Classical Physics
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Lecture No. # 8

We will begin today with the problem of charged particle in an electromagnetic field. But before I do that, I would like to make sure that we have clear, our ideas about basic electricity and magnetism. So I am going to start with a bit of lengthy digression fifteen minutes to start with on basic electricity and magnetism. It might go into fifty minutes, but it depends on what you already know and what you do not.

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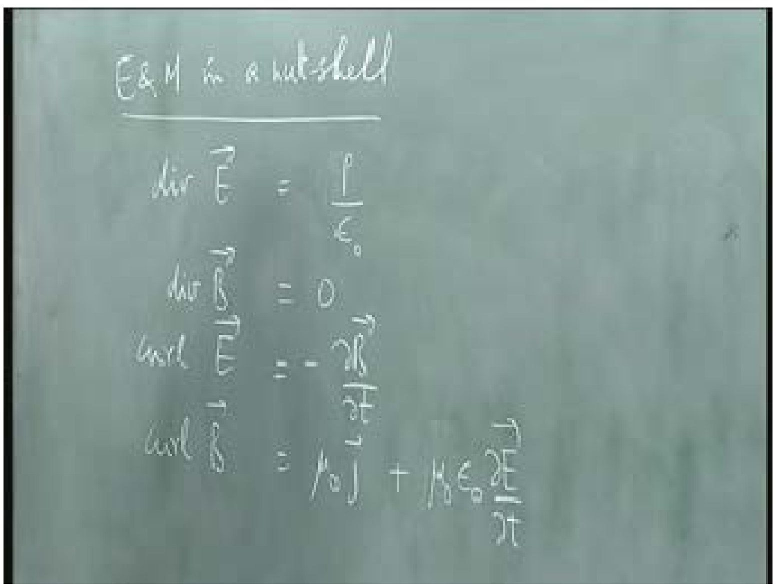

So let us start with the basic equations of electromagnetism. And here is E and M in a nutshell. Let us do that first; E and M after a lot of experimentation, it turns out that all of electricity and magnetism at the classical level can actually be compressed into four equations, the four Maxwell equations. So let us write them down and then let us see what we can do from them. So I start of by saying the first equation is, the divergence of the electric field. This is the charge density. I will give standard international units so it is rho over epsilon naught. The second equation is the divergence of B that is equal to zero. The curl of E, this is equal to minus delta B over delta t; and the curl of B, which is mu naught j plus mu naught epsilon naught delta E over delta t. These are the four Maxwell equations in free space. I am going to only deal with the free space case. As you can see they are two equations for the divergence of the electric and magnetic fields and two equations for curl of these two electric fields.

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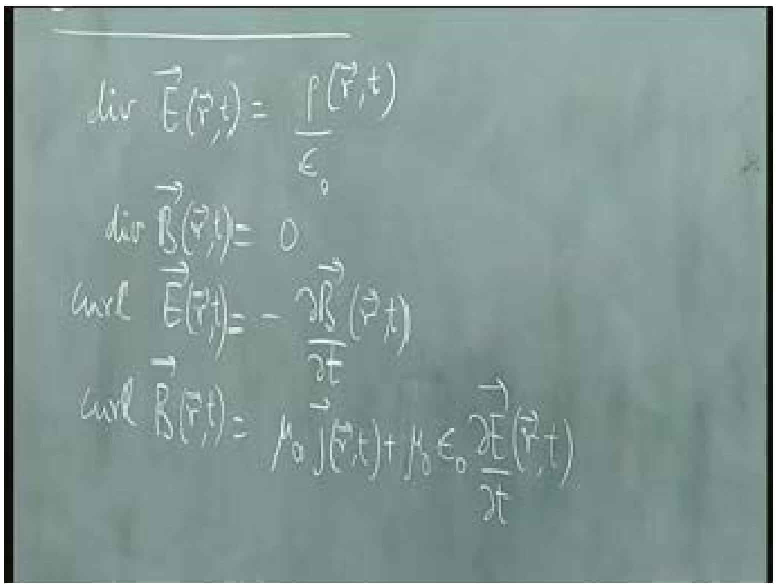

Now the first question, I am going to do this in a sarcastic method. So speak up, give loudly your answers, loud and clear. Is this equation valid, even if this field is time dependent?. Yes indeed, it is. So these are actually field's space and time. So this equation is valid for arbitrary electric fields and similarly this too. How about this equation?. Is it true only for steady currents or is it true only for steady magnetic fields?. Is it clear from the right hand side that, it should be time dependent. Can the field change from point to point, is this true in general? Yes indeed, it is so. These are field equations.

Therefore, this is a function of r and t and this is a function of r and t as well. And similarly here too completely arbitrary space time dependent quantities here. And these are the four Maxwell's equations. I am going to ask them series of questions about these equations. The first of them is, how many unknowns are there? What are the known and what are the unknowns? rho and j are the sources of the electromagnetic field. They are supposed to be specified. And you are supposed to calculate E and B. How many unknowns are there in E and B? Six unknowns. And how many equations are there? Well, if you have six equations and four unknowns. You are in deep trouble. We do not know anything yet. This is all we know. These are the equations and this is crystallized wisdom starting from Coulomb's law in a generalized form leads to this equation. The fact that, there are no magnetic monopoles leads to this equation. This is Faradays law of electromagnetic induction and this is the Ampere Maxwell's theorem.

In the absence of electric fields, we change with time. And for steady currents, this was the original Ampere theorem and now it is Ampere Maxwell's theorem with this displacement current. That is it. These are the equations that I have given to you and how many of them are there, how many equations are there? There are eight equations, because these two are vector equations. These are scalar equations. So if you go by components there are eight equations. But then that is an embarrassment of which is you have eight equations and only six unknowns.

In general, you do not have a solution then, too many equations, too few unknowns. No reason why there should be any solution at all. And yet we know there are electric and magnetic fields revolve around us. And these equations were so, what would you say is happening. There are some redundant equations or at least there are connections between these equations, which are not apparently. So they really are six pieces of information for six unknowns, but they are jumbled up and they are mixed up in a crazy fashion.

And we are going to unravel that, going to see, in what fashion they are really mixed up. So there are really only six independent equations. And the reason is there is a connection between the j and the rho. The source terms here. They are not independent of each other. So, as soon as, you say charges cannot be created or destroyed, there is a connection between j and rho.

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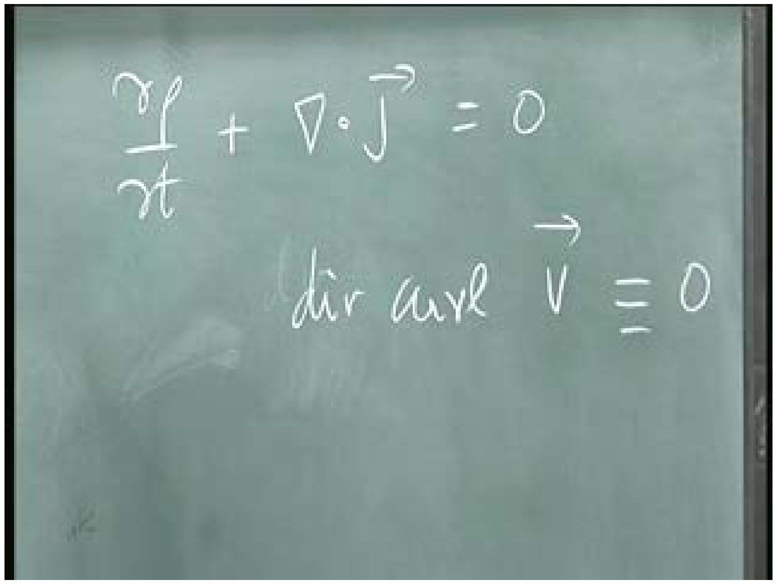

If you do not specify them independently, what is the connection between them? The equation of continuity; this is certainly true, in the absence of sources and sinks del dot j equal to 0.

There is such a connection. There is a similar connection between the magnetic charge density which happens to be zero; and the magnetic current density which should be sitting here under zero. So you really have another constraint equation which is equal in saying zero plus zero is equal to zero and that is a constraint too, because it is not apparently, so you really have only six pieces of information. We will see this in another way very shortly. What is the difference, you can see immediately between these two equations on the one hand and these two on the other.

Well, given these equations there is no magnetic charge and so on. Given these equations, there is a basic intrinsic difference between the first and the fourth equations on the one hand and the second and third on the other. There are no sources. These two equations involve the sources and these equations do not involve sources at all.

