**Classical Physics**
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**Lecture No. # 15**

The question is how do you find the equation of phase trajectory in phase space from the Hamilton equations?

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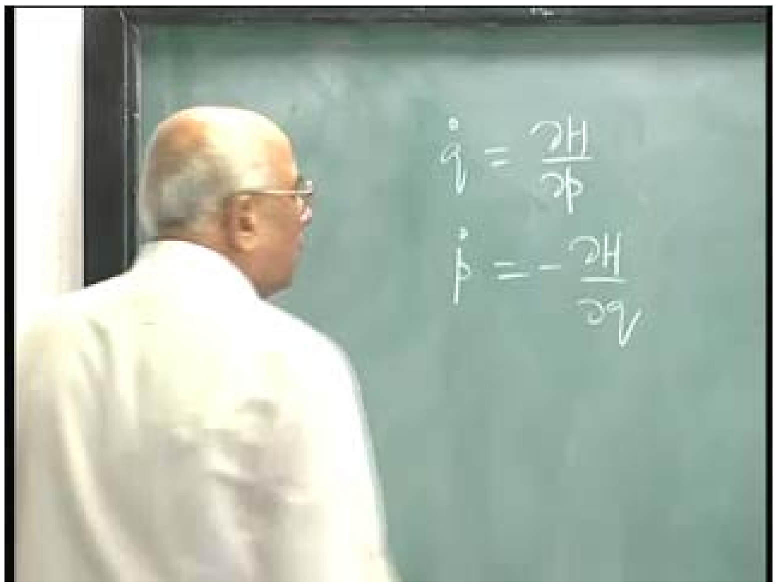


This question is there are two such equations, so in general you have a thing like q dot is delta H over delta P and P dot is minus delta H over delta q. And the question is how do you find the equation to the phase trajectory in n dimensional, 2n-dimensional phase space. Of course, you have to solve these equations, before you can do that and as I said that is not always easy to do. Numerically when integrate the set of equations, but then if it is not an integral system, then you cannot write explicit solutions in general. But if it is a one degree of freedom system, in other words, the phase space is two dimensional then there is just a phase plain and then of course, we know the equations of motion.

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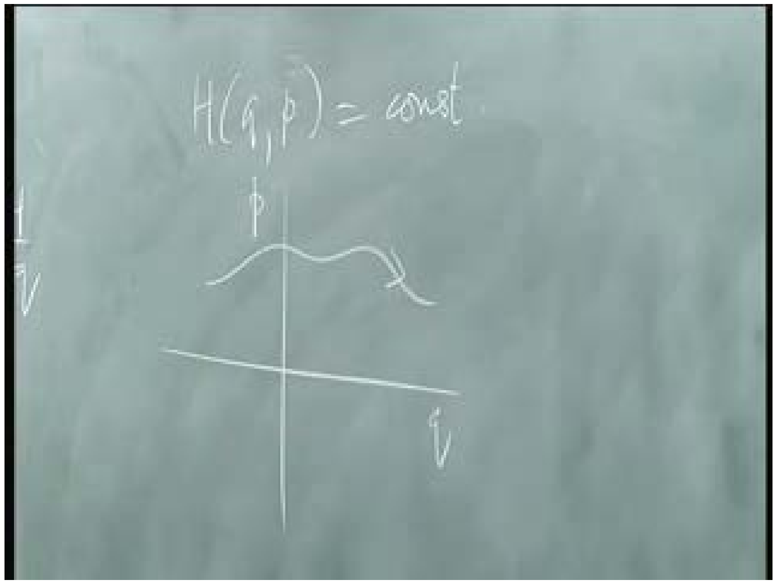


They are simply given by H of q, p equal to constant; that gives you the trajectory in the q p space, in the q p plane, some trajectory of this kind from a single constant of the motion. The moment you have more degrees of freedom then of course, you cannot do this you need more constants of the motion and then the question of intergrability comes in. But in a one degree of freedom system, finding the phase trajectory is quiet trivial for an autonomous Hamiltonian, is just H of q p equal to constant and that is it.

Now, of course, as soon as you have more degrees of freedom not only do you need N constants in N involution with each other; but then you should also be able to find the canonical transformation which would take you to action angle variables and then solve the problem which is not always easy to do. But what we are going to do now is to start from this point and ask what can happen in general?

Because, the number of solvable problems, the number of integrable problems is quite smaller, is quite restricted especially if the degrees of freedom are interacting with each other the decoupled of course, then it is become a becomes a trivial problem. So, what does one do in general? Well, the important thing is to ask, what is the qualitative behavior of phase trajectories in phase space? And now I would like to show you that there is in fact, a kind of hierarchy of randomness and this will take us on to chaos. The situation is as follows.

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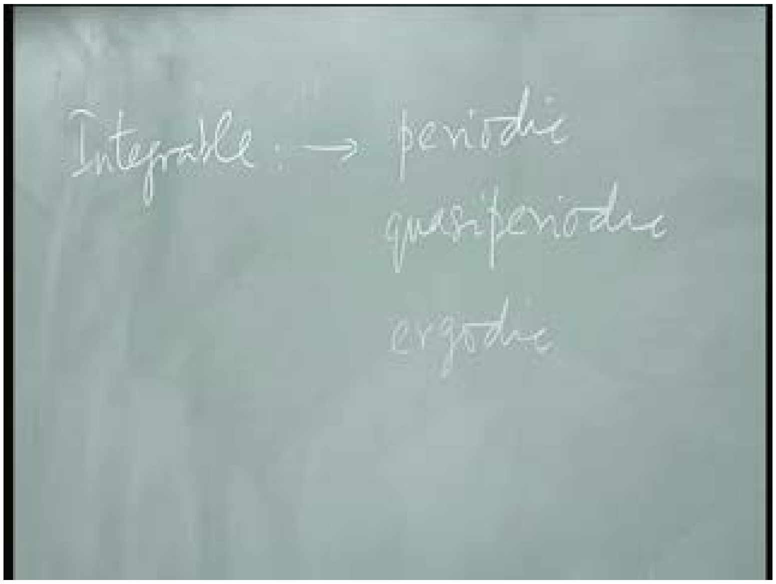


In the simplest instance you have integrable systems, this would mean of course, the motion is regular and you can find the constants of the motion; you can write the solutions down for all the dynamical variables explicitly as functions of time. And for arbitrarily long times you can predict what the feature is going to do given initial conditions.

This would be the small set of integrable systems, of which the harmonic oscillator, the Kepler problem and so on, central force problem these are all examples. Next in complexity, well among integrable systems itself they are restricting ourself to bounded motion not motion in infinite phase space, but bounded motion generally periodic motion of some kind. Then, suppress such motion is of course, such kinds of motion is periodic motion. A little more complicated then periodic motion is quasi periodic motion were you have several periods in the problem, but they are not commensurate with each other. So, the next increasing complexity is quasi periodic motion. The next step in this increasing order of complexity is when the motion is not even quasi periodic, but it is ergodi. In the sense that a set of initial conditions comes arbitrarily close to every point in phase space as a given in enough, given sufficient time.

So, the next one is ergodic motion, quasi periodic motion is ergodic on a torus of some kind we saw that the torus get is densely filled and so on, but you could have something which is simply ergodic on what, on what sort of phase space.

Well, if it is a Hamiltonian system then we know the Hamiltonian is a constant of the motion and if you do not have other constants of the motion which are analytic functions; then this system in general, the motion would be ergodic on the energy surface. So, if you have one or two or three constants of the motion then a arbitrary set of initial conditions, initial volume element in phase space could visit all regions of phase space given sufficient time.

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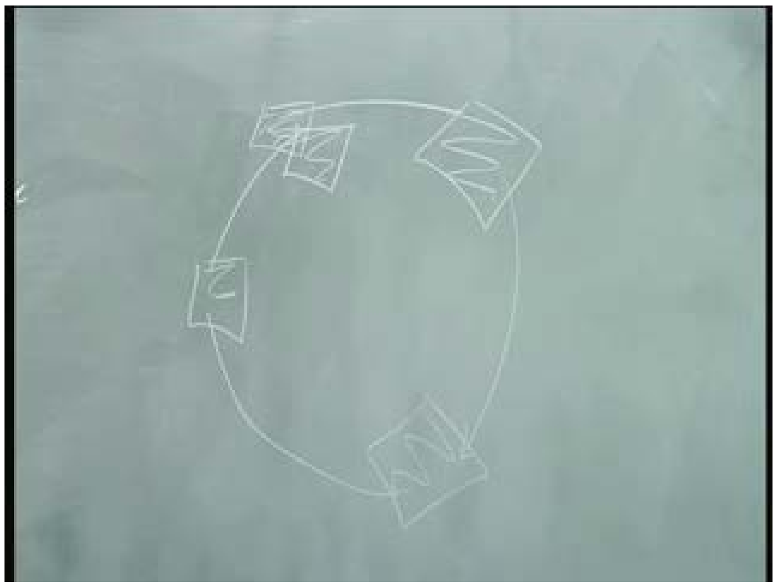

So, for example, if this just to illustrate, if the board is your phase space and I start with a set of initial conditions here. Each of these points goes into a trajectory, it is part of a trajectory, after some time this set of initial conditions could be there and after some time it could be here and so on and so on; does not have to come right back to it is starting point because then it would become periodic motion.

So, if the motion is such that an initial set of a set of initial conditions, an initial volume element in phase space visits the neighborhood of every point of some part of phase space given sufficient time; then we say the motion is ergodic on that part of phase space in general perhaps the surface of constant energy or something like that.

So, this is the next step in complexity ergodic it is not periodic, it is not quasi periodic it could be verse, it could be ergodic in this. But now you could ask well, if I put a drop of ink in a beaker of water and I now look at motion not in phase space, but in real physical space I look at this ink which is a die, is this ergodic in the sense that you put this in the beaker that is you have full space is this motion going to be ergodic in some sense?

You wait long enough and you realize that this piece of little drop of ink actually put is out tendrils become very complicated we assume no chemical reactions; and eventually it get is completely mixed up with the whole of the phase space, with the whole of the space that is available to it.

