Structural System in Architecture

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Lecture – 11
Bending Stress in Beam - I

Welcome to the NPTEL online certification course on Structural Systems in Architecture. Today we are going to start the 3rd module, the week 3. In this module we will be talking about the structural mechanics. In continuation it is the lecture number 11. In this lecture we are going to discuss about the bending stress in beam; that is in the part one. We will continue the same in this lecture part two and part three.

So, these are the concepts that will covered during this particular lecture:

- Introduction
- Centre of Gravity
- Moment of Inertia
- Theory of Bending
- Assumptions in Theory of Bending
- Pure Bending
- Stress-Curvature Relationship

We will introduce the bending stress and the theory of bending. Then we will discuss about the concept of the centre of gravity and the moment of inertia. We will also discuss some of the assumptions in the theory of bending and the way it can be handled for the formulation of the bending formula. We will discuss the pure bending and also the stress curvature relationship of bending.

The learning objective includes formulation and calculation of centre of gravity and moment of inertia. These are the two most important sectional property of any section. They are regularly required to find out the stresses in the beams. Moreover, here we will also derive the theory of bending.

So, before we go to this particular module, let me tell you one very important thing; that is, the 3rd module talks about the structural mechanics and here the bending stress is one of the very important concepts. This concept of bending stress will take you to design and analysis of beams, and finally to the bending equation, the bending stress and bending theory. All of these has enormous importance in the domain of structural engineering.

Any structure which is designed, typically the beam or the slab or anything that can take care of the bending moment, are finally designed based on bending stress. So, initially in the last week what we did, if you remember, we did in the second week is the strength of material, we discussed two or three very important things.

One is the Hooke's law and how the Hooke's law is implemented for the axial compression. Also, it will produce some kind of axial compressive stress. We also did some explanations and lot of examples on how to solve the bending moment of a loaded beam. So, if a statically determinate beam is loaded by uniform distributed load or maybe the pointed or concentrated load here and there, we can easily compute the profile of the bending moment diagram and the profile of the shear force diagram. Then, from the profile we can understand and can eventually capture the highest amount of bending; and bending moment and the magnitude of that. We will take that from there. So, we know a beam is having that much amount of bending moment, that means that much amount of bending moment is existing in a beam and due to that the beam behaves in a way that it will give you some deformation.

For example, let's consider a rod or a bar. When it is comes under compression it will give you an axial deformation, which may be shortening or may be elongation. A beam is also going to behave in a similar way; and under the loading and bending it will give some kind of deflection. So, in the introductory notes I have written that, under axial external loading the beam will bend and the bending of the beam will be shown as a bending moment and each cross section will have different moments.

This bending moment induces stresses. Yes, definitely! Because of this kind of a bending there will be deflection and the stresses will actually be implied or embedded in various layers of the beam. In other word we can say that this bending is due to the bending moment and this bending moment is due to the external loading.

So, beam will try to bend and try to counter-attack the external loading. It will develop the stress in the beam and try to counteract the external loading by that. These induced stresses are called the bending stress, and the principle of the bending stress distribution is required to understand to design the beam. So, before we directly go to the solving of bending equation, try to clearly understand the developing of bending equation.

First of all, let us discuss the centre of gravity and moment of inertia, the two-sectional properties in any section.