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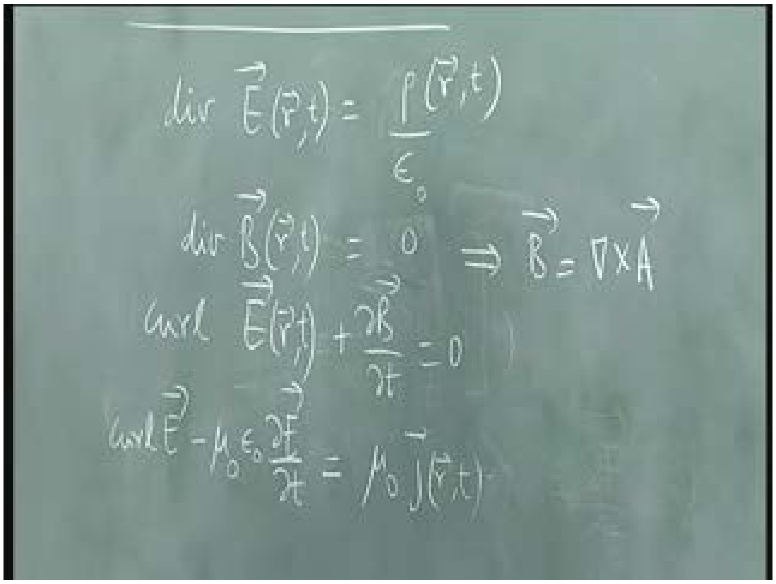

Therefore these equations are valid independent, of the sources. All electric and magnetic fields must satisfy these equations. So let us make that clear by writing this. Bringing this to the left hand side and writing this plus delta B over delta t equal to 0; and this of course continuous to be zero.

So these two equations slightly different from the other two equations and let us bring this E to this side. And let me write this E minus mu naught epsilon naught delta E over delta t. Now there are no sources in these two equations. So there is another way of saying that, these equations are different. What is that? The right hand side of these equations are zero. Not so for, other two equations, so what do you call these equations, in which Michener's equations.

These are homogeneous and these are n homogeneous. Now that we have homogenous equations let us exhaust the contents of these equations first. Put in all the information that you can get from it and then substitute in the first and fourth equations. Now what is the first conclusion, when you draw, when you hear that a vector field B is identically divergence less. What conclusion do you draw from it?

We know that the divergence of a curl must be zero identity. So you have this vector identity, which says that divergence of the curl of any vector field, let me call it, V is equal to zero. This is identically zero. Therefore this equation implies immediately, that B can always be expressed as a curl of another vector field A.

You exchanged your ignorance of B for your ignorance of A, but at least you exhausted the contents of this equation. Because you know, that the divergence of a curl is identically zero. And since the divergence of B is identically zero, for all magnetic fields. B must be the curl of some other vector field always, with no exception.

We do not know the other vector field. But it is always. So you will see the advantage of introducing this A. A has a name. It is called the vector potential. Let me not call it, the magnetic vector potential. Because you will see that A will play a role in E as well. It is just a vector potential. Why do I call it a potential. It is not potential energy. It is a historical hang up. The fact is that, you know, that in the old days, if you had a scalar potential V as a function of possession and you took it is gradient. You got a force and this very often in mechanical instances, had the implication of being the potential energy.

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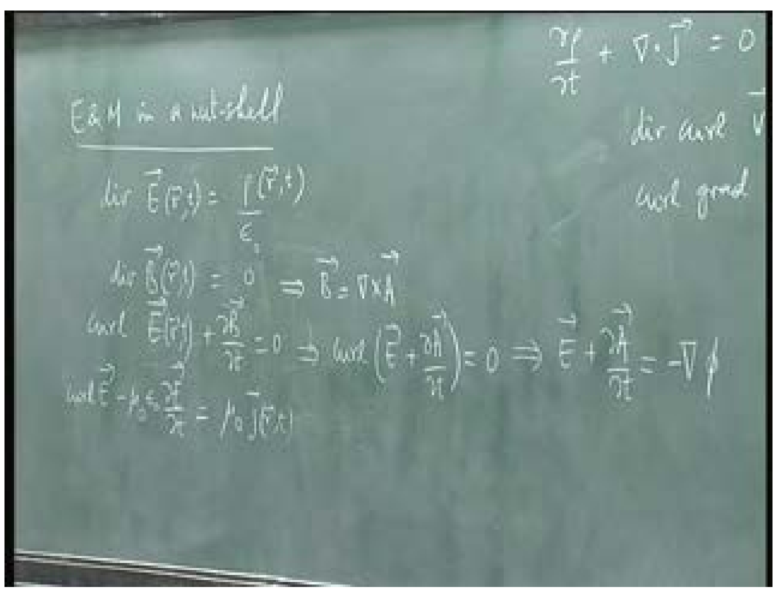

And you differentiate it in order to find the force the vector field. So that name got carried over and since you take this vector field A and you differentiate it. After all the curl operation is some kind of derivative and you produce another vector field B. You call it the vector potential, vector, because A is a vector. That is the only reason for you.

So anything you differentiate to produce another field, you call it the potential, does not have the connotation of physical potential or anything like that. It is not a potential energy, but it is just a auxiliary field which you introduce, whose curl is going to give you B. Now I put into you, that once you have written B in that form, A of course is also r and t dependent. You have exhausted the contents of this equation; that is it. There is no more information that this equation affords. But you put that in here. And you promptly discover that, this implies that the curl of E plus delta A over delta t is identically zero once again.

Now notice that I have taken the curl operation del cross and delta over delta t. And I have commuted the two. And the reason is, these are partial derivatives. Therefore you can differentiate it in either order does not matter. So curl of delta B over delta t is the same as delta over delta t of curl E. So I mix the two up, then I get this. I interchange then, I get this. But this again affords us some simplification. It says this vector field E plus delta A over delta t has no curl. It is identically curl free or irrotational and what quantity has the gradient of a scalar field is identically has a zero curl identically.

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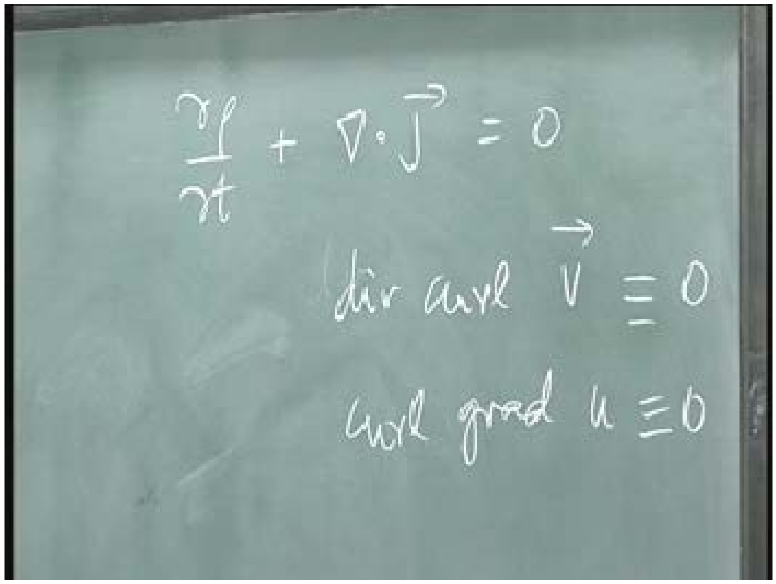

So of the gradient of any scalar quantity is identically zero. Therefore the contents of this equation are completely taken care of, because once you write E plus delta A over delta t, as the gradient of some scalar field. That is it. This equation is taken care of completely. So this would imply that E plus delta A over delta t equal to the gradient of a scalar field. But let me put a minus sign here, purely as a matter of convention. The reason is, in electro statics you know that E can be written as minus the gradient field phi scalar field and this phi has the connotation of being the potential energy, of a set of charges. So we keep that minus sign for that reason. No other reason and write E plus delta A over delta t in this form. What would I call phi? I call it, the scalar potential. Not the electro static potential. This has nothing to do with electro statics. It is always true. So this becomes. Therefore let us write our basic equations.