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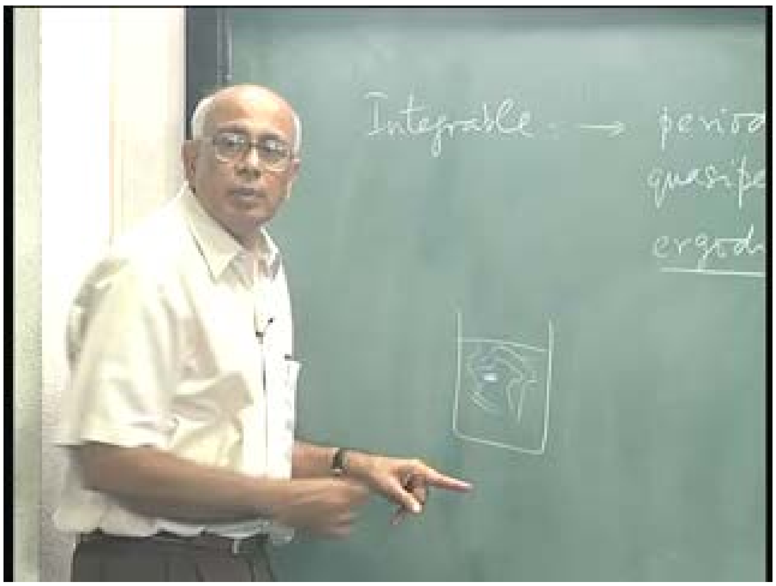


So, in that sense if I start with in this beaker of water, if I start with a drop of ink after some time is does that and then it does that and so on. So, you think potions of this initial ink, this initial volume element are to be found arbitrarily close to every point in the available space. And in this sense it is ergodic, because given enough time there are little pieces or bits of ink, little molecules which come from the initial volume element and arbitrarily close to every portion of this volume you can actually find some parts of the initial ink drop in that sense it is ergodic.

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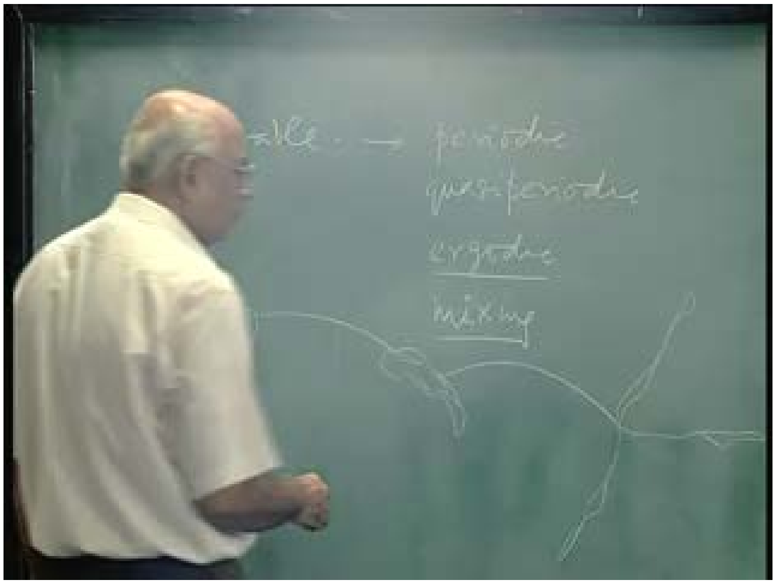

But it is verse then that what is actually happening is that as time goes on an initial volume element could become distorted. Even if the measure is preserved, even if the volume is preserved as in the case of Hamiltonian flows in the case of dissipative flows of course, the volume itself could shrink. But, the point is there it could, it could in this case actually have a distortion a very bad distortion in shape.

And a little later, it might do this even if this initial volume is equal to this final volume here, you can see that pieces of this initial volume are getting separated out. And this could go on and on and on and in fact, it could end up, such that pieces of this initial thing would be arbitrarily far apart; as far apart as the system size itself as time goes on. Such motion is of course, ergodic because little pieces of this go through everywhere, but it is more than that this thing as got mixed up with the rest of the phase space completely. So, what could you call such motion, mixing, it is called mixing. This is this kind of motion is called mixing and that is the next step.

We need to make this concept of mixing a little more precise and let me do that in a little formal way. All though, there are many, many definitions of mixing you have to ask mixing in what sense, how do I define distance between two points in phase space and so on. So, we can make this more precise strong mixing is the following.

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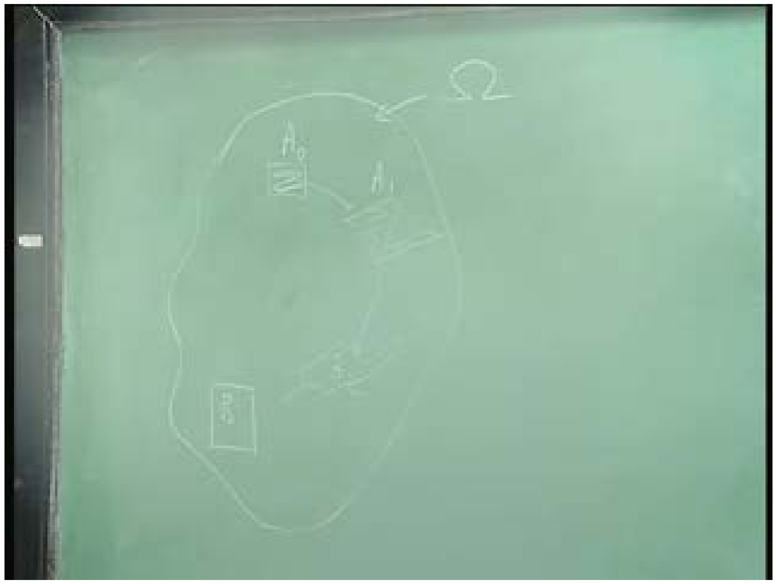


Suppose, we have a phase space and let us call the volume of this phase space omega this full phase space. Let us call that omega the full set of points in phase space and let us take a small initial set here, and let me call this A. So, the set of points it belong to this volume element I call A in full space I call omega all the points of the phase space I call omega. After some time this A if this is A 0 at time zero at the end of one time step this set of initial points as moved out and perhaps become something like this and let us call this A 1. So, all these points are moved into that and a little later perhaps they have become this and that is A 2 and so on.

Now, let me put a window here a reference volume and call it B if the whole thing is stirred up completely and points of this A after n time steps are completely mixed up then how much of A 0 would be sitting inside B. How much of this would be sitting here, exactly as much would sit there as is the initial concentration right.

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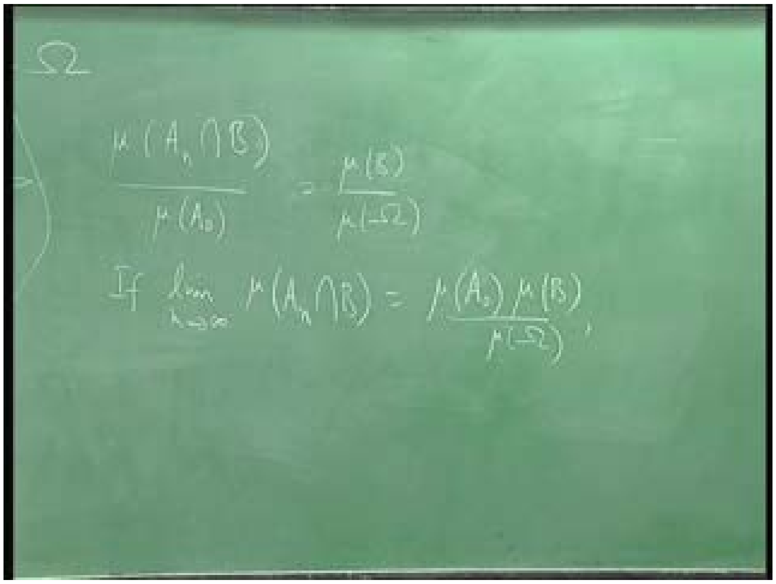

So, what would you say is the measure let us put measures here, I would like to know what is the measure of A inter section B; after n times steps A sub n is the set of points we started off as A 0 and I ask what is the measure of the inter section of the set A n with B how much of A A n is sitting inside inter second inter sects with B is common to B, what would that be.

Well, the inter section of that mu A 0 ratio of this inter section to how much you started with if this is equal to mu of B over mu of omega itself, it tell us you how big the window you have compared to the entire volume if this is true in the limit in which n goes to infinity, then I would say it is completely mixed totally. So, if limit n tends to infinity mu of A inter section B is equal to by mu I mean the volume or the measure whatever you would like to call it the reason I introduce a mu will become clear a little later, when we do statistical physics then.

Then the dynamics is set to be strongly mixing; the intuitive ideas is very clear, if I mix a little bit on the famous example is ink in water or. But I still rum and cola. So, you have a lot of cola and you put in a little bit of rum and you stirred it Farley then if the amount of cola with the amount of rum that you have in any reference volume is satisfies this ratio then you would say it is been completely mixed up.

So, in that sense strong mixing is a very familiar property in dynamical systems and it goes one step beyond ergodicity. I would like you to appreciate the fact that mixing implies ergodicity, but ergodicity does not imply mixing.

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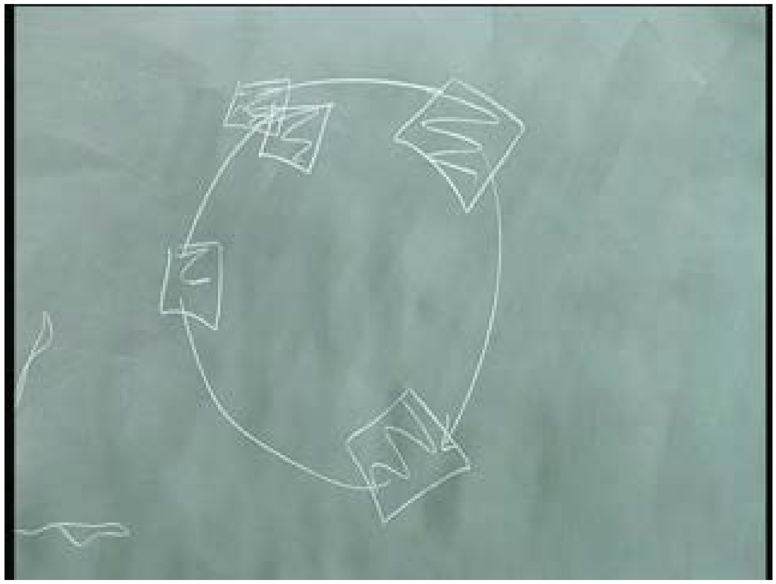

Because you could have ergodicity without distortion of the volume elements where as mixing implies certainly that it distorts badly. So, the implication is in one direction mixing implies ergodicity. The converse is not necessarily true; now when you take a liquid in stirred two liquids into each other and you are actually helping the mixing process.

And the next question to ask is, how fast does it mix, how rapidly does it mix? Because that is crucial, if it mixes. So, slowly that it takes the age of the universe to get Farley mixed up then it is preferred all practical purposes not mixing at all.

