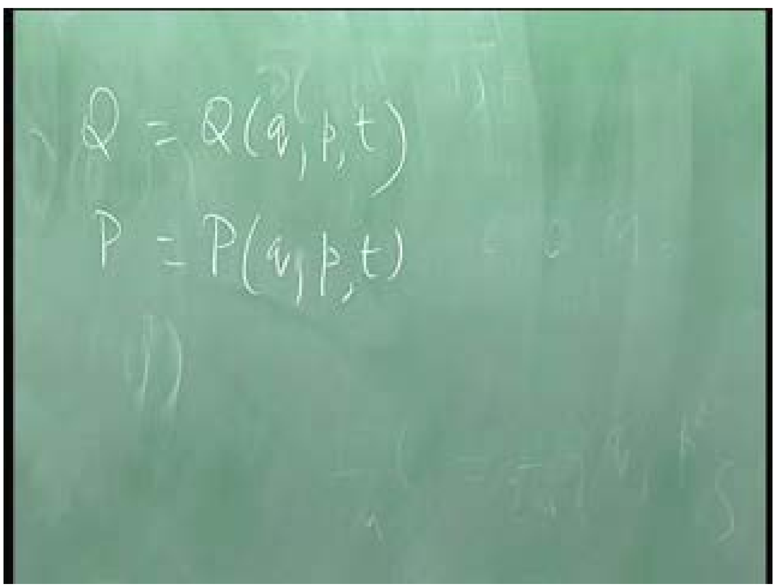


What is really happening is that Q is a function of all the little q's p's and t's and P is function of all q p t. This is my change of variables. I would like to make this change of variables in such a way that the Jacobian matrix has determinant plus one. And I would like to preserve the structure of Hamilton's equations. This is guaranteed in two different ways.

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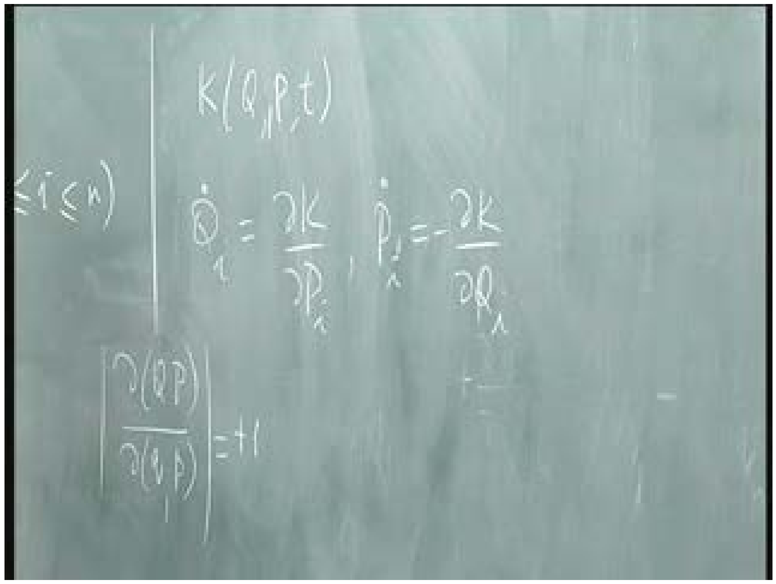


One of them is to simply say that if this set of equations is true, I would like to have a new set of variables such that the following is true, such that on the one hand I have this. On the other hand I have not H, but some K of Q P and t. Some new function K, which is a function of the new coordinates, new generalized momenta and may be time, such that Q I dot equal to delta K over delta P i. P i dot equal to minus delta K over delta Q i.

And we would like to have delta of Q P over delta over q p equal to plus one. Incidentally we would like the transformation to be invertible. Because, if I am going to solve the problem by going to the new variables after finding the solution I would like to get back to my old variables and ask what are they as functions of time. I would like to solve my original problem.

So this transformation must be invertible and of course if this determinant is non zero then this transformation is at least locally invertible. So I would like to have an invertible transformation from little q little p to capital Q capital P such that this property satisfy and the structure of Hamilton's equations is also satisfied. After that I need to tell you given capital H, the Hamiltonian in the old variables, what is K in the new variables. I need to tell you this. I need to give you a prescription.

It is not obvious that capital K is just, capital H in which little q is expressed in terms of capital Q's and P's and T's. It is not obvious. That is true. That is only true, if you did not have at dependence here. Then this is true.

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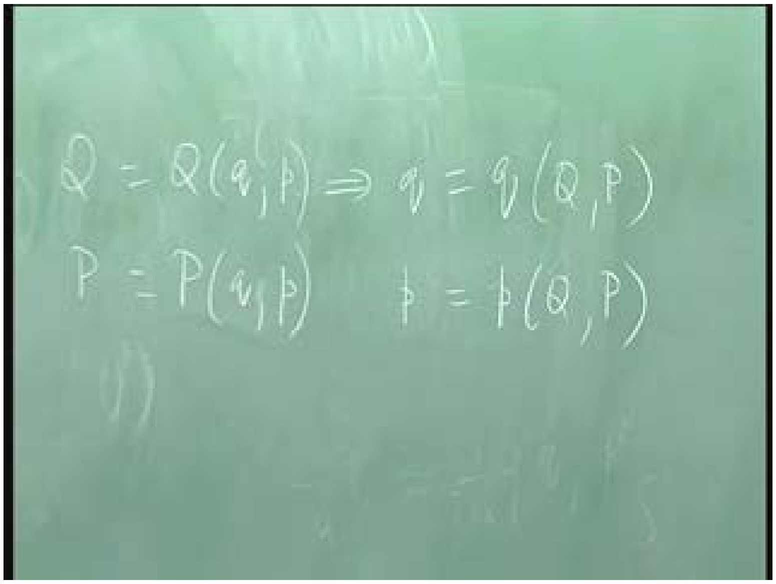


You could simply solve this. Invert this set of problem. Invert this set of transformations. So this would imply that little q is equal to a function q of capital Q's and P's and little p is a function of capital Q's and P's. And then put that into this H and call that function K. That would be true provided you did not have t dependence. When you have t dependence, then the algorithm for finding capital K is a little more complicated than that for finding the original, in this autonomous case.