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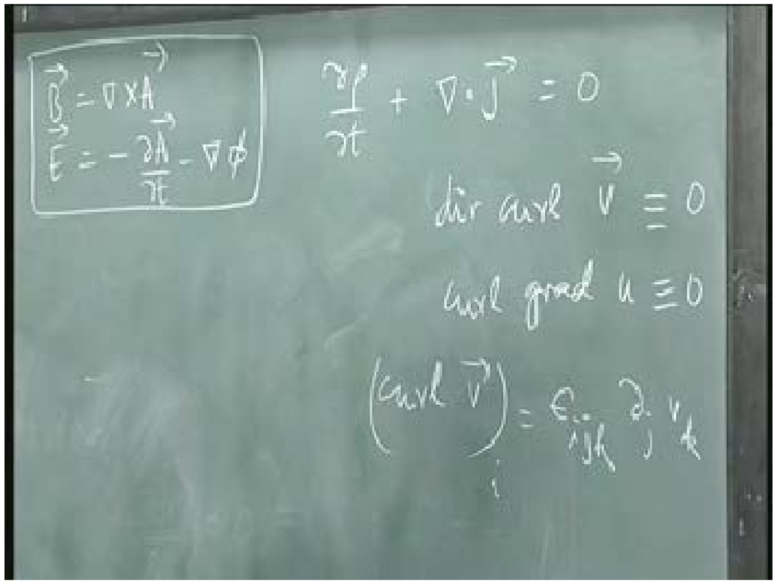

One of them is, B equal to the curl of A and E is equal to minus delta A over delta t minus the gradient of phi. That is it. The fact that I can write these two equations, that these two fields in this form derives directly from the two homogenous Maxwell's equations and that is it, their contents are completely taken care of. So erase these two equations and I am left with those two. What have I done so far?

Well, I got rid of B and E in terms of auxiliary quantities A and phi. And the idea is, if you solve for A and phi, you can solve for E and B, by differentiation by these rules. But I have gained an advantage. The fact is, these are two vectors. So that, really six numbers sitting here but A and phi between them are count for only four of them. So it means that, these homogenous Maxwell's equations have ensured that all E and all B can be found from four points, not six. That is enough to find all of A and B. Incidentally these identities are completely trivial to prove either component wise or in index notation.

Because in index notation, the divergence of the curl of V, the curl of a vector field, it is i th component would be epsilon i j k del j V k where this is delta over delta x comma x sub j, the x component and then the divergence of that would be epsilon i j k del i del j V k with the contraction over i and j. Because the i th component is like del dot this. That is del i and then the i th component of this curl. This is it. But this is a symmetric in i and j. This is anti symmetric in i and j. And you have to sum over all possible i j k, in the contraction of the symmetric with the anti symmetric identity is zero. So it is a trivial statement that the divergence of the curl of the vector will be zero.

Likewise the curl of the gradient of u stands for the i th component of this curl, for an instance would be epsilon i j k del j that takes care of this and then del k on u. That is the k th component of the gradient of u. Once again this is anti symmetric in i j and j and k. And you got this is symmetric gradient and therefore it is zero. This also tells you that, there is no other possible vector identity. This is it. These are the only two identities that you have to really remember. And we put that in and this is what we get. Now let us go back a little bit and ask these four Maxwell's equations. They are first order partial differential equations. They are first order for a very deep reason.

I will get back to it later. But they involve the divergence in the curl and I want to explain to you, why they involve only the divergence and the curl. So please remember that, we also had this equation divergence B is equal to zero and curl E plus delta B by delta t is equal to zero.

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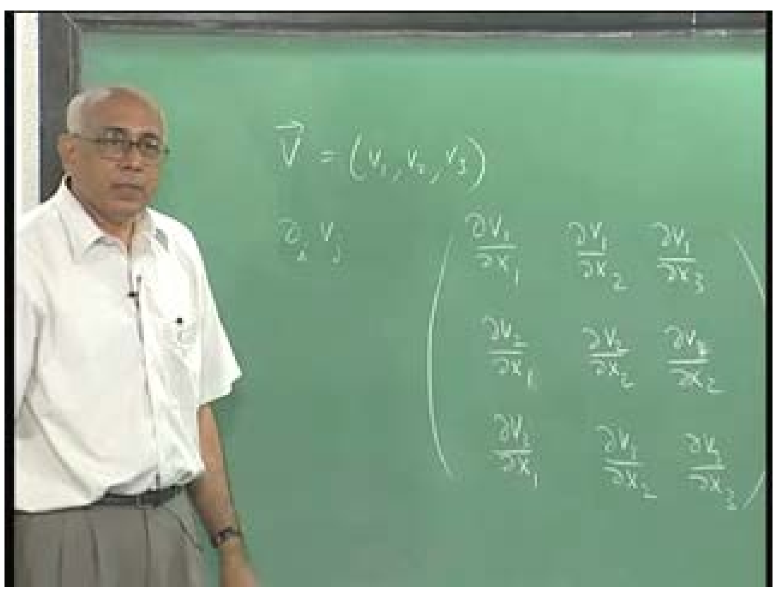

My question is why do we specify what is so special about specifying that the divergence and the curl of a vector field?. Why not anything else?. Because after all, if I have a vector field V which has got components V one V two V three, then you can differentiate each component with respect to each of the coordinates and really you have nine derivatives possible. So you will have del i V j or, if I write this out you have delta V one over delta X one delta V one over delta X two delta V one over delta X three. X one X two X three stand for X Y Z and V one V two V three stand for the X Y Z components then you have delta V two over delta X one delta V two over delta X two delta V two over delta X three. So you have these nine components, you could actually put in a little matrix of this kind.

The nine possible partial derivatives and what does the divergence correspond to?. The sum of these diagonal elements, what does the curl correspond to?. It is a vector. Well, it is, this minus that and that gives you the three component of the curl. This minus that gives you the minus two components of the curl and this minus that gives you the one component of the curl. Now why is that out of the set of nine quantities. You choose the diagonal free. You call it something. Call it the divergence. And it is a scalar and why do you choose the anti symmetric differences of the out diagonal elements. And you call that a curl, one of the vector field. And the answer is that, it turns out that, when you take the tensile of rank two of this kind.

The sum of the diagonal elements as you can see is going to be invariant under similarity transformations. It is not going to change. We know that the trace does not change. And this means under rotations this quantity would not change. That is the reason. The divergence is a scalar and it turns out that the differences between the off diagonal elements, is a vector transforms like a vector.

So very specific transformation quantity law is implied, we would like to have all physical quantities expressed in terms of objects, whose transformation properties are known to you. That is why objects can become physical laws, can become form invariant, the same for all observers. And this is the reason why you choose these two particular combinations, but a deeper question still occurs. That is how do we know we do not need any other combinations, any other derivatives. Why are these alone enough, and the answer is buried in the theorem due to Helmoltz, which says, if you give me in rough form it says in a region of space. If you give me the divergence and the curl of a vector field, and you specify the normal component of the curl on the surface of this volume element.

Of this volume then the field is uniquely determined. So we take request to this theorem. In this vector analysis which says, essentially that specifying the divergence and curl is in a sense enough. It is sufficient to determine the vector field completely but having said, that you still have to understand physically, what does it mean to specify the divergence and curl. And here let me go back a little bit and give an example where this comes from. What is the reason for it? So that is important to understand. Because every equation in mathematical physics, that you see would always involve vector fields with divergence and curl. And the reason is buried in this theorem. And you would also like to understand why this is sufficient.

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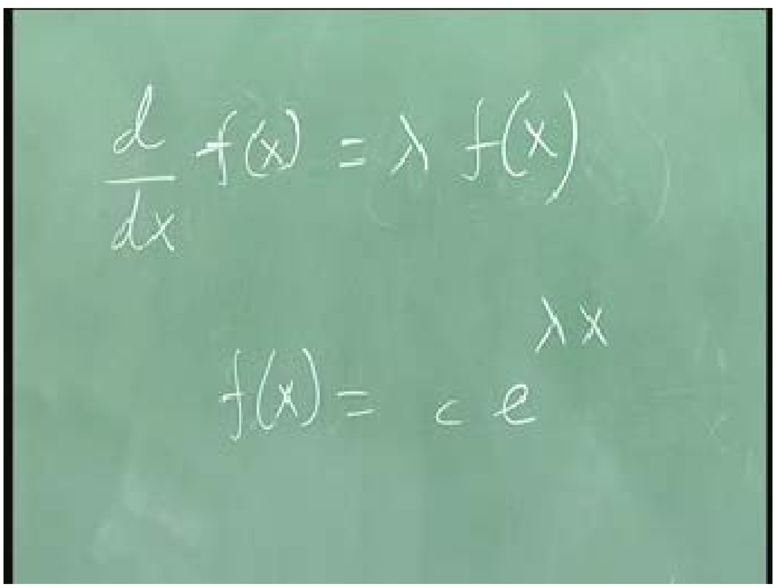

So let us understand why it is sufficient to specify the divergence and curl let us go back and ask what is the solution to an equation of the form, d over d X f of X equal to lambda f of X. This is a eigen value equation in which d over d X is the operator differential operator, f of X is the eigen function, lambda is the eigen value and f of X is again the eigen function. What is the solution to this equation. The solution is f of X is apart from a constant. It is e to the lambda X. So you agree that e to the power lambda X is a eigen state or eigen function of the derivative of X.