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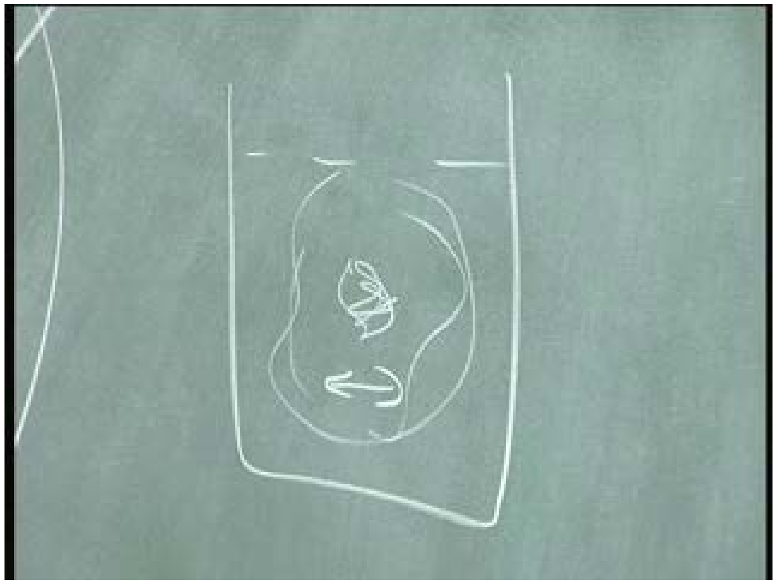

So, the question is how fast does it do it? If I take this familiar example of a drop of ink in water, how fast do you think it mixes.  So, I take a little piece of die here, and then of course, it will start putting all tendrils and by molecular diffusion it gradually con the concentration equalizes everywhere how fast do you think that is happening. It has to do with what is the rate at which this increases. In other words, what is the rate at which these points separate out.

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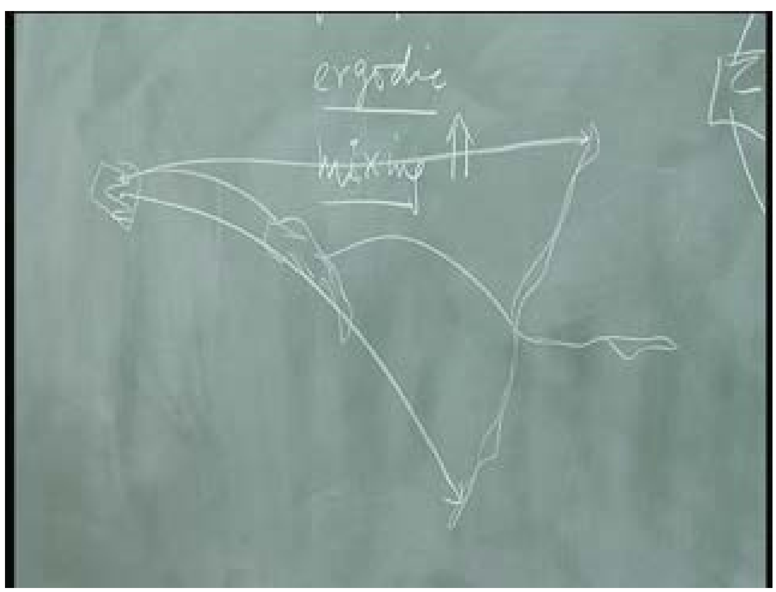

So, mind you a point we started here has ended up there and a point we started here might have ended up here and the question is how fast is this separating.

So, now we are getting quantitative we would like to know how fast is this happening and we would like to know the rate at which it is happening.Of course, it would depend on where you are what the dynamics is what is the system is and so on. But, what could you say is the case here in the case of ink.

So, I am asking on the average of course, you know that this tendril is highly irregular it is not a regular process at all. So, what is starts out as a nice little drop soon puts out all these filaments and so on. So, we are asking on the average how does it separate, how rapidly does it separate? And this happening due to diffusion.

It could take a long time you can see these beautiful patterns for a long long time. So, it is; obviously, a slow process and it is a complicated answer. If you heat this liquid and there are thermal currents then of course, convection currents would set it would be set up and that would mix things fairly fast you put a spoon in it and you stir it that makes it even faster.

But suppose you do not want to stir the spoon and the sugar in your coffee, but you would like to have this mix very, very slowly perhaps, because you would like to enjoy company for what a little longer depends on the company of course, now what would you do, how slow do you think you can make this, how fast do you think it is happening if it is entirely due to molecular diffusion

No thermal currents, no gravity, no advection, no forced convection very slow of course, how slow is very slow how does it go like a power of t this is what you should ask?

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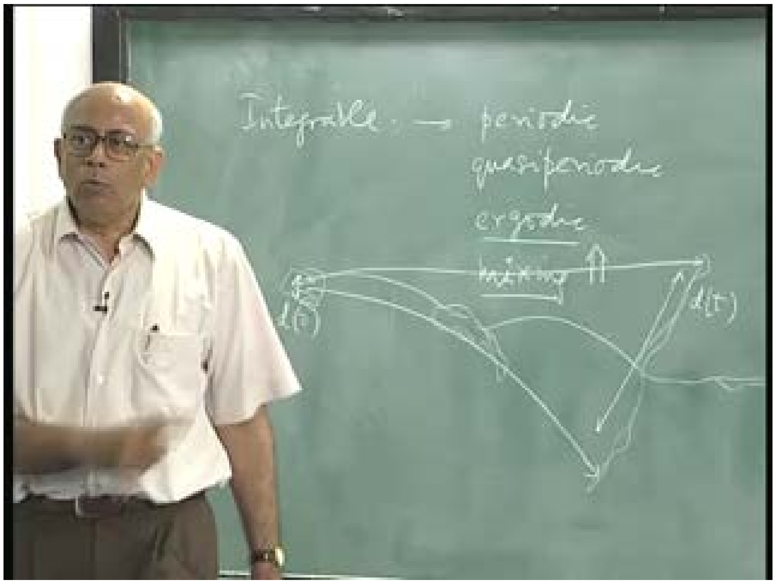

An initial distance a little distance here d at time 0 as compared to a distance d at time t the question is how fast does d of t increase with t we will make this very precise now, but how would you say it is happening in the case of diffusion. Well, diffusion is like random walk. So, it is like this molecule goes here and then it does purposelessly moves around and then it does this and so on and slowly things spread out. So, if you are doing a random walk, as oppose to a purposeful walk at constant speed then in time n n times steps. The distance I cover is proportional to n if I cover it at uniform velocity in a given direction. But if I mainly do a random walk in this fashion and I walk around like this and I retrace my steps I all directions then the mean n to n distance goes like square route of this time square route of the number of steps.

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So, this thing is mixing like d of t is increasing like t to the power of half times d of 0 that is very slow, very, very slow, you have to wait four times as long to go twice as far. So, it is very very slow on the other hand, you can increase the speed you can give it ballistic motion and that of course, means you are moving at constant speed and then it will became propositional to t you can make it super diffusive, super ballistic you can accelerate it in which case it will go like t squared and so on and so far.

So, you can start increasing thing, but the big surprise is generically typically, in phase space this mixing in most dynamical systems occurs exponentially fast; apart. Initially we have to make this very precise we make this concept precise. So, the separation is exponentially fast. So, typically it would this would not be the situation at all, typically this would go like e to the power lambda t d of 0 with a positive lambda, because it cannot keep doing that because if the phase space is bounded soon on a later things are going to come back. But the fact is the initial separation if it is exponentially fast we have what is called exponential sensitivity to initial conditions.

This means a small initial difference in the initial conditions can lead to an exponential exponentially growing separation any initial error therefore, would proper get exponentially fast. Any imprecision in specifying the initial condition would proper get exponentially fast and this is what chaos means this exponential sensitivity is very, very crucial this is really the characteristic, now how would you quantify it and we will do that.

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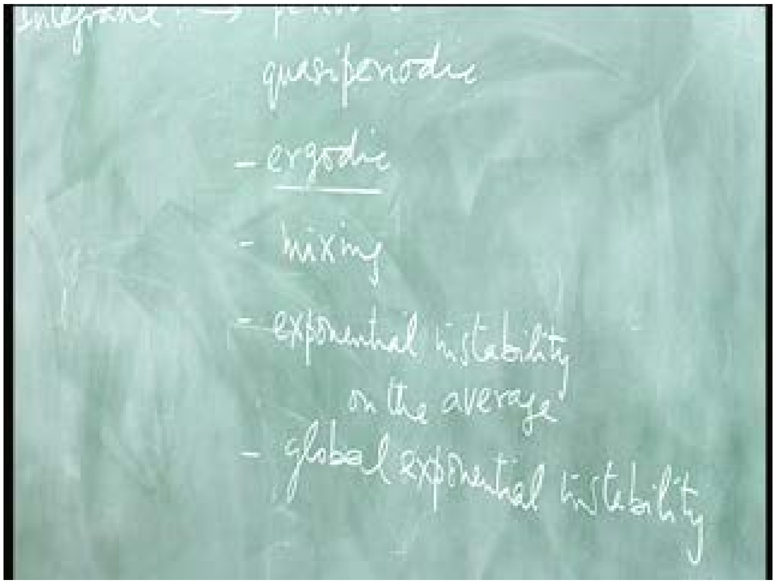


So, the next step after mixing is exponential instability I call it an instability, because really is things are moving out flowing away from each other exponential insten instability. On the average, by on the average, I mean they could be regions of phase space by you have such behavior, but there could be other regions of phase space, where you have regular behavior.

So, some initial conditions could remain on periodic trajectories, some could remain on quasi periodic trajectories, others could become part of this exponential sensitive region, where initial conditions would separate out exponentially fast. And then you could have much verse situations where almost all initial conditions would lead to this kind of separation.

So, the amount of regularity that you have in this sea of chaos is getting smaller and smaller. So, what is start out as islands of chaotic behavior in a sea of regularity gradually, as parameters change can become an island of islands of chaos in sea of regularity can become the other way about islands of regularity in a sea of chaos it happen. So, this is what this going to lead to chaos exponent global. Now, the definition of dynamical chaos a several definitions of it possible and there is no real unique absolutely unambiguous definition depends on what you would like to look at, but there is a reasonably agreed up on definition with some caveats which I am going to talk about and that is the definition I will use.

One thing we would like to understand immediately is that if you have an infinite phase space the idea of an exponential instability is not very strange. Here is a very simple example I have one dimensional phase space not even two just a line.