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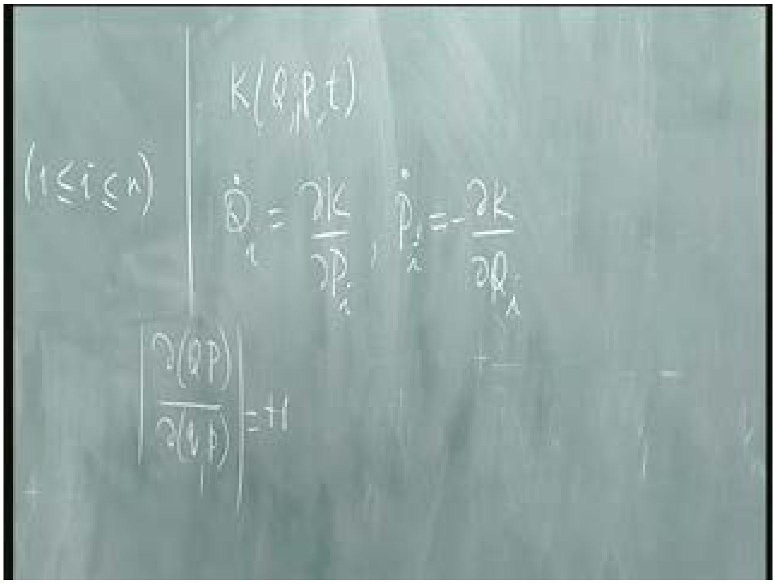


But for the moment you will do this in complete generality. This some function K can be found from capital H, such that this set of equations is satisfied. And this is true. A transformation which does this is called a canonical transformation. And I am going to abbreviate it CT.

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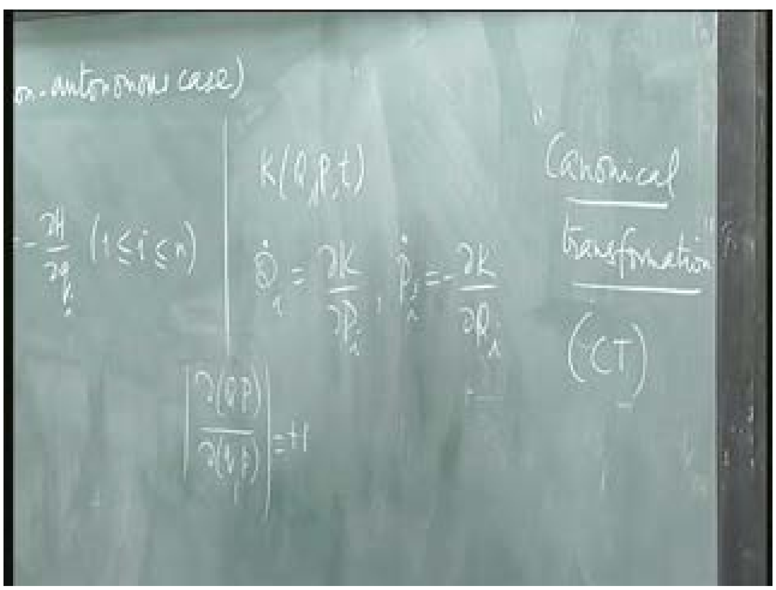


We are going to use canonical transformations at length little bit. So it is important to understand what is being implied here. And what is the motivation for doing this?. The motivation is, I would be like to be able to solve Hamiltonians equations. There are a set of two n coupled differential equations.

Very non linear, but I might be able to exploit certain symmetries in the problem. And reduce the problem to a lower dimensionality by solving some of the equations in particular, if it should so turn out that in the new variables some particular Q is missing. And that derivative is zero here and this is immediately a constant of the motion. So this is the motive. Now, what is the best I can do?. How many such constants of the motion, can I find?. n of them.

I mean all, if K does not depend on any of the Q's, then all these fellows are constants of the motion. And I will straight away find n constants of the motion immediately and it turns out that it is necessary and sufficient for the problem to be implemented. So, the whole point of doing canonical transformations is to check, if I can make canonical transformation such that the Hamiltonian, the new Hamiltonian does not depend on the new generalized coordinates.

But I would like to preserve these properties of a flow in phase space which is volume preserving which preserves the structure of Hamilton's equations as well and that is done. Not always possible. When it is possible it turns out. You can go the whole all and you can make all the Q's cyclic coordinates. I will give a criterion for this. When this can be done?. But before that, I would like to point out that there is another way of defining a canonical transformation, an equivalent way which does not say anything about this structure. But it says this property is true.