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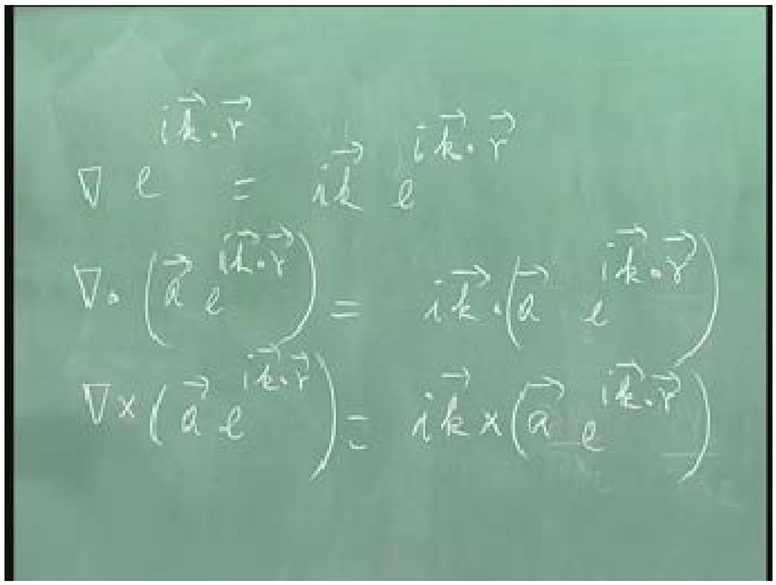

Now let us generalize to three dimensions and instead of lambda, I would like to take something like i k. So I ask if I took e to the power i k dot r, but k is an arbitrary constant vector and r stands for the three position components X Y Z. This is a scalar function. I put an i , because I am thinking in terms of waves, in terms of oscillatory functions cos and sine functions and so on, which i can combine into e to the i k dot r. This stands for e to the i k one X plus k two Y plus k three Z and I ask what happens, if I took the gradient of this. That is a scalar function.

I am asking what happens if we took to the gradient of this scalar function. What is the answer, i k times r. So you could now say that e to the power i k dot r is an eigen function of the del operator with eigen value i k, that is a vector operator. So it is eigen values also vector. That is a very simple result. You can see what is the physical meaning of this result.

Well, if you put e to the i k dot r equal to constant, the surfaces you get correspond to k dot r equal to constant that says k one X plus k two Y plus k three Z is equal to constant. What sort of surface is that? It is a plane in space and the gradient is normal to this plane, because the gradient of the scalar field is always normal to its level surfaces. So it gives you the direction normal to this plane. That is what this is, but what is this, del dot some constant vector.

Let me call it, a e to the i dot r. What is this equal to, a is a constant vector. k is another constant vector, the r dependence the coordinate depends in e i k dot r and I do del dot this. What happens? So you must take the X component of this guy which is, a one e to the i k one X plus k two Y differentiated with respect to X. And Similarly for Y and Z and add up these three. This is a simple matter to verify. It is a two line thing to verify that, you are going to get i k dot a e to i k dot r. k is a vector and a is a vector. So you find the dot product of k dot a and it is multiplied by the scalar function e to the i k dot. So in a sense it is turning out that the eigen function of del dot is a e to the i k, for arbitrary a and arbitrary k.

And likewise it is not hard to see that del cross a e to the i k dot r written out to be i k cross a e to the i k dot r. So even that cross has exactly the same kind of eigen vectors. Now that is marvelous. Because it is really telling you that if you give me a vector a e to the i k dot r, that is a vector field, where a is any constant vector. Then this vector field is acting as an eigen function for del dot and del cross, the divergence and the curl. So what it is doing is, to take the divergence and curl operators and converting them to numbers. It is converting into algebraic quantities. Differential equations are getting converted to algebraic equations. And what is that trick called? When you convert to differential equations from algebraic equations, it is called Fourier analysis. This is exactly what you do to convert the derivative operator to multiplication by k, by the Fourier variable. And this is what has happened.

So now we know that, if you give me a arbitrary function of r, I can expand it in a Fourier basis, if it is a respectable function. And since these are linear equations, I can superpose the equations for each component and reconstruct the field. And for each component those things are valid. So what is this telling you? It says that the divergence of this vector field is, i k dot this guy. So it says the divergence, tells you the component of this vector field along the direction of k. And this is the curl of this vector field, this i k cross and therefore this curl operator, this guy here is telling you this is normal to the direction a. So it says in sum that the divergence equation is going to give you along any arbitrary direction you choose.

The divergence equation is going to tell you how much of that vector field points along this arbitrary direction and the curl equation is going to tell you, how much is in the normal direction, in the transverse direction and together they determine the entire field. So this is the reason why all these equations involve divergences and curls, independent of coordinates, independent of observers, independent of, what your coordinate system is.

So it is carrying a dictionary which is independent of the observer. No matter what k direction you choose. The divergence will give you the longitudinal component and curl will give you the transverse component. And that is true for every k and you superpose all possible k's, you get your fields. So this is the reason why the Maxwell equations specify. This is the reason for the theorem, the Helmoltz theorem which says, if you tell me the divergence and the curl of a vector field diagram and suitable boundary conditions, I tell you the field uniquely.

The reason is the divergence equation gives you the longitudinal components and the curl equations give the transverse components, independent of the direction of flow for every k. So is it clear? That is the real reason why you have divergences and curls everywhere in Physics.

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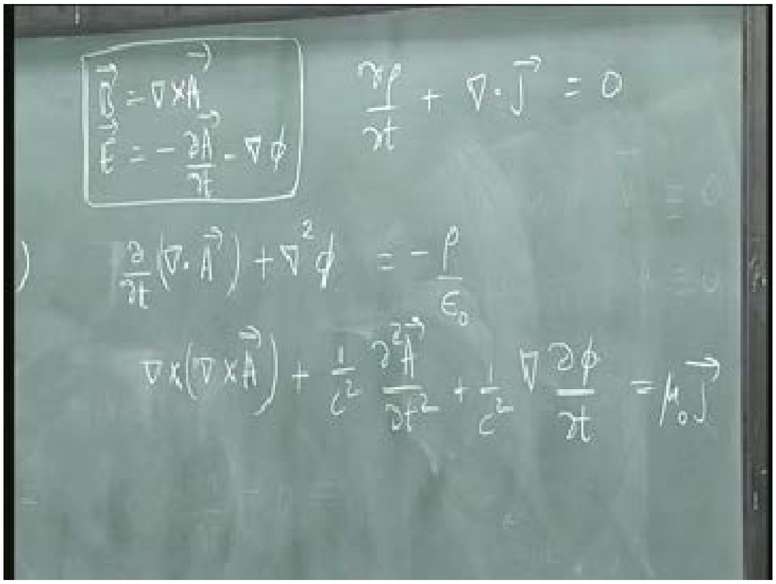

Now, we know that let us put these things back and ask what do these equations tell us. I am going to pluck this equation back in here and what does it give you. It says del dot E so it is minus delta over delta t del dot A. That is the first part minus del dot del phi, but that is equal to minus del square phi.

Let us call the Laplacian of this phi and it got its own interesting set of properties. We will come to that later and that is equal to rho over epsilon naught. So let us put the minus sign here and put epsilon naught here plus this, plus and that is the first equation. We took care of these two equations. We need to take care of this equation here and it says the curl of E is of course minus del. The curl of B; so it says del cross del cross A minus mu naught epsilon naught.

By the way let us anticipate things a little bit mu naught epsilon naught is the dimensions of one over the square of the velocity. And conventionally it is called C square, speed of light in vacuum. So we can just put that in. So this is equal to one over C square time derivative. We have to differentiate the second time that becomes a plus one by C square delta E over delta t gradient of delta phi over delta t equal to mu naught j.

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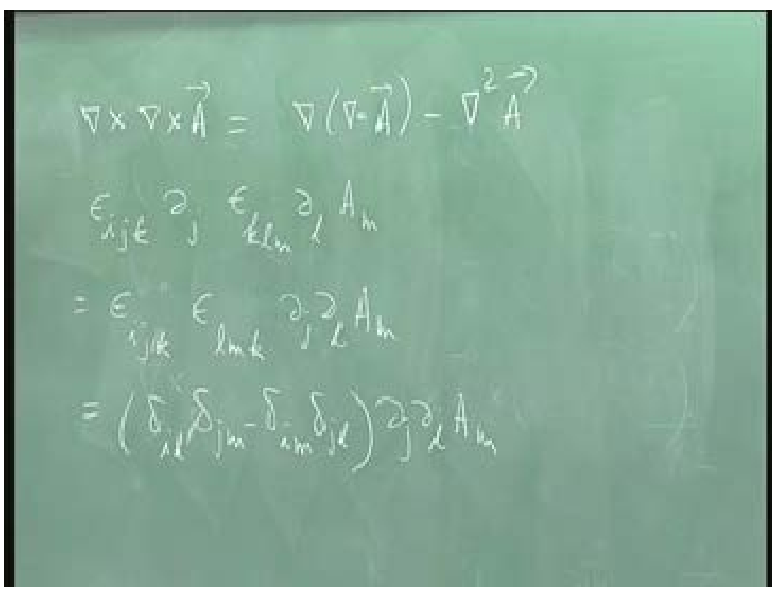

Now we have this famous vector identity for the curl of the curl of the vector field and what is that identity gradient of the divergence of A minus del square A. So curl A del cross del cross A equal to the gradient of the divergence of A minus del square A. Del square is a scalar operator it acts on each components of A, so produces a vector field, which in Cartesian coordinates has components del square A one del square A two and del square A three.