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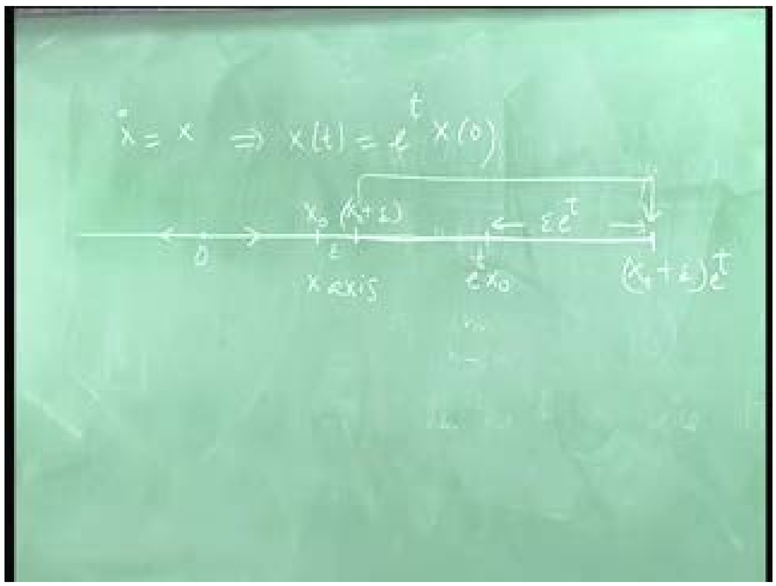


And I have this equation X dot equal to X on the real axis this displace exponential sensitivity in this sense, in the sense we have just mentioned because what is the solution to it, this implies x of t is e to the t times x of 0. So, on this x axis which is your phase space here is the point 0. And we know that in this case the point 0 is a critical point because that is why the x dot vanishes and it is an unstable critical point because it says any x of 0 to the right of this point flows away in that direction and anything to the left of it flows away in this direction. So, it is a repaller.

We also know that if you start with a point here and a point here. So, this is x 0 and this x 0 plus epsilon after time t this x 0 find itself here, which is e to the t times x 0. But this same fellow finds himself here which is x 0 plus epsilon e to the power t. So, the initial difference epsilon has now grown up to a difference which is epsilon e to the power t in that sense of course, you have exponential separation. But this is a very trivial statement. This problem is integrable completely integrable and this not what we mean by chaos.

This is happening because, the phase space is infinite because it is unbounded. But this is not what we are interested in, we are interested in situations where the phase space is bounded and you still have this exponential separation almost everywhere. And this is a little hard to conceive it is like having a separatist at every point, because we have seen lots of examples where if you have a hyperbolic point.

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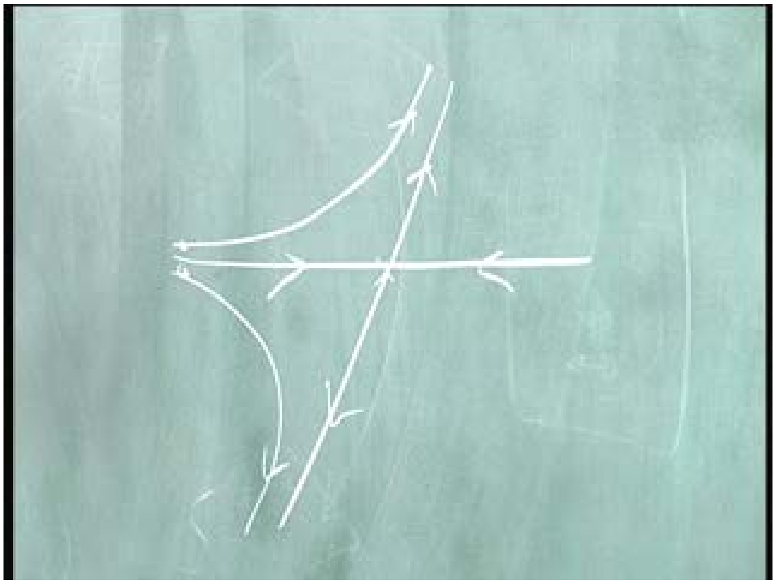

If this is a hyperbolic point and this is the stable manifold and that is the unstable manifold then we know that trajectories would look like this; and then you see on either side of the separate trix, if I start here or start here you can see that these two have very different futures. And in that sense the separation is enormous I mean it could even be exponentially fast, but the fact is this is not sufficient to produce chaos again. But something like this is going to happen, you have a separate trix in the middle, but now imagine a situation where it is happening almost everywhere what would that imply?

What would that imply, how could you possibly have separate is everywhere in phase space almost everywhere?

It is clear that if you have an unstable periodic orbit somewhere which is unstable. So, that things to one side of it move in one direction, the other side move out in another direction then this phenomenon can happen. That indeed if you have unstable and I am going to demonstrate this if you unstable periodic motion. So, that on that trajectory you have periodic motion on the periodic trajectory. But slightly to one side of it you have things flowing away and the other side of it you have things flowing away another direction then you can have this phenomenon of chaos.

So, I am going to write down the necessary conditions what looks like necessary conditions and then we will justify this using simple module. I also need to make precise the idea of exponential diversions which I will do in a short while. So, this brings us to when chaos can happen it is got many names it is called deterministic chaos. The reason being that the rules at you write down a deterministic there is no actual noise we have not put in any stocastical element or noise or anything explicitly. It is also called chaotic dynamics it is also called dynamical chaos and so on and so far.

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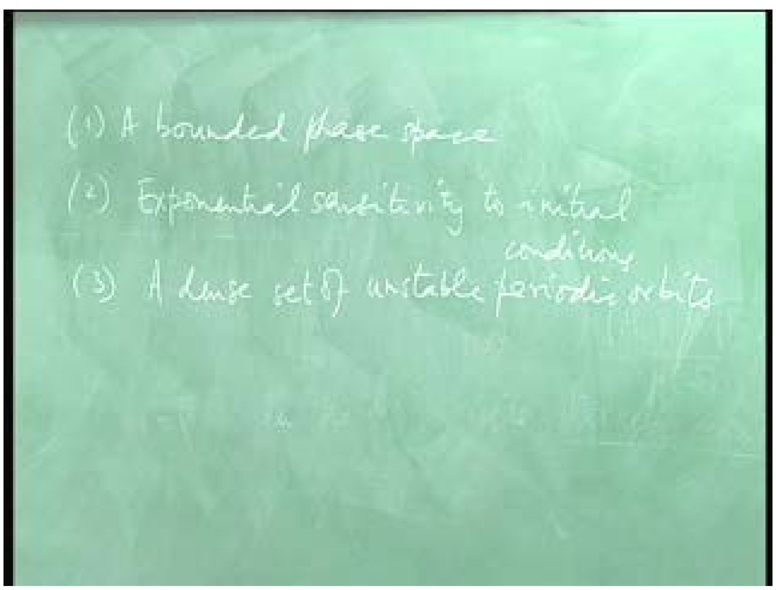


We will just call it chaos; it happens, when you have one a bounded phase space, two exponential sensitivity to initial conditions in a manner, which I will quantify in a second we are going to define a quantity like a lambda, I talked about here. And three this seems to be necessary for chaos the one that I am going to write down now, there is no rigorous proof where it is always the case. But I do not know of any instants when it is not there. So, in that sense it seems to be absolutely essential a dense, set of unstable, of unstable periodic orbits embedded in the phase space. Now, once you have these three conditions met then the system displays chaotic behavior.

So, let us take this systematically first what do I mean by exponential sensitivity to initial conditions well let us do that let me define that here, or I should mention and I would come back to this beyond this global exponential sensitivity you could actually have dynamics which can be put into to one to one correspondence with random, with random dynamics.

So, the dynamics even though it is determined by even though it is specified by deterministic law could turn out to be such that the outcome is like tossing a coin raise random as coin toss also called the Bernoulli trial. So, we will come to that and I will show you how very simple modules can generate that kind of dynamics. Notice also that I have not made any distinction between I have not said anything about whether this is Hamiltonian dynamics or dissipative dynamics conservative or dissipative it is completely general as it stands here. Hamiltonian systems also display chaos. But the route to chaos is very specific dissipative systems display routes to chaos much more reach structure of routes to chaos many more routes are possible and a lot of study has been done on this.

But Hamiltonian chaos is very complex the way it originates it is fairly complex and we will try to look at some aspects of it today. So, first this business of exponential sensitivity what do I mean by that well.

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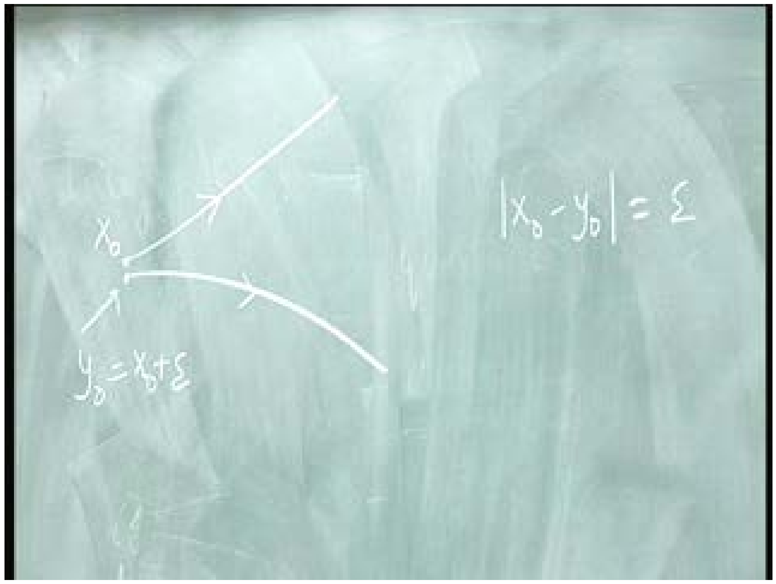


Suppose here is phase trajectory and here is another phase trajectory and let us suppose that initially you start with a point x 0 and another point y 0 which is x 0 plus epsilon and x 0 goes off in this fashion and y 0 goes of in that fashion. I would like to ask how is this separation increasing as a function of time my suspicion is that it increase exponentially fast and if so, I would like to know, what the exponent is, this is what we would like to do.