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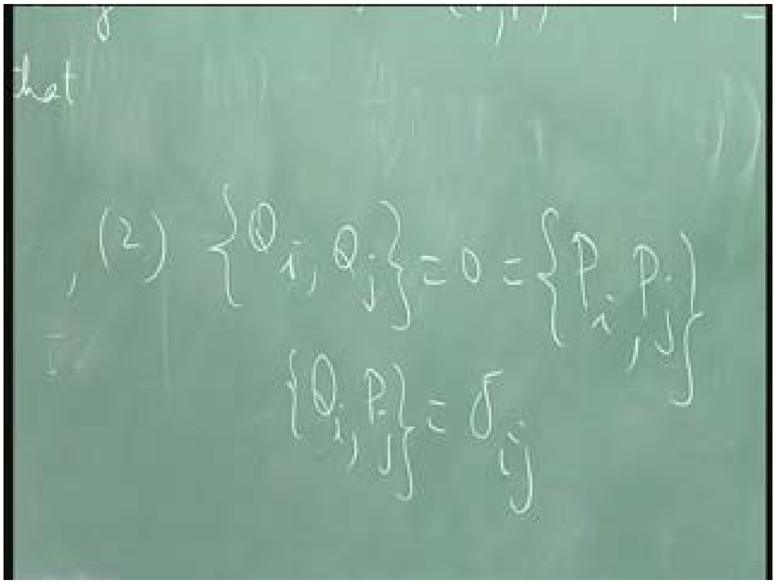


I started writing that down and two the canonical Poisson bracket relations are true. In other words you should be guaranteed that Q i Q j equal to zero equal to P i P j. And Q i P j equal to delta i j. So that is true, then also this magic is obtained namely the transformation is guaranteed to be a canonical transformation. One often uses this. And the reason, you often use this is, because you do not want to write restrict this to any specific Hamiltonian.

We are talking in generalities now. You are saying you give me a system with n generalized coordinates, generalized momentum and some given Hamiltonian, then I make a canonical transformation such that the new Hamiltonian is simplified. And I do that without reference to the Hamiltonian by simply saying these conditions are true. Now there is no reference to any particular Hamiltonian here.

But you guarantee that, if this and that are true then for every canonical system, Hamiltonian system with n degrees of freedom with those generalized coordinates of momenta, the transformation is guaranteed to be canonical. So you can easily see that there are some transformations which would be canonical for a given Hamiltonian. But there are other transformations which would be canonical for all Hamiltonians independent of what the Hamiltonian is.

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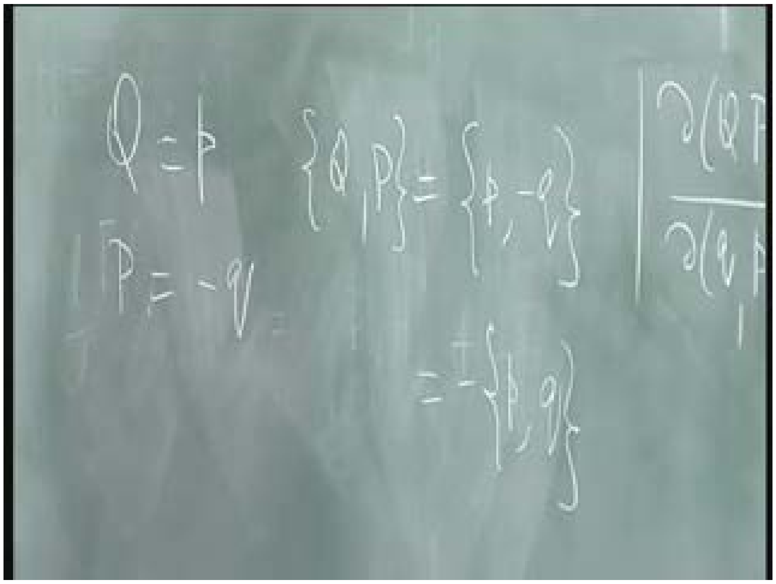

Here is one. Suppose I set Q equal to p and P equal to minus q with n degrees of freedom, then of course Q i is equal to P i and so on but look at it one when degree of freedom. What happens now? What happens to this guy? What happens to Q P? That is equal to p with a minus q. That is equal to minus but p with a q. But, we know that the Poisson bracket of a with b is the minus the Poisson bracket of b with a and therefore this is equal to q with p is equal to one.

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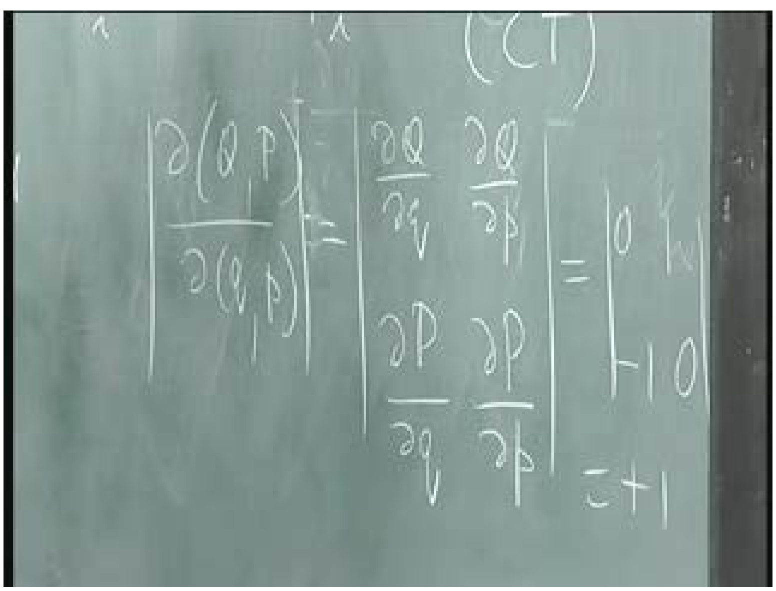

So that satisfies and what is the Jacobian matrix do delta Q P delta q p. This is equal to, we want the determinant. We need the determinant of delta Q over delta q delta Q over delta p delta P over delta q delta P over delta p. What is this equal to where Q does not depend on little q here.