That is what the del square stands for. In case you have forgotten this identity it is not hard to derive, because we want the curl here. So put an epsilon i j k and then del j of del cross A, the k component that is equal to epsilon k l m del l A m. And then you move this thing back here and write this as epsilon i j k epsilon l m k del j del l A m and that by the identity is delta i l delta j m minus delta i m delta j l del j del l A m.

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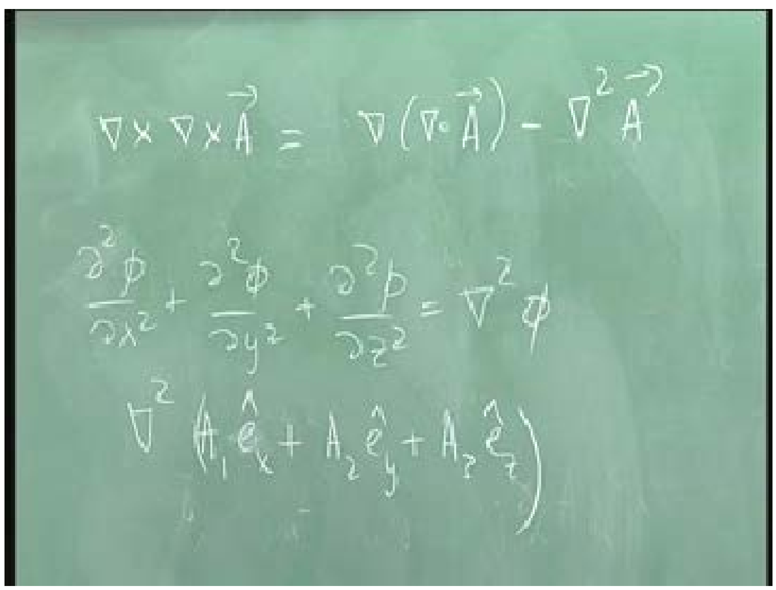

Now you use the chronicle delta to simplify this matter and then you get this. So it is very simple thing you do not have to remember anything. You just have to remember the fact, that when you contract this epsilon symbol once, the answer is in terms of delta. So let us put that back in here and you have got del of del dot A. There is another del which is sitting here. So let us call it one over C square delta phi by delta t, if I took care of this term. And the first term here plus one over C square d two A d two minus del square A equal to mu naught j. These are the two equations. I erase this and I have to deal with these two equations. Now that exhausts all four Maxwell's equations. The price you paid is that, the equations order is increased to second order differential equations.

The Laplacian of a scalar function simply says we have got phi. It is d two phi over d X two plus d two phi over d Y two plus d two phi over d Z two. This is what del square on a scalar function is, but then del square of A, a vector. A would be A one unit vector in the X direction plus A two unit vector in the Y direction plus A three unit vector in the Z direction and these are constant vectors in Euclidian space. So they do not get differentiated and therefore it stands for a vector with components e X times del square A one then e Y times del square A two times and so on.

So it is just a short hand notation for three del square acting on the A one A two A three and it is a vector created out of these three. You do not have to do that. Divergence is really not defined as del dot A, the divergence of A for a vector field A. The divergence is defined as a flux per unit volume at each point in a limiting case and then algorithm for calculating that quantity turns out to be del dot A.

Similarly the curl is not defined in terms of this ridiculous determinants or anything like that it is not del cross A. But rather the circulation of this vector field over small line each point divided by the area. So it is the circulation per unit area in the limit, in which the area go to zero and algorithm for calculating it is del cross A. So this question is, is there a similar thing for del square? What is the significance of del square and so on.

I would not answer that question directly. It is a scalar operator del dot l del, but I would like to mention though, that the reason the del square operator appears is very profound and very deep. It is really buried in differential geometry. The del square operator or rather the generalization of del square operator complicated spaces is a very basic problem. It probes very basic property, is of the space itself.

So I know it is an unsatisfactory answer at the moment. But let me give you a physical picture of why the del square operator has the kind of role that it plays. Let us go back to electro statics, what it does. What the del square does a good digression, good to understand that. Incidentally, if you did not have time dependent fields, but you had only electrostatics. Then of course you know this goes away and you have got del square phi is minus rho over epsilon naught.

Imagine for a minute, rho is time independent j is time independent, what you call that equation, Harsens equation. That is the fundamental equation of electrostatics and what is the primary property of charge free region, where del square phi is zero. There is no absolute maxima or minima. This is the primary property and you have the mean value theorem. And it says the value of the field at any point, where del square phi is zero. The value of this phi at any point is the arithmetic mean of all points, values of all points symmetrically situated above it.

Whenever you have Laplace equation; that is true that is the statement that, I am making. Of course, you put a source so along true. What I am trying to bring out here is, the significance of the Laplacian operator. When you say del square phi is equal to zero. It says phi is a function and eigen function of the del square operator with zero eigen value. It is called a harmonic function and the question is what is so harmonic about it. And the answer I am giving is that, this function is such that every solution of the Laplace equation is such that, the value of the function at any point, is the arithmetic average of the values at point symmetrically situated about it, whatever be the number of dimensions.

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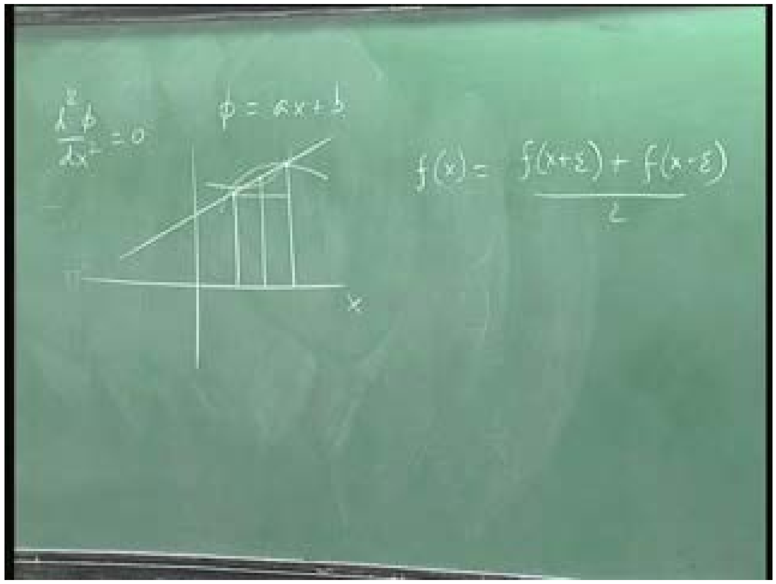

Let me go one step further back. Let us look at Laplaces equation in one dimension. That is my Laplace equation in one dimension, just the second derivative is zero. Now what is the solution to this equation, a linear equation, linear function. So it tells you phi is equal to a x plus b. So let us plot this phi as a function of x. This is what it is. What is so special about a straight line. I want the value at every point. The slope is constant what does it lack. Well the first derivative is non zero. It has no maxima or minima. That is very true, but even functions which are not straight lines will have known, what does it lack.

It is first derivative is non zero in general. But the second derivative is identically zero. So what does the second derivative measure curvature. There is no curvature. This function has no curvature. So the consequence of that, if you took the value at any point here, it is guaranteed to be the arithmetic mean of values symmetrically situated about it, of this value and this value.

So this is x and this is x minus epsilon x plus epsilon. You are absolutely guaranteed that f of x equal to f of x plus epsilon plus f of x minus epsilon divided by two. You guaranteed that. On the other hand, if the function at curvature, then the curvature like this, the value here is smaller. The value at this point is larger than the arithmetic mean and it would curve the other way, the value is smaller than the value than the arithmetic mean, only if the curvature is identically zero. You get this mean value.