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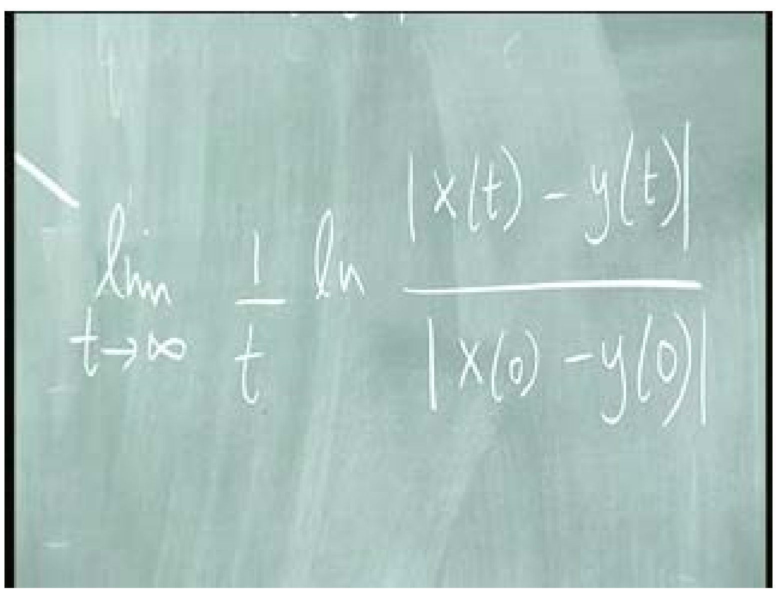

If that is a suspicion then let us call we know that x 0 minus y 0 equal to epsilon this is the difference between these two guys. I ask what is  x at time t minus y at time t  if this goes like e to the power lambda t times the initial distance then it is clear that if I took this quantity and divided by  x of 0 minus y of 0  they should go like e to the power lambda t right I would like to extract this lambda.

So, I take a log, then I get a lambda t I would like to extract the lambda part. So, I do a one over t means and I take the limit as t tends to infinity.

What do you think will happen to this, what do you think this quantity will be if the phase space is bounded? You agree this is going to detect then e to the lambda t I wait at long enough. It is gone off what do you think is going to happen if the phase space is bounded.

It will up at 0, the reason is x of t minus y of t can never become bigger than capital L the size of the system and then you have a one over t down here. So, the moment you take this limit you are going to end up with a 0. How would you avoid that 0, how would you avoid it? I would like to find out by the way what is this whole quantity going to depend on? It depends on the dynamics it depends on how I specify x as a function of time right it depends on the rules of the game the dynamical evolution equations what else does it depend on.

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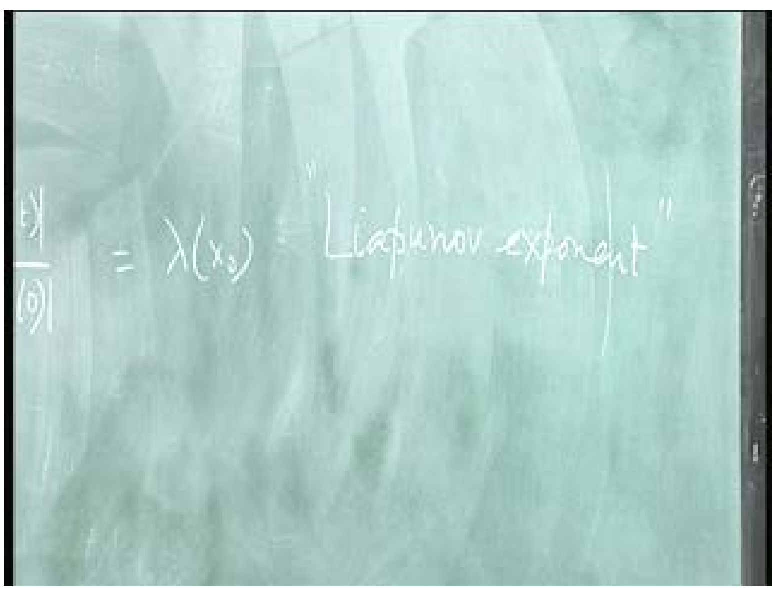

On the initial condition it depends on where x 0 is it is certainly depends on where x 0 is and I would like to have a property of x 0. So, I would like to poll this whole thing lambda of x 0 it exist, and this thing is call the Liapunov exponent and it depends on x 0, but now the way I have written it it also depends on y 0.

And I have to get rid of that depends on y 0. So, what should I do?

[FL]

And then what should I do, delta x it is a epsilon I put a epsilon already. So, what should I do?

Epsilon tends to 0 epsilon should tend to 0. So, I should take a limit epsilon tends to 0.

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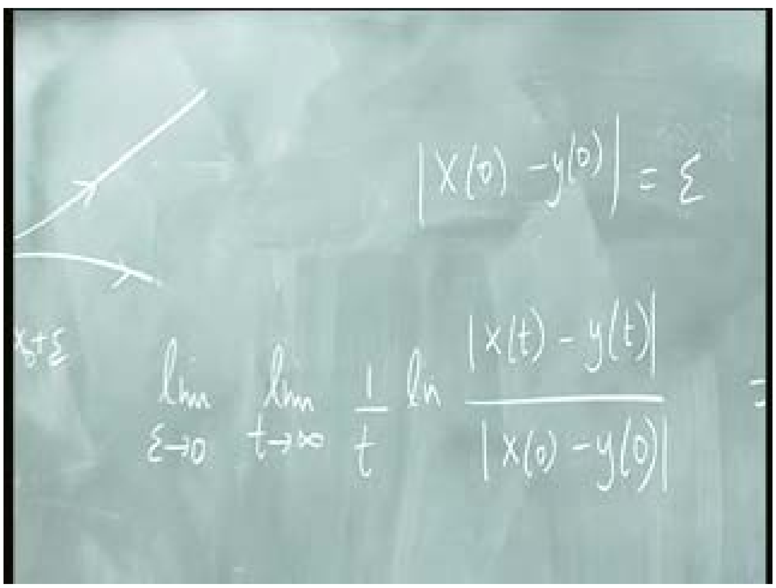


Where x (0) minus y (0) equal to epsilon and I would like to let epsilon tends to 0. So, do you think this would do the trick?

So, what should I do, what do you think will be the answer here?

Will be 0. So, you get a non trivial answer what should I do?

Change the order of limits, change the order of limits it is a crucial. So, change the order of limits.

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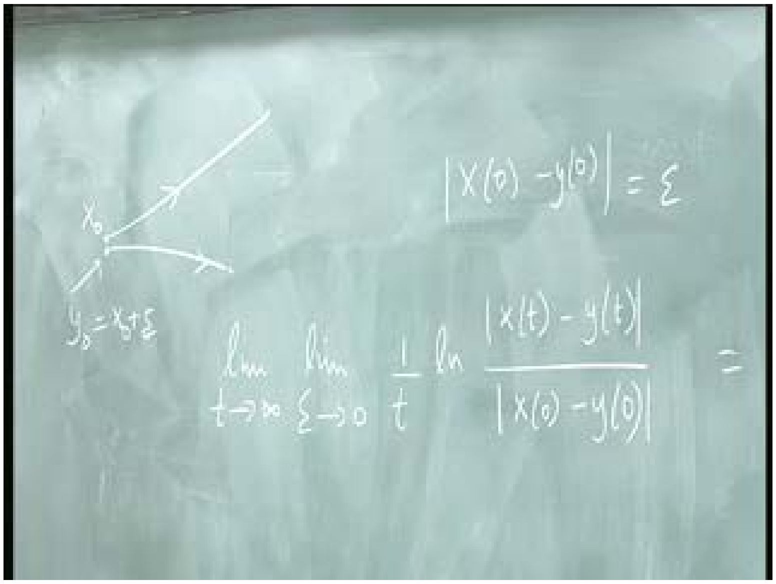


So, limit epsilon goes to 0, limit t tends to infinite. Indeed do the trick that will indeed do the trick this is a little bit like saying if I have a to the power b, and I am going to let a and b both go to 0 from the positive side the answer depends on which one you like go to 0 first if you like b go to the 0 first the answer depends on which one, you let go to 0 first if you let b go to 0, first the answer is 1 and if you let a go to 0 first then the answer is 0 right.

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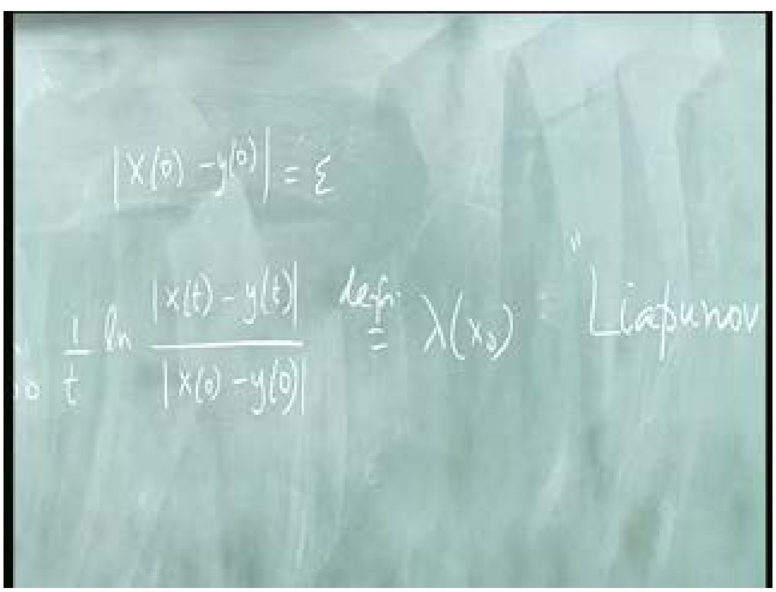

So, the order of limits maters and of course, mathematics will not tell you what the physical situation is it will not going to tell you which order you should choose is the physics or whatever you want derive whatever you want to understand. That is going to detect what order you prescribe and this is the definition, this is the definition of the Liapunov exponent.

I would first like to find out what is happening on that, let me explain this a little bit we would like to find out what is happening to this initial point. So, this initial point is on some trajectory right.

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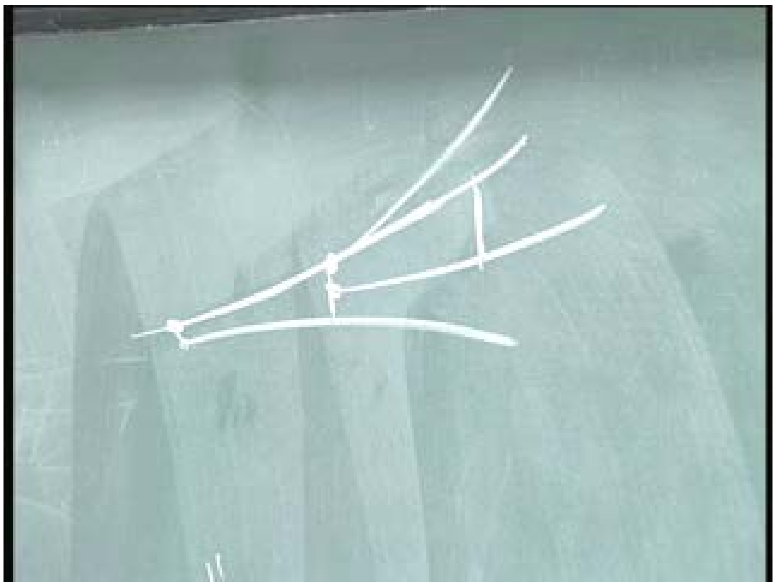

So, a small initial error here, this trajectory would move like that and this would move of in this fashion what I would like to find out eventually is, what is the Liapunov exponent on this trajectory. So, once this initial separation becomes this much I would like to reset the clock here, and then this guys goes of like that this fellow goes of like goes off in this fashion and I would like to find out what the separation is etcetera.