So you get a zero there. It is zero and then a capital Q on a p will give you a one and a p on a q gives you a minus one and then a zero again and that is equal plus one. So this is a canonical transformation and now you begin to see, why I said it does not matter what you call the coordinates and what you call the momentum. This is a system with two n dynamical variables in pairs, the variables are being identified and they have a certain geometric structure and that is it.

And you can see this. This is a mastery classic example of what is happening. So what I call the whole coordinates. I could call minus a new momentum. And what I call the whole momentum. I could call the new coordinates and everything is unchanged. Nothing has changed. That is the canonical transformation, an extremely simple canonical transformation. Yeah. No advantage.

This does not get you any advantage at all, because it does not reduce anything. But the point I made here was to show you that you have such a trivial linear canonical transformation. It does not do anything. It is not going to reduce variables. It is not going to do anything, but the fact is it does illustrate the point that in Hamiltonian dynamics the distinction between momentum and coordinates is rather artificial.

What you start of is generalized coordinates and momentum can be inverted and it is just the same problem all over again. No, I am not talking about physical dimensions. That is a different problem altogether, because I am assuming that all these are dimensionless quantities. So every quantity that I take, I divide by some characteristic momentum or length in the problem, coordinate dimension in the problem and make them all dimensionless states. So I have assumed and I have done that.

And that is always do nothing. There are other such examples of transformations but this particular one does not do anything very quickly. First of all it is a linear transformation. And it is a rather trivial one. But I brought this out just to show you that the distinction between generalized coordinates and generalized momentum is an artificial substance. You actually you have two n variables. And they are paired together in conjugate pairs.

That is important structure but which one you call the momentum and which one you call the coordinate is irrelevant. Is it clear. Of course in real life you are going to look at more complicated transformations. This is not only trivial one. It is not going to help very much and now let us go on. We are going to see a lot more about canonical transformations. Let me go on and tell you when the problem is completely solved and what is the criterion for it.

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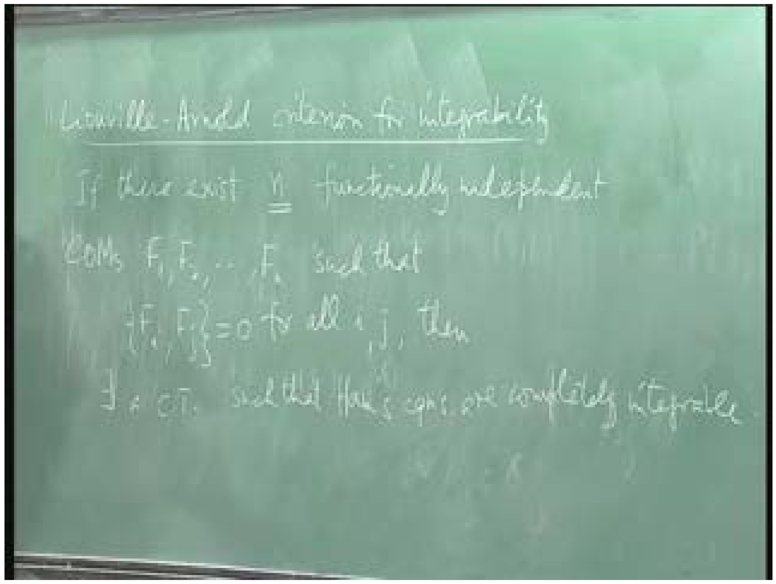


The theorem itself is very hard to prove as to when to solve this problem completely but I will state the theorem. And this is a very famous one. It is called Liouville-Arnold integrability and it says the following. First point is, I will illustrate this. If you can make a canonical transformation such that, all the Q's are cyclic variables, then this set of equations is completely solved. It is really totally solved.

I will illustrate that very shortly. You might ask look how do, I do this. I am giving you n constants of the motion instead of two n minus one and I am claiming the problem as totally solved and that is very trivial to see and I will show you how that happens. But what the criterion for integrability says is the following. It says, if there exist n and this is important, n just n, not two n minus one n.

Functionally independent, by that I mean one of them should not be the square of another. You are going to tell me the Hamiltonian is a constant of the motion but H square H cube E to the power H are also constants of motion. And they are not independent. So you cannot say that the problem is solved just because you have functions of a given constants of the motion. They must be independent. Function of the independent COMs F one F two F n.

Let me just call them F one F two. It is very convenient to choose F one equal to the Hamiltonian. We know that already that it is a constant of the motion in an autonomous system. So, if there are n functionally independent constants such that, there are other conditions one puts on it namely they must be nice functions. They must be differentiable.

They must be twice differentiable and so on. We will keep that in deserved. They should actually be analytic in all these variables. We assume that for the moment such that the Poisson bracket of F i with F j equal to zero for all i j. Every constant of the motion, every F i Poisson commutes with every other F j. Mutually they all Poisson can move, then if this is true, then there exist a CT, a canonical transformation such that Hamilton's equations are completely integrable.