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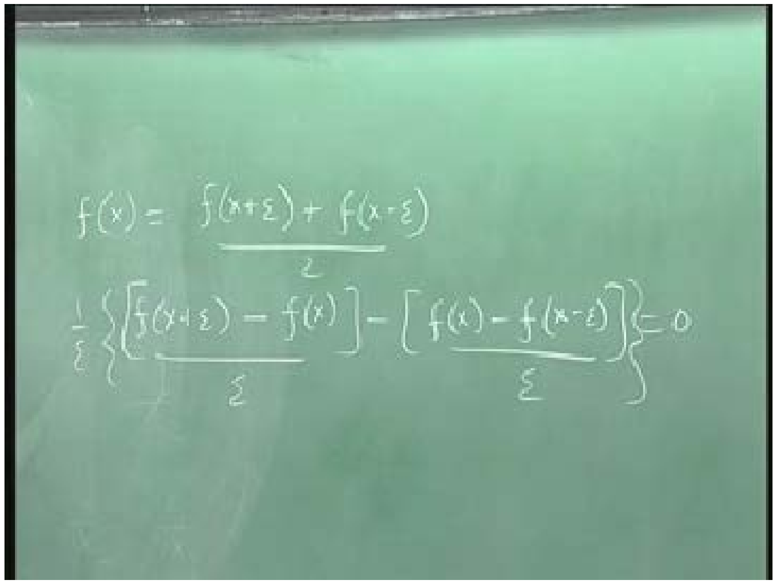

And of course it does not take long to realize that I could write this as, f of x plus epsilon minus twice f of x which, I will write f of x minus f of x minus f of x minus epsilon and that is equal to zero. I divide this by epsilon. And this is, of course the derivative from the right. And this is the derivative from the left. And you divide one more by epsilon and take the limit. And what derivative do I get, the second derivative. And that is exactly this. So you see a harmonic function. This is a harmonic function in one dimension, only a straight line. But guaranteed that value at any point, is the arithmetic average of points either side.

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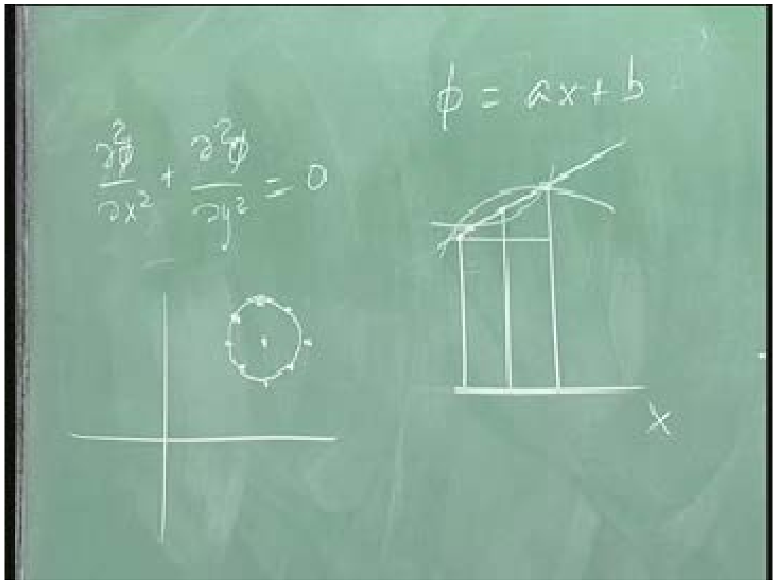

Now two or three dimensions in two dimensions, for example the same property would translate to something much more interesting. Because if you have d two phi over d x two plus d two phi over d y two equal to zero. Then of course, if you put phi equal to a x plus b plus c y plus d. This is true. That is a trivial case. But you have more interesting possibilities. It is possible that, the x derivative d two phi over d x two is some point t and the y derivative is minus that. So the functions do not have to be linear. But they could still satisfy this point. And it is not very hard to see, if I did the same trick as, what I did there, that the value at any point is the arithmetic mean of the value here.

Take these four points x minus epsilon x plus epsilon y minus epsilon y plus epsilon divide by four after summing them. And I would end up to that equation to the limit. But then the del square operator is spherically invariant. It is a scalar operator. So, I could in fact have taken four points this, this, this and this. I still have this.

You take all these eight points and do one eight. And I still have it. And in fact, I could do the whole lot and take all the points on the circle and divide by the length of the circle. And I would have the value; that is the harmonic problem. And in three dimension, it is a sphere. As you know, this is the property of harmonic functions, the mean value problem any number of dimensions, so that is what is so basic about the del square operator.

It is eigen functions, give you the generalization of a function without curvature in one dimension. It satisfies this mean value theorem. So that is why, it is so special. There is other reasons, why this is very particular. In fact if you know the spectrum of the Laplacian operator in a domain actually known everything about the domain subsets.

So that is the reason, it is a very basic probe. By the way in electrostatics, this has the consequence, that a function which satisfies this cannot have a maximum here, because it cannot be larger than these values. And still be it is average, your average. It cannot have an absolute minimum here and the consequence of that is that in a charge free region. You cannot put a charge held together by electrostatic forces alone and expected to be in stable equilibrium. Could be equilibrium but unstable. Some of these terms will be positive, some would be negative. If it is a absolute minimum, all of them would have to be positive and they cannot be add up to zero. That is an absolute maximum. All of them will be negative. We cannot add up to zero. So it always have saddle point some positive, some negative and you have unstable equilibrium. What is that theorem called Poisson's theorem. Just say that, del square function satisfies the Laplace equation, cannot have absolute maximum or minimum. So let us come back. Now we have to deal with these two equations. And the bad news is that this equation which looks almost like Poisson's equation has this A sitting.

Now what does this equation look like, if you did not have this term, it looks like the wave equation. It looks exactly like the wave equation. So there is a one over C square. It is second derivative with respect to time minus the del square the three dimensional wave equation. The source term on the right hand side and there are mathematical methods for solving the wave equation, just like there are for solving of Laplacian equation or the Poisson's equation.

Unfortunately you cannot use them directly. Because the A equation involves phi and the phi equation involves A. It could be glorious, if you could put this equal to zero or you can put this equal to zero. And now I put it to you that you can always put it equal to zero A is arbitrary. Well remember the physical fields are E and B. They are my physical observables. They carry energy momentum and so on. I can measure them.

So these quantities are specified by the sources and the suitable boundary conditions. But these quantities are auxiliary points. And the question is, are they uniquely specified or not? It is quite clear, if I add a constant vector to A, the fields are not going to change, because the derivative is going to act on constants.

Similarly, if I add a constant scalar to phi, nothing is going to change. The gradient is not going to change. But there is more to it, than that, you can add a lot more to it, than that. So let us look at that. This is going to be exploited and I am going to finally tied down to non uniqueness of the Lagrangians.

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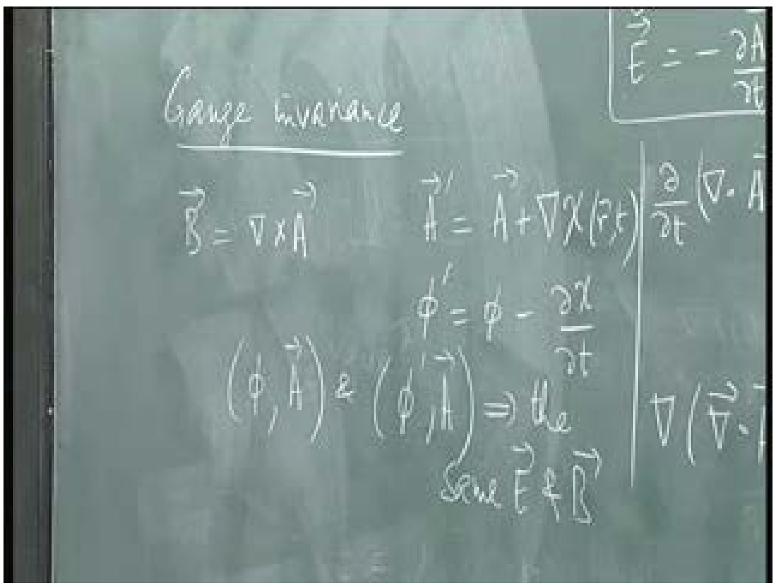

So let us look this carefully this is called Gauge Invariance. Now the question we are going to ask is, I say B is the curl of A. B is a physical field probably unique in every given situation. Is A unique or not? This is the question you have to ask. Is this unique?. For a given B, is A unique or not?

This is the question apart from the constant I can always add trivial constants. But is it unique. I can actually the fact is the curl of any gradient is identically zero. Therefore to A, I can add the gradient of any arbitrary scalar field and B would not change. So it is quite clear that A as well as A prime, which is A plus the gradient of any scalar field of space as well as time does not matter, would lead to the same B. Because the curl of this gradient could be identically zero. So A is not unique.