So, each time I am going to actually reset back to the trajectory, that I am interested in or the region of phase space I am interested in and find a property of a region or a particular point. Now of course, if you do this for a finite amount of time you do not get a limit at all. So, it is not going to give you, the definition of e to the power lambda t. The statement I am making is precisely in the same spirit in which I said that the flow is like e to the power lambda t, when you linearise a flow it is e to the lambda t depends on the linear matrix and depends on the eigen values whether they have got positive real parts or negative real parts will determine the stability, but these are asymptotic statements of course, after some time the system is no longer linear it goes out, but the whole idea is to take a limit inside a region always.

I am not saying this very clearly, but I will give you examples as we go along I realize it is little hazy at the moment, but let me give an example as, we go along you need this limit, but you have to be careful that the limit this limit is taken before that limit because in the other direction just gives you 0 and does not tell you anything about separation at all.

Now, let us ask suppose this separation is like the square route of t like it was in the case of the diffusive motion we looked at, then what would the Liapunov exponent be what would it be suppose it goes like square route of t. So, this over this goes like t to the power half and then of course, you realize that you get half outside and you get one over t log t and what happens to that it goes to 0 because log does not increase as fast as a power t.

Suppose it goes like t to the power a million what happens then? It is still the same, it is still the same. So, you see this test this this definition will distinguish between things which have exponential sensitivity as oppose to any kind of power law sensitivity it does not matter at all. No matter how high the power you will still not find a positively Liapunov exponent. On the other hand the moment is exponential you find this.

Now, I could ask the following question suppose the separation increase like t squared e to the power t squared what would happen, what would happen to this?

It would become infinite. So, this is super exponential it is faster than exponential, but you see we have seen in our experience of dynamical systems that generically generically, it is always unless you have singular points strange kinds of behavior generically, it is always exponential. We saw that when we linearise the matrix in a critical points that you always had exponential behavior, because you have the Jacobian matrix and then you expand in solutions or exponentials.

So, the typical case is in fact, exponential always the only question is it zero or nonzero, if it is is it positive is it negative, if it is negative of course, you have shrinkage no it is no growth at all is it 0 in which case, do not have chaos or is it positive in which case you have chaos. Or the next question you should ask is look all this is nice in one dimension, but really I have an n dimensional phase space in general then the question is in which direction should I choose the epsilon.

And there is no reason why if I start here, and I start with the neighboring point here, this separation can grow at one rate, but that separation can grow at another rate. Therefore, in n dimensional phase space how many possible Liapunov exponent could you have n of them corresponding to n direction. So, really you have a whole Liapunov spectrum and if I define the distance between two points by the Pythagoras theorem or something like that, then what I am getting by this definition will be the maximal Liapunov exponent; obviously, if some directions come together and some directions grow it will only the growth that will matter right.

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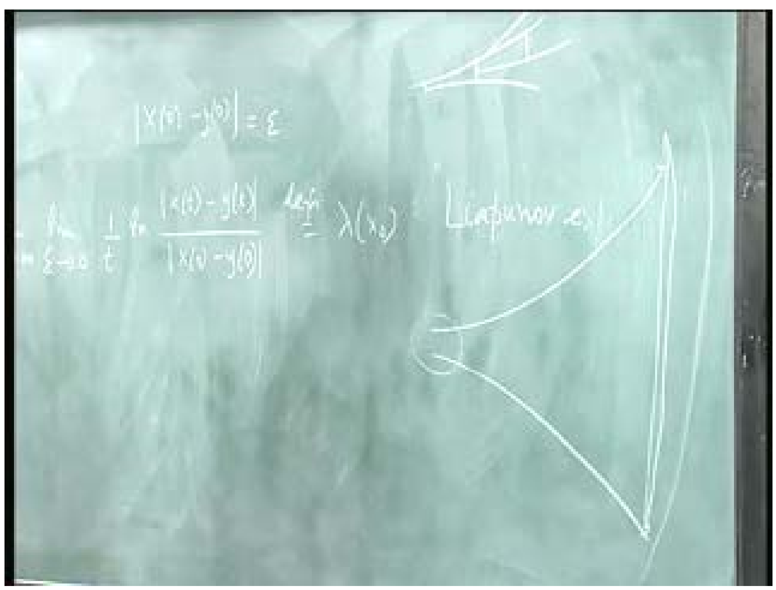

Because take this suppose you start with initial points in this circle and these points, let us say after some time become like this we occupy may be the volume is a same or even the volume is shrinks you do not care, they occupy space like that then this point could go there, this point could come here.

But you see any separation in this direction has shank to practically nothing, but a separation in this direction has increase to this and when you compute this distance here between these two points it is only the increase dimension that will dominate and the other direction will not dominate at all. So, you would really find the maximal Liapunov exponent and this also tells, you that if there is shrinkage of phase space volume if there is dissipation such that in area becomes a line after in enough time or a higher dimensional volume becomes lower dimensional object.

It still does not matter because you still lossed information. So, this error has grown exponentially even though the error in this direction has shrank may be even to 0.It does not matter, but the initial error has grown and this is what the Liapunov exponent is guaranty to do. So, having set this, we need know to have some systematic way of understanding this whole business and we need a simple model like we had the harmonic oscillator in the case of integrable systems, we need a harmonic oscillator for chaos unfortunately unfortunately, there is no such a simple model.

And the reason is simple, the reason is if you have differentiable dynamics if you have dynamics described by the differentially equations set of differentially equations then in one dimension there is nothing to solve you have x dot equal to f of x you can always solve this in principle.

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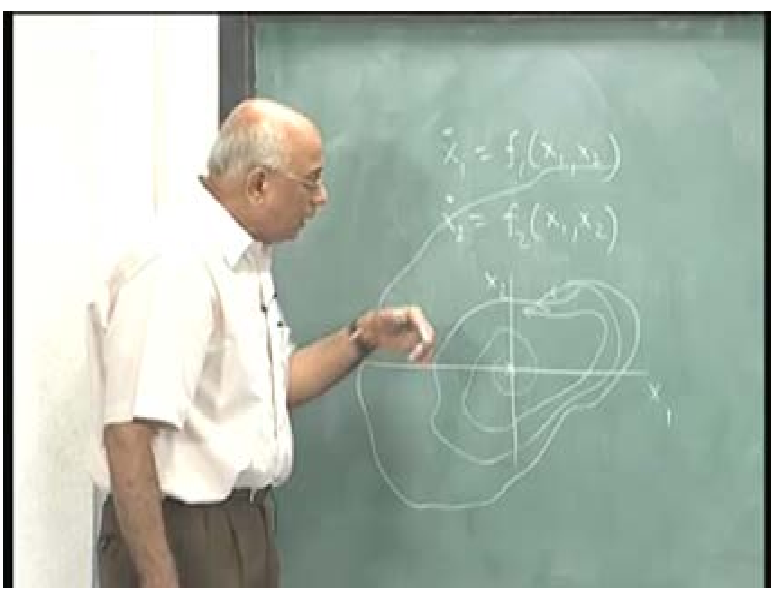

In two dimensions we have non-linear systems which would look like, x 1 dot is some f 1 of x 1 x 2 and x 2 dot is f 2 of x 1 x 2 and your phase space is a plane x 1 x 2. Then you cannot have chaos in this problem at all, because the only kinds of critical points you can have here, would be where the right hand side vanishes and they would be isolated critical points and we have classified the simple critical points.

There is one more possibility which we did not look at and that is what is called a limits cycle it could. So, happen and it does not happen in Hamiltonian systems in dissipative systems it could. So, happen that you have an isolated periodic trajectory.

A single isolated periodic trajectory such that, you may have a trajectory like this which is periodic such that, there could be for example, an unstable spiral point here and things could go and flow into this as time permits as time increases. So, this is a stable periodic trajectory. And wherever you start it is possible you may flow into this given enough time then an isolated periodic trajectory of this kind is called a limit cycle.

And it is a hard problem to discover limit cycle. We generally have to do this numerically. I will give some simple examples, where you can actually work it out explicitly this is possible this kind of attractor is possible and it is the next in the hirearchy of attractors the first one we had was attracting fix points or critical points, and then next one is going to be limit cycles, stable limit cycles and after that there is just nothing in a plane that is it. But if you have three or more variables in three dimensional space, it is possible for the phase trajectory to wonder round forever and ever, in a region like a ball of wool never intersecting itself, but not periodic not living that region either. In two dimensions that is not possible at all, you cannot have big triangles.

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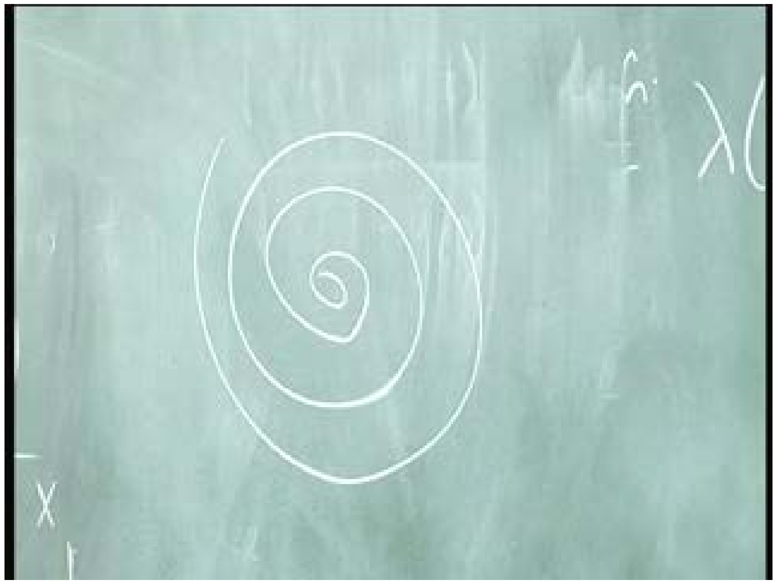

Because anything that is come in if it cannot intersect itself it has fall in or else become part of a periodic trajectory. But in three dimensions you can see that the trajectory can go around like a ball of wool. So, it is exceedingly complicated very strange peculiar object called a strange attractor does not intersect itself, not periodic motion, not quasi periodic motion. But does not live that reason either and on in that region called the attractor. To initial points where did you start with would exponentially separate.