By that, I mean you can explicitly solve all these equations. Write down all the Q's and principle as function of P Q's and P's as functions of Q. This is a necessary and sufficient condition. We need these guys to be functionally independent and you need them to have zero Poisson brackets with each other. Now two quantities a and b or F one and F two, their Poisson bracket is zero. I have been saying their Poisson commute, but I also pointed out that there is another term they are said to be in involution with each other.

So what is needed is n constants of the motion in involution with each and the first thing you have to understand is that this operation of commute two quantities with zero Poisson bracket is not one of those operations which would have this property of associativity namely of reflexive whatever you call it.

There is, if a Poisson bracket b is zero and b Poisson bracket c is zero it is not guaranteed that a Poisson bracket c is zero. This is not guaranteed. It is very easy to see that this is not true but you can use a Jacobian identity.

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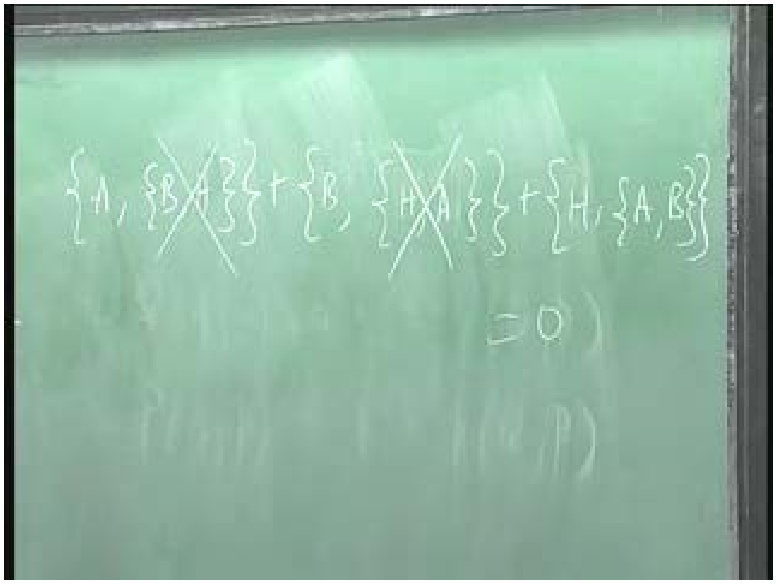

Then of course when you have two constants of the motion you can discover a third because it is clear that, if I took A with B C plus B Poisson bracket C A plus C. This is equal to zero. This was the Jacobian identity. Now suppose A is constant of the motion and B is a constant of the motion and C is chosen to be the Hamiltonian. That is all, so constant of the motion.

So C is the Hamiltonian. B with H but if B is a constant of the motion and they are all time independent then this is zero. Similarly C was the Hamiltonian. So you have H with A and this is also zero and C is the Hamiltonian.

So what does that imply? It says that H Poisson bracket with the Poisson bracket of A with B is zero. What does that imply? The Poisson bracket is itself a constant of the motion. So it says if you have two constants of the motion other than the Hamiltonian then the mutual Poisson bracket is another constant of the motion and then you can use that to create more Poisson brackets and so on and therefore an algebraic structure is evolved. So it is possible that in some problems you may be able to exploit this to go on. And find other constants of the motion. In fact if you look at angular momentum L sub x in many problems is the constant of the motion. L sub y is the constant of motion and this will imply that their Poisson bracket which is L sub z is also a constant of the motion. That is why if two components of angular momentum are constant so third is also a constant of the motion.

It follows from this. Of course it might fissile out. It may turn out that this guy here is either zero or a number. If it is equal to six. We know six is already a constant. It does not matter. So it could turn out to be trivial in some cases but it is also useful in many other cases because it helps you generate more constants of the motion. What is worst is here is n of them. Functionally independent and after that the claim is the problem is solvable. This is the purport of the theorem Liouville-Arnold theorem.

Now solvable in what sense, and what does this canonical transformation can do. What it does, I am not going to prove this theorem by the way, but I am going to show you how the problem become solvable if that is true. Let us assume that this theorem is true. This is given. Then it says there exists a canonical transformation such that the problem is solvable.

It does not tell you most notably how to find these constants of the motion. And it does not tell you even worst how to find the canonical transformation. So this is more like an existence theorem. It really is guarantying you that there exists something, but it is hidden and you do not know where it is but it exists. You know the problem is solving. Now, what does it imply and how does it get solved.

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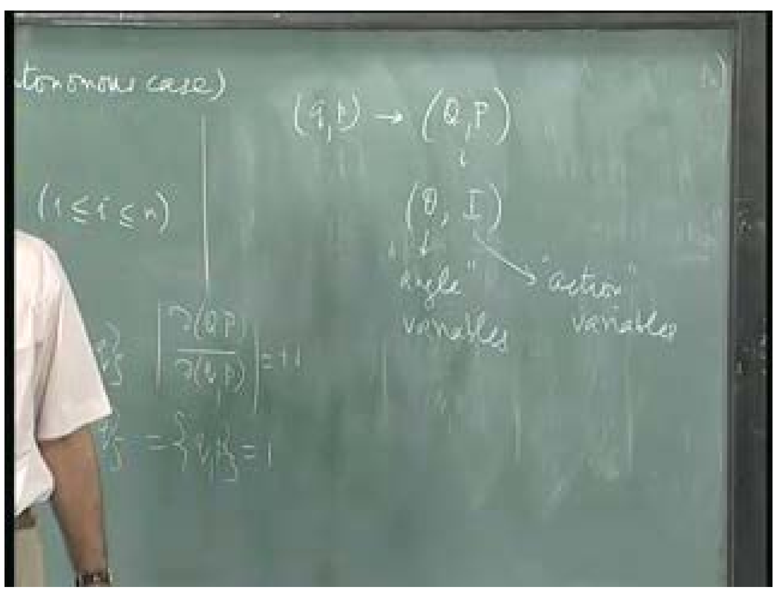