In classical physics it is A is not correctly physically measurable. It is not unique at all. B very much is, but then, we have to be careful if I change A to A prime, I may also change E unless, I compensate for it with phi. So what I should say is that, simultaneously we are changing A to A prime, by adding the gradient of the scalar. I should also change phi to phi prime which is phi minus delta x over delta t minus, because there is a minus sign.

This compensate and it is now trivial to check that B and E would be the same, if you had the scalar and vector potential specified by pair phi and A or the pair phi prime and A prime make note of this. This is called Gauge Invariance. The word is historical. It was pointed out in the early part of the twentieth century. It comes from something called high invariance which from a while introduced got translated into English as gauge like the meter gauge or the broad gauge and so on. That is a work on measure of some kind. Do not worry about the historical antecedents of this word. But it is stuck. It is called Gauge Invariance.

At this level, it looks like a trivial mathematical ploy. But this principle of Gauge Invariance, when you translate it to quantum mechanics and then quantum field theory turns out to be perhaps, the most basic dynamical principle of all. Because all the fundamental forces of nature are taken to be mediated by quantum fields, which are called Gauge fields.

And it arises from a generalization of this idea of electromagnetism. So it is not a trivial mathematical artifact. It is very much rooted in reality. We will see even at this level what it implies. So, I put it to you once again, that A phi and, well let me write it the other way. Because there is a reason for it phi prime A it implies same E and B. Yes it does, because you can see E is going to change. If I change A to A prime, I have added an extra term which is minus delta over delta t gradient of phi. But B must be unique and the only way, I can get rid of it is, to simultaneously change phi. Then you would have discovered that, d is equal to minus delta A by delta t minus grad phi plus delta phi over delta t. Quite right. Could used A prime there.

You would have added this delta phi. It would have turned out definitely, which two equations. These two we have not even come to it yet. We have not even come to that yet. These equations would of course change. But these are not the equations for E and B. The point is, should these equations also remain unchanged and my answer is, no they would not change. They would not remain unchanged, they will very much change and that is the whole point I want them to change, because they are equations of A and phi. But the physical fields are E and B.

I simply have to make sure that, I no matter what pair A and phi, I choose subject to this freedom. The same E and B are produced at the end. No need at all. No need why A prime and phi should satisfy it. So A and phi satisfy this equation, I do not know what A prime and phi prime satisfied at the moment. And I do not care. That is the whole point, I am going to ensure that, I choose my ki, in such a way that A prime and phi prime satisfy a simpler set of equations. But since I am guaranteed that, A prime and phi prime do the same job as A and phi as far as E and B is concerned. I would like to solve an easier problem. It is a little bit, just to give you a analogy. It is a little bit like, saying I give you a set of vector equation.

I could solve this in Cartesian coordinates or spherical polar coordinates. There may be some symmetry in the problem which causes the problem to be simpler and spherical polar coordinates, I use that and after I do that, I translate the solution back to Cartesian coordinates. I do not loose anyone. So this is exactly what is going to be done. I am going to exploit this freedom. Rho and j are physical quantities absolutely A and phi are auxiliary quantities.

No, let us please understand these two are specified. You are not solving these equations, in order to find rho and j. You are solving these equations in order to find A and phi and therefore E and B. So rho and j are given to you. They are not changed. They are gauge invariant. No question. E and B are gauge invariant, but A and phi are not. They are intermediate quantities. They are auxiliary quantities and my claim now is that, this freedom is what I am going to exploit in order to simplify that set of equations. So now here is what I do, you have given me this freedom and I come back to this equation. And I am going to do exactly what she suggested. Put this in there and see what happens.

Now I put it to you that, I can choose my chi in such a way. So let me not clutter up the notation. But I will say it in words, it is enough to understand, I will choose my ki, in such a way that, this equation becomes del square phi prime equal to minus rho over epsilon naught. Without this term, how do I do that? Well, I first pretend that, I have been stupid enough to choose some A and phi and that is my equations to start. And let us suppose that del dot A is not zero,I have chosen an A and I calculate del dot A. And I find it. It is not equal to zero. What can it be. Well it is equal to this del dot A, is a scalar quantity.

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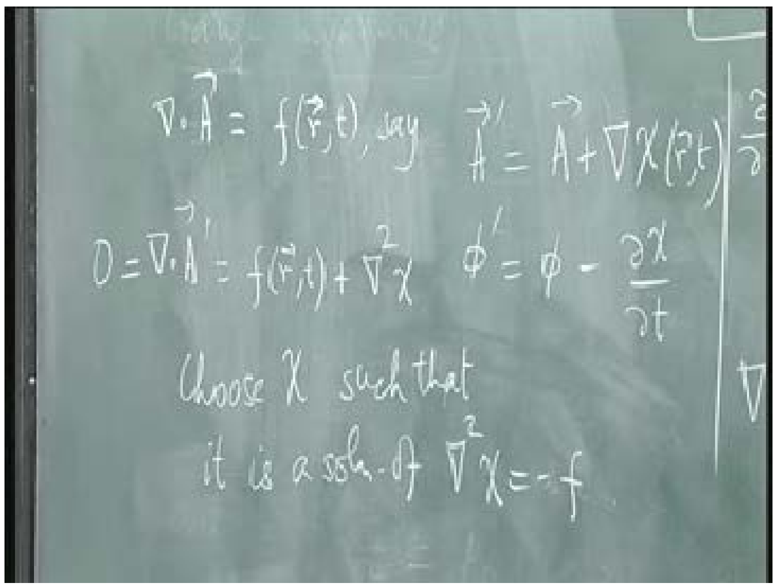

So it is equal to some f of r and E then, what is del dot A prime. It is equal to del dot A which is f of r and t plus del dot del ki, which is equal to del square ki. And I set this equal to zero. I set it equal to zero. How would I do it, by choosing ki, to be a solution of del square chi equal to minus f. So no matter, what A you give me. I find out what it is, divergence is and I find it some function f of r and t. Then I immediately switch to A prime, which is A plus the gradient of ki, where I choose ki. And that is up to me to choose this ki, such that it solves the equation del square chi equal to minus f. That is Poisson’s equation and I know how to solve it. So choose ki. chi is called the gauge function, such that it is a solution of del square chi is equal to minus f. Very good question, can that equation be solved.

Now I have to put conditions under which Poisson's equation can be solved. This is a well posed problem in the area of partial differential equations. If you specify appropriate boundary conditions, Poisson's equation can be solved. It is guaranteed that the solution exists. I would not go to the technicalities of when it exists and what conditions and so on in all the normal electromagnetic problems this would be solved. In fact we are going to write the solution now, which second term this I have not done anything. I have not gone here at all. I am just saying you, give me a del dot A. You give me an A. I calculate del dot A. I discover it some function, not zero. I promptly choose an A prime related to A through this, such that del dot A prime is identically zero. That means you give me f and I tell you what I is. Not necessarily. Yes, in a closed analytical form.

Let me give you an instance. If this you have to specify things, is it differentiable, is this f very bad. I have to after all integrate this equation. So is that f integrable? So, I have to answer technical mathematical questions of this kind. But we are going to write physical arguments to write down a solution to Poisson's equation. I will do that from coulomb's law, which you already know. So we write down a free solution to the Poisson's equation. But take it from me. At the moment, as an assertion, that you can solve Poisson's equation. There is a table mathematical algorithm to solve Poisson's equation. Therefore in principle, no matter what A you give me.

I can rewrite this equation in terms of A prime and phi prime, where del dot A prime is zero. Yes, I can make del dot A equal to some constant. Yes instead of zero and then this is again zero here. And therefore, this is certainly true. Yes I can do this. But if I can make it a constant, why do I want this constant. Why do not I make it zero?. I can do just as easily. You are right. There is nothing unique about the choice in this case. Yes. We are trying to find E and B. We have written these equations, in terms of A and phi but A and phi are not unique, because it is some derivatives.

Of them, which gives you E and B and you can put various arbitrary gradients and so on added to A, such that A and B do not change, but the auxiliary equation for A and phi. I would like to use this freedom to write in, as simpler form as possible in particular. It will be very nice, if I could decouple the phi equation from A equation. And that is, what is achieved. If I put this equal to zero and there is no A dependence. Here it is only phi. I should call it phi prime.

So I am going to take this, as I have given. And I can always choose my gauge such that del dot A is zero, in which case the first equation becomes Poisson's equation. Again for phi or phi prime, but I will drop the prime. I will assume that, you have chosen del dot A is equal to zero. This is called the Coulomb gauge. Yes. No let me put that in. So, I choose this. I choose this A prime and instead of A, I now write A prime minus grad phi grad chi and instead of phi, I write phi prime plus delta chi over delta t. I put that in here and then the statement is del dot A prime is zero.