So, this is possible when you have third variable here, but then we do not have any very simple analytic solutions, such systems will not have analytic solutions. And therefore, accept numerically I cannot illustrate this to you and all of you by now have seen this famous icon of chaotic dynamics the butterfly the Lawrence butterfly which is available on all kinds of websites where if you run this for a long enough time you see the solution do this differential equation just moves around in completely chaotic manner in a very, very strange factor object called a strange attractor.

But there is no way you can write down explicit solutions. So, in that sense it is difficult to understand accept numerical what we would like to do is to understand it in an analytical fashion at least a semi analytic fashion and here is where discrete time dynamics comes in it terns out. That while in the case of differential equations you need at least three variables couple non-linearly they must be non-linear in order to have chaos possible at all.

If the dynamics is not differential dynamics namely it is not differential equations, but difference equations then even one variable is sufficient.

So, this is the big advantage you have you have simple modules possible with just one variable a scalar variable; which you can write down generally a variable are number between 0 and 1, such that there is a prescribed rule by which this number changes from time step to time step, but in discrete time steps and then I can produce chaos. In such a objects are called maps. So, let me write that down.

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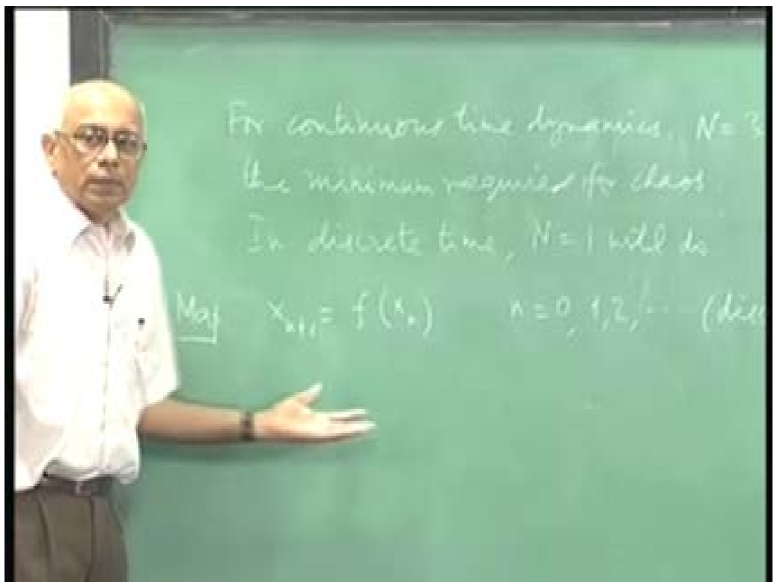


For continuous time is the minimum and equal to 1 will do. And instead of calling them differential equations we call them difference equations and just as we call differential equations flows difference equations are called maps. So, let me focus on maps from now and I have single a single variable and I variable I call x a real number; and what is the map? It says x sub n equal to some function x sub n plus 1 x n. So, this is called a one dimensional map.

That is my rule, that is my dynamics, because it is not real life it is just a very simple module is what carry catcher of what happens, but in a sense you can actually map very physical problems to it for instants, the insect population, each year or each season would depend on the population on the previous season in a certain non-linear fashion and that would be an example of this kind of map. In fact, such maps of first studied in ecological in the ecological context and they realized.

In particular several people did this the history of chaos is very, very interesting Lawrence E.N. Lawrence I should write this.

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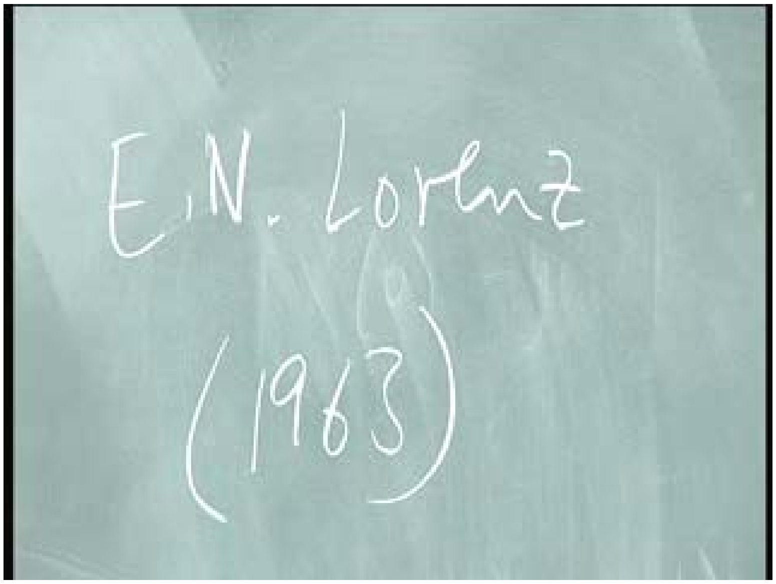


So, we will assume this a variable if this thing is real and let us ask what happens if I look at the very simplest of maps a linear map.

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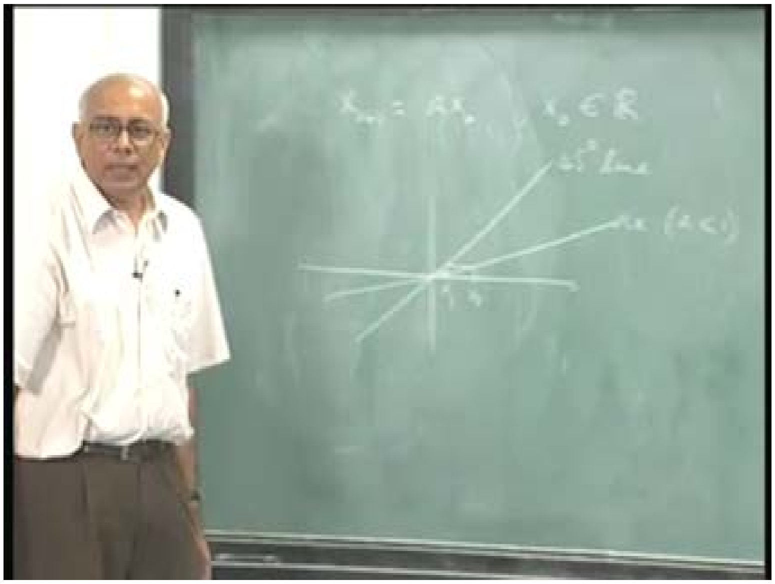

So, let us put X n plus 1 equal to a X n, where is a real number positive number say and I specify x 0 is an element of the real numbers. For what what happens to this map? So, on the x axis I stop with some point x 0, and then I find out what is x 1 and then I find what x 2 and so on, how do you do this, solution is alter trivial write I mean x 1 equal to a times x 0 x 2 is equal to a squared x 0 and so on and so for. So, how would you do this numerical to this pie this method of successive approximation is not it. So, what you do in general is plot the forty-five degree line and you plot this function y equal to a x. Let us suppose, this slope of this function is less than 1. So, this is and I start with an x 0 here, then graphically how do I find x 1; this is called the method of successive approximations right.

So, what you do is go to this point, and then go horizontally to this forty-five degree line a call that x 1 and then you go to this point here, call that x 2 and so on. What is going to happen wherever you start, you going to flow into this origin right. What could you call this origin? It is not a critical point this no differential involve here, what would you call it? What would you call a point for which it terms out that x n plus 1 is equal to f of x and itself is equal to x n itself?

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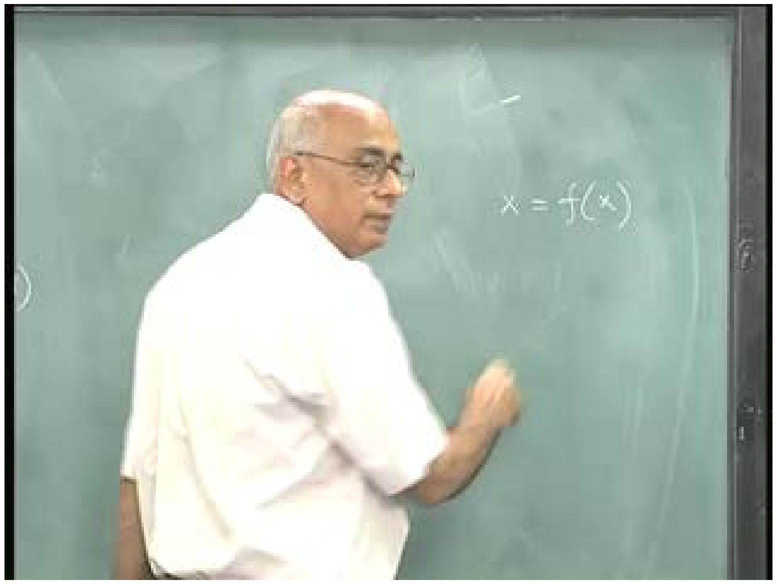


So, this mean x is equal to f of x. These are the fix points right these points do not change, because if you under iteration under this map it does not change at all they called the fix points. So, for maps fix points would plays the same role as critical points played in the case of differentiable dynamics. So, this linear map y equal to a x the fix point is solution of x is equal to a x which is x is equal to 0 is a stable or unstable.

It is of course, stable you can see that it thing comes in by started here.

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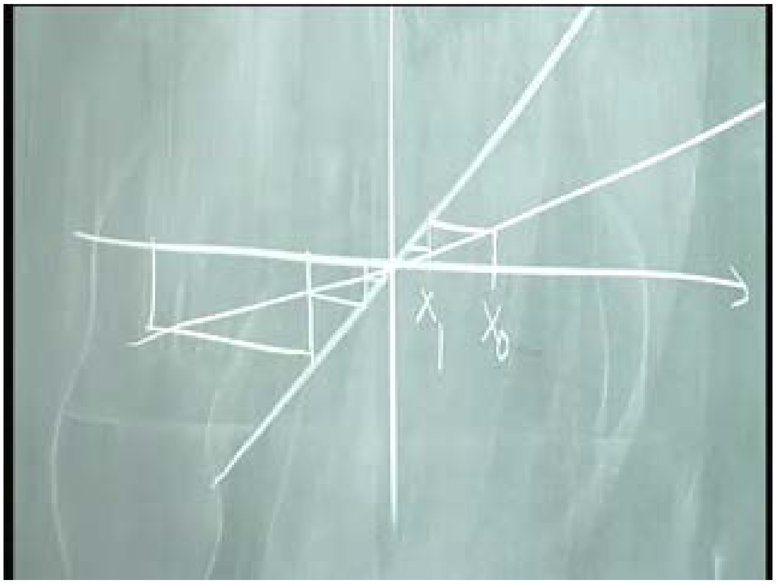


For example, the next step at go here then at go here and then go here and flow into this point here.