Well, in such cases it turns out that the canonical transformation has a special name. And it is called action angle variables. So really instead of q and p you go to capital Q and capital P. And the standard notation for this is not capital Q capital P but rather theta. And I and remember there are n of them. Little i and subscript i has to be put in and these datas are called angle variables and the capital I's are called action variables.

I will explain why this is true. We will also look at some simple examples. These are called angle variables. This is called action variables. And this transformation, this canonical transformation which does this magic is called the transformation to action angle variables, but the way the magic operates is as follows. It turns out that once.

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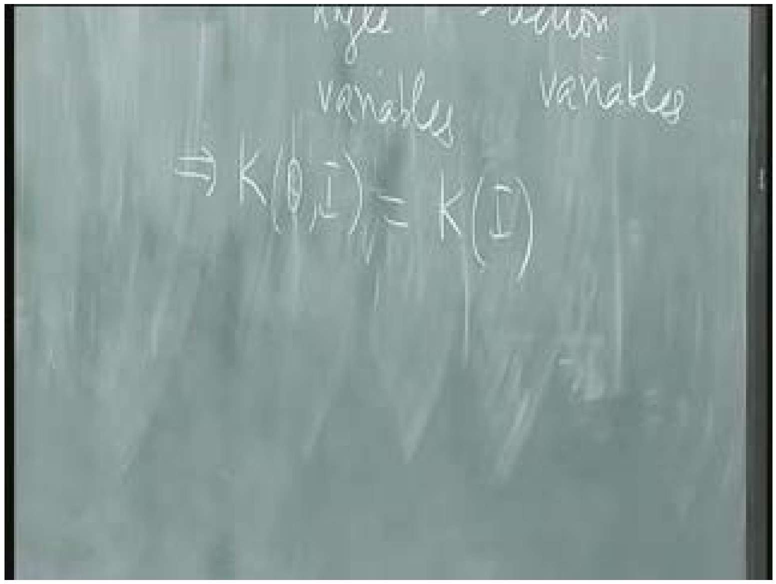


Once you do this canonical transformation K of theta I becomes equal to K of I alone. No dependence on half the variables, on the new general coordinates. No dependence. What does that imply, because we can write the equations of motion down.

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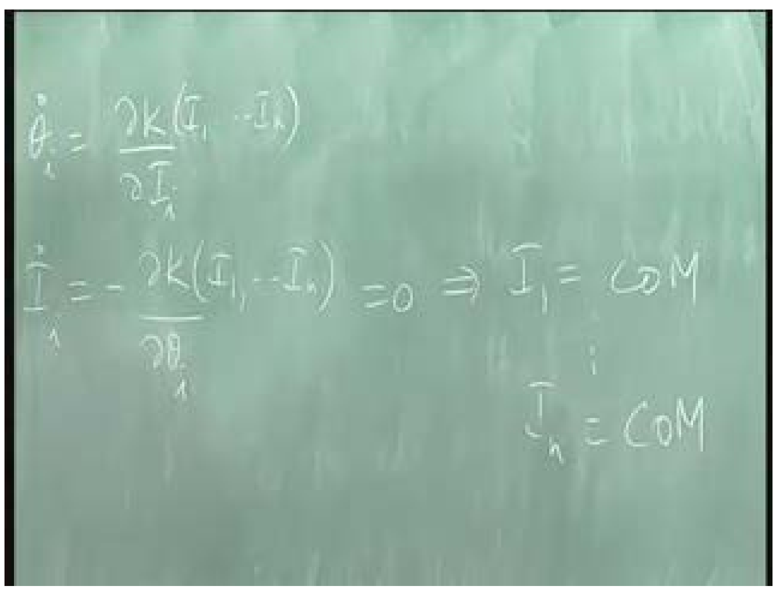

It implies, I know that theta dot theta i dot equal to delta K which is a function only of I one up to I n over delta I i. Instead of capital Q and capital P, I am going to use theta and I here. And I i dot with a minus delta K of I one I n over delta theta i dot. For little i running from one to n, if this is possible, if this happens, then what is the consequence of this equation?.

This is immediately zero, because this has no dependence on the thetas. So this is equal to zero. Now what does that imply?. This implies is I one equal to constant of the motion, I n equal to constant of the motion. All the i's are constants of the motion. The new generalized momentum which, I have called action variables they are constants of the motion. If that happens, then what is this guy equal to?. What does that imply. What is this equal to. What is that give you. Well, what can this be a function of?.

After you differentiate, what is it a function of of the i's. Some function of the i's. So all these guys are function of the i's. What shall we call them what shall we call these functions. Some function of I one up to I n, each of them, each partial derivative with some function. I already call these angle variables.

For reason, which I will explain later and this is the rate of change of an angle. So what should I call it some omega. Let us call it some omega looks like an angular velocity. So we will call it omega sub i. Do the omega i's depend on time?