So, where does it leave you? It leaves you with Poisson's equation for the scalar quantity. So the sum of this entire thing is, you can use gauge invariance. The freedom in the choice of the vector potential, to ensure that the first of these equations; this equation deduces to Poisson's equation for the scalar quantities. Because the statement is, you can always choose your gates such that del dot A is 0. If you want to call it A prime, that is ok. Good point, once you simplify the first equation, you may be in deep trouble, in the second equation.

But let me say what happens. That is a good point. You have to worry about that. But then, what I am going to say is, permit me to get rid of this. And this is called the Coulomb gauge. So if I work in the Coulomb gauge, then the first equation becomes Poisson's equation and I claim. I solve Poisson's equation. I am going to show you how to write down a solution, then phi becomes known to you. This is known to you and then you put it in here. This is known to you and this is zero.

So move this known to the right hand side and then it says, A satisfies a wave equation with a known right hand side. And if I can solve the wave equation, then the problem is done in principle at least in principle and after that, if you find A and phi, then you differentiate it. So all of the electromagnetism is solved in the Coulomb, but the important point is, I want to assert this again and again, that you can always work in the coulomb gauge.

This is always possible, whether it is advantage to do or not, is a different question, whether you want to do it or not is a question. But it decouples the two equations the A and phi dependences and it reduces the problem, solving the Poisson equation for phi and the wave equation for A.

And these are well studied mathematically, that solves all of Maxwell's equations. But you can take another tact you can say I do not like, that can I not use the gauge freedom, to put this equal to zero. If I did that, I would just have the wave equation for A. Can I always do that. The question is can I always do this? And the answer is, what is that you have here. You have got del dot, if I put A prime here and I got a del square chi minus one over C square d two chi over d two equal to what ever this quantity is. So I give the same argument and I say.

Suppose you find that del dot A plus one over C square delta phi over delta t equal to some g of r and t. suppose it is equal to some g not zero. But some zero, then I go to phi prime and A prime and I choose my chi, such that one over C square d two chi over d t two minus del square chi equals minus g.

Just as I chose my chi, my chi, here to be a solution of Poisson's equation, I choose my ki, there of the wave equation and the assertion is, I can solve the wave equation subjected to conditions and so on and once I do that I am guaranteed that del dot A prime plus one over C square delta phi prime over delta t is zero. So in that gauge this quantity can be set equal to zero or rather with A prime phi prime. My assertion is you can always set this to zero always. Can you do both. Can you set this to zero and that to zero. No, the reason is gauge freedom gives you the freedom to handle to choose only one scalar quantity chi. So if you choose that chi, to be a solution to the wave equation, there is no reason why it should also satisfy the Poisson equation and vice versa.

So you have only one piece of freedom, can I choose therefore a gauge in which A is always zero, the vector A. No, physically you know, there is a magnetic field and it need not be zero in general. So you cannot do that. Because A is a vector and chi is a scalar function.

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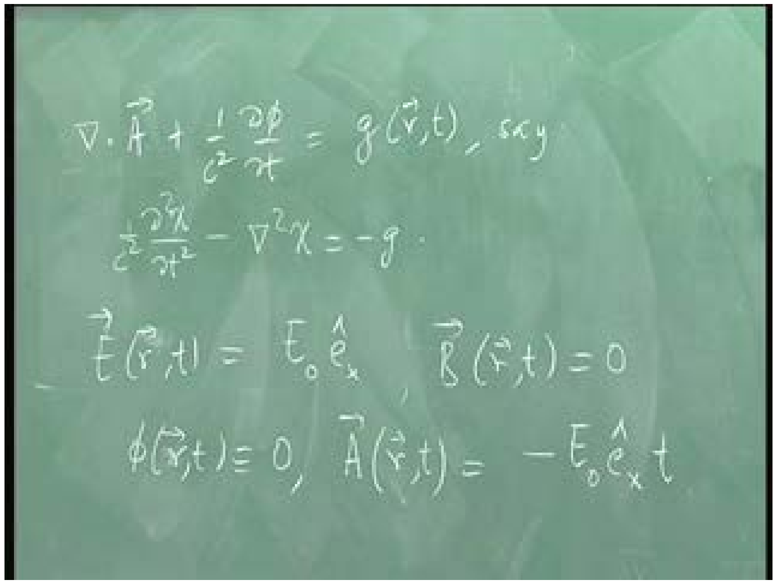

Incidentally gauge in which this is zero is called the Lawrence gauge. There are cases, where this is more advantageous than this gauge. Can you tell me why you would think that setting this to zero is better than that to setting to zero?. You can always do, that in any given problem. You can do either this or that. Why would one be preferable to another. Well ok. Again the same question.

Suppose, I solve the equation in the Lawrence gauge, then I find A; I put that in here. So this becomes a known. I take it to that side and solve Poisson's equation. But why is it conceivable that the Lawrence gauge is any better than the coulomb gauge, then I have no problem. I will just decide phi equal to zero and I am done.

If I have electrostatics, can I do this. I do not have a magnetic field. I have a constant uniform electric field in the x direction independent of space and time everywhere in space, I have constantly. Can I choose phi equal to zero. You really think you have to have faith in you, because I just put into you, that there is one piece of information ki. That is the scalar function, you can choose that in many ways and the claim is, yes you can always make phi.

You can always choose a gauge such that phi is identically zero. May not be very good thing to do. But let us look at this example. We talked about, I have E of r and t to be equal to E naught e x. That is it, where e naught is a constant. No space dependence. No time dependence. And B equal to zero and my question is, can we choose a gate in which phi is zero. Yes, I can choose phi of r t to be identically zero and I choose A of r and t is equal to minus E naught e x. So I have a vector potential, no scalar potential, no magnetic field and of course del cross A is zero, because this is a constant vector. This involves only time. So there is no magnetic field.

But then minus delta A over delta t that gives you this field. So, you can do electrostatics using only a vector potential and no scalar potential, even in the simplest of the electrostatic problems. You can use a scalar potential to be identically zero. It would, of course be very stupid to do this and the reason it would be stupid is, because the whole purpose of the game, is to find the fields. And now I am giving you a vector potential which constructed to give you this field. So I have to tell you the field before I tell you, what the field is. Then I go through the potential. This is not much sensible. But the point is in principle. This is ok. So please remember that, all electric fields are not minus the gradient of the scalar field, because these are not all irrotational. The moment you have the magnetic field which is changing with time, you immediately will have a electric field, whose curl is not identically zero. Therefore which cannot be written as the gradient of the scalar and even in cases, where you have electrostatic fields of this kind. You could work completely with a vector potential and no scalar potential at all.

Similarly you can choose a gauge, in which delta phi over delta t is zero. You can choose a gauge where phi is zero. You can choose a gauge where del dot A is zero. You cannot choose gauge in A is zero, because that is a vector. You cannot choose a gauge, in which it is not common. You cannot choose a gauge in which the magnitude of A is the given constant, because this would involve technicalities, because the magnitude involves square root of A x square plus A y square plus A z square and once you bring in square roots and singularities, problem becomes different. We have been talking about non singular gauges. There are singular gauges. But the simplest possibilities are, as I said most standard possibilities, either the Coulomb gauge and this is zero or the Lawrence gauge and now what I am going to do, is to write down next time, the solution to Poisson’s equation.

We will work backwards from Coulomb’s law which we already know plus the superposition principle and then I show you how to write equations. So this wave equation, so once we do this, then we have to address the real problem and that problem is what happens, if you have a charged particle moving in a electric gauge. What is the equation of motion?.

And our target is, to find the Lawrence force equation. This is the equation of motion and the question is, how are you going to derive; so physical arguments to write down the Lagrangian and then use that in Euler Lagrange equations, you derive the Lawrence force. And finally, I will come into why the particular charge in Largrangian. It is again buried in relative, all of electromagnetism is buried in that. And finally the last point I want to make is, that these are first order differential equations, unlike Newton’s equations of motion.

The original Maxwell equation is, the first order differential equations and the structure of this equation is, such that there is Lawrence invariance built in same for all, that requires this set of equations, we have at this moment. You can see this in another way, in a very heuristic and crude way. Once you have relativity spatial and temporal coordinates, get mixed up. So if you have first order derivatives in space, you cannot have second order derivatives in time. So that is the reason why time equations are also first order partial derivatives. This is fine as long as the problem is mathematically well posed. So I hope, I have given you a sort of in a nutshell summary of, what basic problem is in classical electromagnetism. Now we are going to couple it to the magnetic fields to the charged particles. We will do that next class.