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So, since I flow into the into this points form both sides x is equal to 0 is a stable fixed point. Is it true for all values of a? If a is greater than 1 is just the opposite what you think is going happen?

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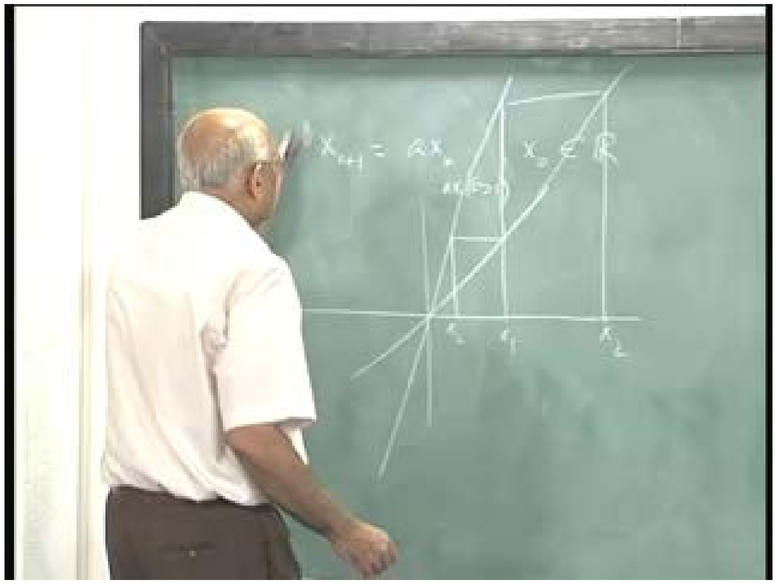


You flow away from it right. So, it is clear if it had a greater than 1 you just the opposite and a greater than 1 would think something like this. If I start with an x 0 then I go to this point call that x 1 this ix x 1 then I go to this point. So, this is x 0 this x 1 and then I go of here to x 2 and. So, it is clear away from this critical point.

What happens if a is exactly equal to 1? Then it is says map is x is equal to x that is the identity map and that is a d generate case because, an every point is fix point wherever you are do not move at all it is a completely d generate case. So, it is trivial d generate case and it is clear that everything is going to depend on what this f of x axis in the vicinity of a fix point. Let us look at a like slightly more complicated case.

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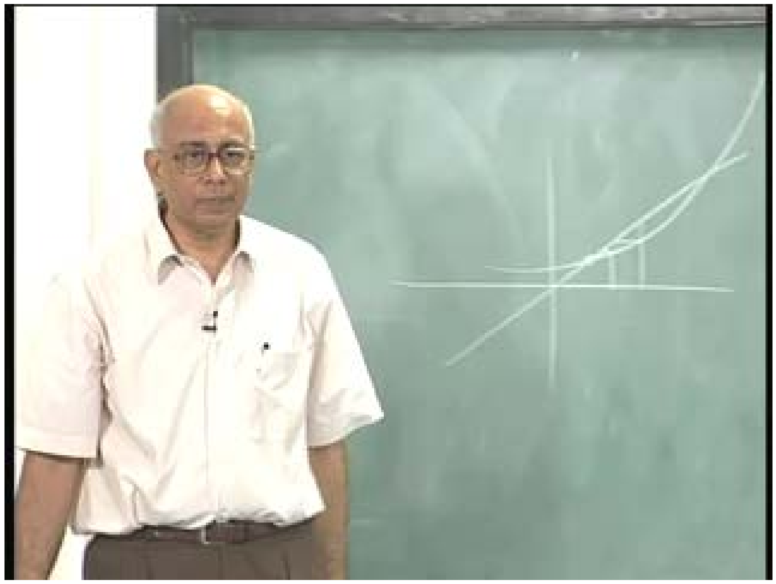

What happens if I have map it is looks like this, say that is my map, what is going to happen? The two fix points there as one here and as one here. So, if I start at any point here I go here find the solution and I flow into this. So, wherever I start if start here I find this and I meant to this if I start arbitrarily close to this I am going to flow away from it, and if I start arbitrarily close to this I am going to flow into it any region any point in between these two places.

These two fix point by start I am going to flow into this this would therefore, be stable critical fix point and that would be unstable fix point and off course you can see that they must alternate. And it is also clear that this is unstable, because is slope at this point is bigger then 1 and this is stable, because is slope of the map of that this point is less than 1. So, that is the simple criterion for whether a fix point is stable or unstable.

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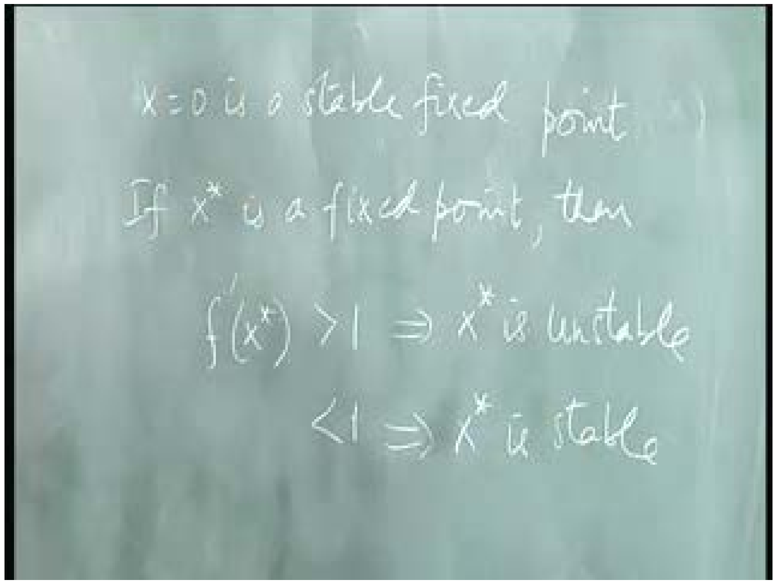


If x star is a fix point, then f prime of x star greater than 1 implies x star is unstable less than 1 implies x star stable by there is nothing with says that the slope cannot be negative that certainly possible, because you can easily see, the map where a function like this then two it would flow away if the slope of greater than 1 on the other hand if the map is like it would flow into it.

So, it is not f prime of x star that is important, but the magnitude of it is imported. So, you need this, what happens of the magnitude is exactly is equal to 1, what would happen?

If it is plus 1 would be same, but if it is minus 1, it will go in loop around it. So, all those of you have studied the methods of successive approximations which is all all I am doing here because what I am doing an each time trying to find the roots of the equation x is equal to f of x by taking a trial solution finding f of x calling it the new x, find f of x call it the new x and keep doing this to see if I conversion not and whenever does is stable fix point it is going to it converge to it provided you start in the right base in of attraction of this point.

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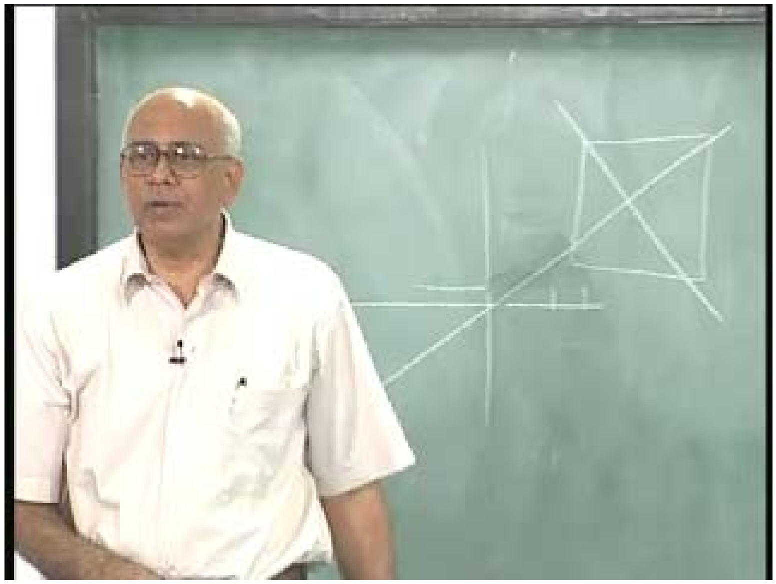

So, you could have for example, if this where slope minus 1 exactly that of kind then if you start at some point you would go here, you would go here, you would go here, you would go back. And you keep doing this loop here if the slope is exactly minus 1. So, it need a go in nor go out in that case and such a thing is called in indifferent fixed point a marginal fix point.

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What would happen if you had a map which did this at some point there is a tangency, what would go on now? That is a fix point because that is where the map intersect the 45 degree by sector, but now the slope is exactly 1at that point would you call this stable or unstable.

What would you call it? Well, it depends if I start here form the left from below the slope is less than 1 and from above this greater then 1. So, it is clear that if you start somewhere here you just flow of but, if you start here you going to flow in. So, you have stability from one side and instability from the other. So, it is really unstable this lot of thing is called marginal fix point or indifferent fix point.

And what can happen even better is a following and this happens way of quite often if you had thing like this a fix point is missed out there and let us say, the slope is really bigger then 1 else where and then it is chaotic in some fashion. What can happen now is it that you find yourself in the vicinity of this region then once you are here you do this and then you flow of and you can do this for an arbitrarily long time if you closer and closer to that point which means for long time you think that the system is regular.

But it really is not it is chaotic it moves off and then if the dynamics such that map brings back to the vicinity of this point of this is tunnel region. Then you would again spend a long time imagining that the system is regular and this called intermittency. It is really chaotic not periodic, but there apparently long periods or burs of regularity followed by burs of irregularity and so on. Re classic system this this is only one root intermittency the many other this is system mathematical module the dripping of a for set the leaky tap very often intermittent.

Because we watch it long enough you see the drops building and then in that falls very complex problem it is classic chaotic problem because the surface tension, the gravity, the viscosity everything plays a role and that falls and you think it is falling in regularly way. But actually it is not falling regularly it may fall regularly for an hour and after that you find is several drops at the same time then very slope again very periodic and so on. So, this really goes intermittency dispensation.

Now, these simple modules can actually module we use to mimic much more complicated situations and what I am going to do next time tomorrow. That is I am going to start with a simple one-dimensional map and show you how you have full bloom chaos in this problem. Then we look at system which has whole root to period by doubling root to chaos. So, it will give us some handle on how to compute they have to exponents and  we do that tomorrow.