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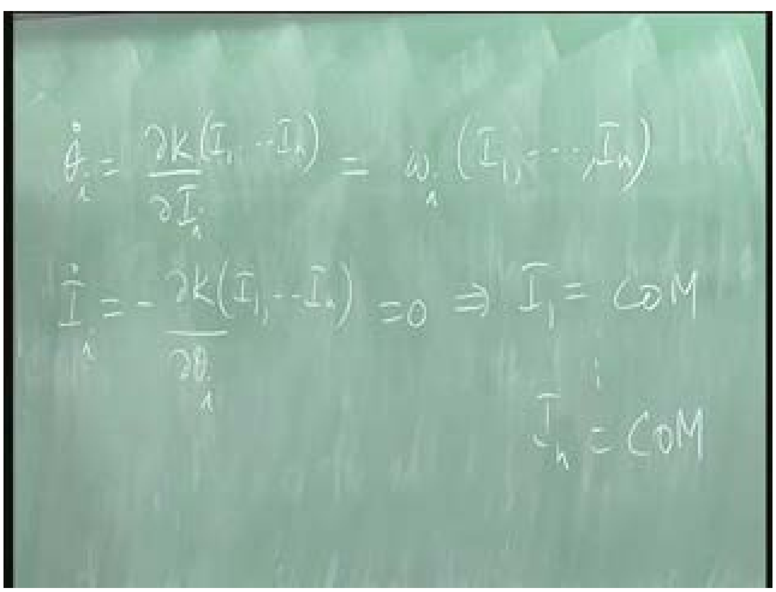

No, because the i's are all constants of the motion, for a given set of initial conditions they remain unchanged in time. So what is the solution to this equation?.

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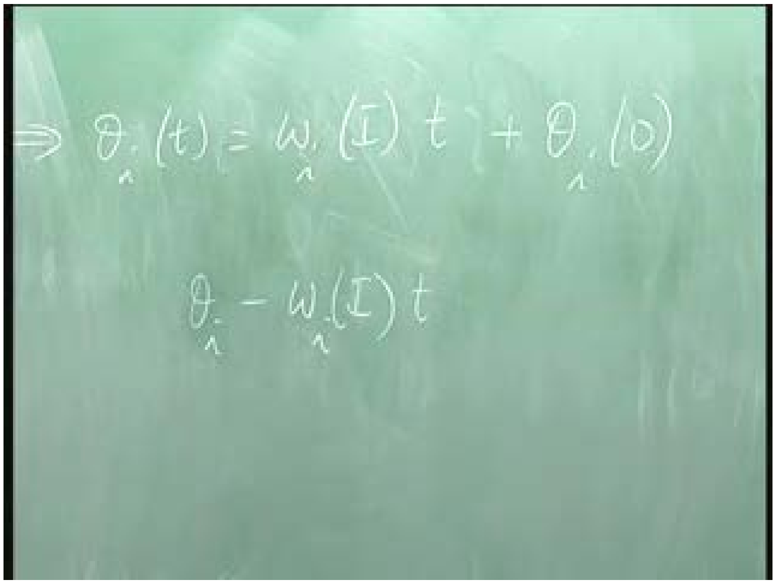


Of all I's t plus theta i zero and the problem is solved. What are the new constants of the motion?. You need two n constants. You need in two n dimension phase space. You need two n constants. We found n of them and they are I one to I n. Actually we started by saying there exists f one to f n already and these I's need not be the same as the f's but you absolutely guaranteed that the I's are functions of the f's combinations some functional combinations.

There are n of those and n of these and we already have n constants of the motion which are time independent, where are the other constants of the motion. The theta i's of zero are all constants of the motion, so all these quantities teta i minus omega i. These are all constants of the motion. These are functions of I's by the way. They are time dependent. Just like our one dimensional example they had time dependent constants of motion.

In this case in a fully integrable Hamiltonian system. This is called a fully integrable Hamiltonian system. You have n constants of the motion which are time independent the action variables and n more which are time dependent explicitly but the time dependence is linear.

Extremely simple and you are going to see what is going to happen. Why I am going to call this an angle variable. It will become very clear, because we are going to look at bounded motion.

We are going to look at bounded phase space essentially oscillation of some kind, generalized oscillation, not free and bounded motion. Then these thetas will indeed turn out to be angles. And the problem is fully solved, you can see. Now the only thing that changes are n angles, and if I tell you now by anticipating myself that, these angles go from zero to two pi, then what sought of phase space do I have?. I have a phase space, which is determined by n angles from the two n dimensional phase space.

I change variables to a new set of variables. And this set of variables is just n angles, each of which goes from zero to two pi independently. If you had one angle, what would the phase space look like a circle, if you had two angles what would they look like?. No, because if you have two angles on the surface of the sphere in three dimensions, you have an asymmetric angle with longitude which goes zero to two pi but the polar angle go only zero to pi. But, I have two angles going from zero to two pi, a torous.

So, you have one angle like that and another angle like this. Each point on this circle you have attached the circle at right angles to the right hand. That is it torous. That is topologically equivalent or donut. The surface of a donut, if you have n angles you have n torous. So this is why we say that Hamiltonian dynamics for a fully integrable system takes place in a phase space some n torous, n dimension torus and that is starting to play a very fundamental role here.

We will see in a little more how this really comes about, but the problem is actually solved. Now this happy situation, I am sorry to say applies only for true cases. The harmonic oscillator, and the Kepler problem in three dimensions. In general, this is not true. But, it is very important to see, where this will breakdown. Because that is what, the whole thing is about.

Most system will start with something solved. It is like quantum mechanics. You start with a hydrogen atom, but how many situations you have in which you have one hydrogen atom and nothing else in the universe. But you need the hydrogen atom, in order to do everything else by perturbation. That is exactly what is going to happen. I now leave you finally with the following exercises when we take up from here, tomorrow, if you took the simple harmonic oscillator.

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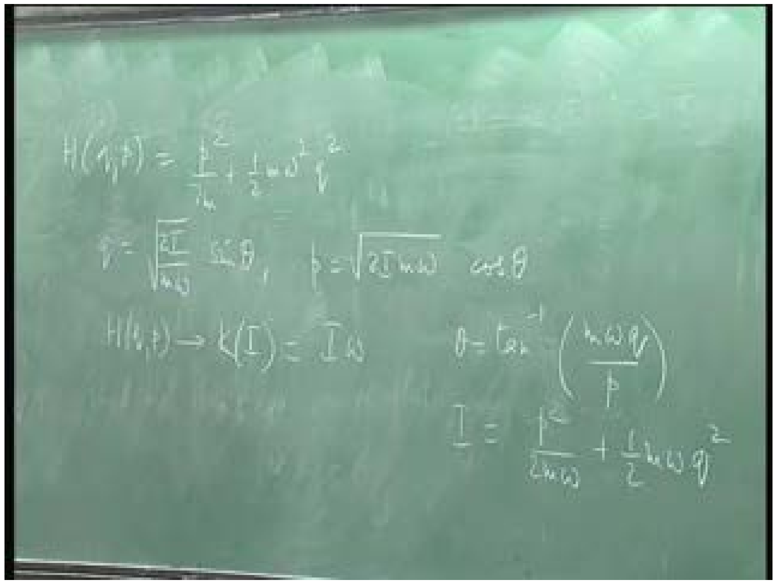

So example linear harmonic oscillator and the Hamiltonian in q p is equal to p square over two m plus one half m omega square q square. Of course you can solve this problem without action angle variables. This is a trivial problem. Even for the JEE you need this problem, solution to the problem.

But now let us use a sledge hammer to crack a nut and solve this by going to action angle variables. Now I have to tell you what the answer is in order to tell you what the action is. There is a prescription for it if you look later on but change of variables is the following.

This is q equal to square root of two, I over m omega sine theta. I do not square to these things, because I usually use units in which, I put m equal to one omega equal to one. But it gets dimensionally right. p equal to square root of two I omega cos theta. This is not going to pull the coordinates by the way incidentally, if you are on a plane a system with two degrees of freedom x and y and two momentum p x and p y going to pull a coordinates on this plane, is not a canonical transformation.

Why is that?. Right away you can say choosing that in real space going from x y to r theta is not a canonical transformation and correspondingly going from p x p y to p r radial momentum and angular momentum. That is not a canonical transformation. Area is not preserved. The Jacobian is not one magnitude is not one. So that is, right away rules it out. It looks like going to pull a coordinates, but it is not. It is in phase space. And it is got the square root and things like that it is not hard to show that H of q p goes to K of I which is equal to I omega.

Check that out. It is very trivial, because you just have to write q square and p square and plug that in here. And immediately you get. So this new Hamiltonian does indeed have data as a cyclic variable. It is only dependent on I. Of course, you can invert this whole business. So I is equal to this divided by omega and you can also see that tan theta. If I divided one by the other, then there is a one over m omega. So this is equal to m omega q of theta and inverse. And I equal to p square over two m omega plus one half m omega q square.

So these are the new variables theta and I. I need you to verify explicitly that the parse on bracket of theta with I, is equal to plus one, should be. So these are the action angle variables and the problem is trivial because now all I have to use is Hamilton's equations here. And it immediately tells you that, I is a constant and state us. Yeah, no it depends. It does not help very much.

When you solve numerically, it might help to go to a canonical transformation, in which part of the q's are cyclic coordinates. It is not making it simpler, because it is very hard to do numerical integration, in which you preserve the volume. So this structure of Hamilton's equations has to be preserved. And that is not trivial numerically. So the numerical routines for solving Hamilton's equations would have to be, such that the integrators are simplistic integrators that, you really preserve the structure of Hamilton's equations that, the volume element is preserved.

The canonical structure is preserved. And this is a non trivial task, very non trivial task. It is important to do this, because if you look at accelerators, you have these particles zooming around, then you would like to solve the equations of motion numerically. System is very complicated. You would like to solve it numerically. But then in a minute or so you would have large errors multiplying system in your calculation unless you are very careful to preserve the Hamiltonian structure.

So it is a very non trivial problem in accelerator physics to get numerical packages of, integration, where you preserved structure of Hamilton's equations. Highly non trivial problem, a lot of work has cornered on that, but my point here was to tell you to start telling where this whole thing is going to break down, where (( )) is going to come from for that this is needed.

This portion is needed. You must appreciate the torous structure of phase space for integrable system. Then slowly we will get rid of it. Next step is to ask, what happens if I put two oscillators and this would be on a two dimensional torous. In four dimensional phase space and it is little hard to visualize, we can still do it and surprises are stored. But this is the action angle variable transformation.

For a linear harmonic oscillator, I urge you to show explicitly solve this go through the routine and solve it. But above all show that the Jacobian of this transformation is plus one. And the canonical Poisson brackets are preserved. That is useful exercise in handling Poisson brackets. Let me stop here. And we take it from this point tomorrow.