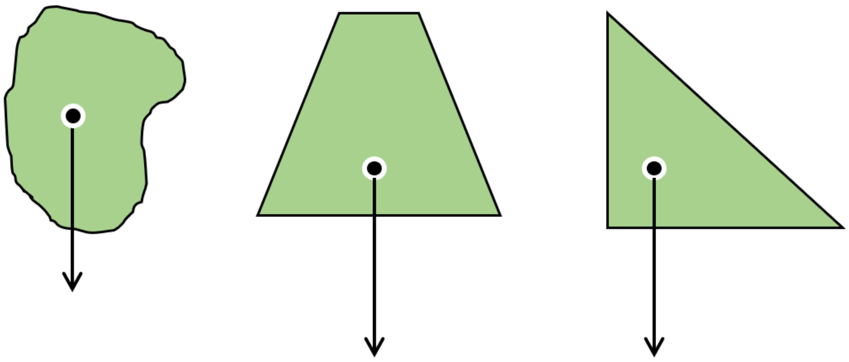


Figure 1: Centre of Gravity of different bodies

So, centre of gravity, as you must know about it because the class 12^{th} physics gives you some idea about it. Centre of gravity is an imaginary point in section plane where, the total weight of the body may be thought to be concentrated. Suppose, this is a rectangular kind of a section, and its imaginary point from where the total mass of this particular section or body is concentrated is the centre of gravity. So that is one of the ways to understand centre of gravity. So, we assume that the whole mass of the particular section or the laminar section is concentrated about the point, an imaginary point inside, as shown in Figure 1, and that point is the centre of gravity. On the other hand, if irregular mass is there and we have to find out the centre of gravity, and if mass is somewhat distributed little heavily in the downside, as shown in central image of Figure 1, then it may come down. If the body is triangular, as shown if right hand side image of Figure-1, its centre of gravity goes toward side, because the distribution of mass is heavy in the left side.

There are small formulas that expresses that the centre of gravity of triangle lies above H/3 from the base. In the same way, there are other various formulas to find out the centre of gravity of critical sections.

The next is the moment of inertia. It is basically the product of the area and it is perpendicular distance from a fixed axis. It gives the quantum of effort required to rotate that particular section with respect to the specific axis.

Let us imagine a door, and when you just put a handle, you can swing that particular door. So, this particular door will actually rotate about the hinge points. So, depending upon the total area of the door, hinge locations, the masses and all, definitely the effort required to open will be different. So, moment of inertia depends upon the area and the particular fixed axis; and the fixed axis is sometimes the centre of gravity axis or CG, or sometimes it is an axis parallel to the CG axis, or maybe anyway.

So, in Figure 2, I have considered a body with an axis. This considered area or the lamina has to rotate about this particular black dot axis.

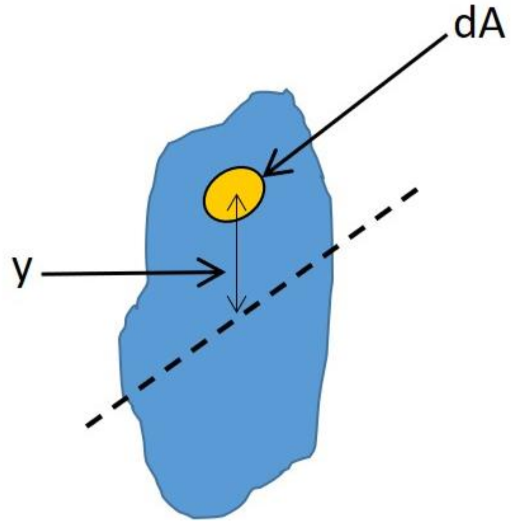


Figure 2: a body and its moment of inertia

So as per the definition this area has to be multiplied with the y square. What is y? This y is the distance from CG of the body to the considered axis.

$$I = \sum d.A \times y^2$$

So, this is going to be your moment of inertia of this particular yellow coloured area. Now if you sum it for the whole, then you will get the moment of inertia. That is the one single way we can understand moment of inertia.

So, now if we see the moment of inertia of a rectangular section with depth D and width B, then it is given by:

$$I_{xx} = \frac{1}{12} BD^3$$

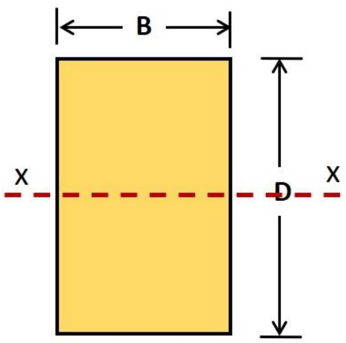
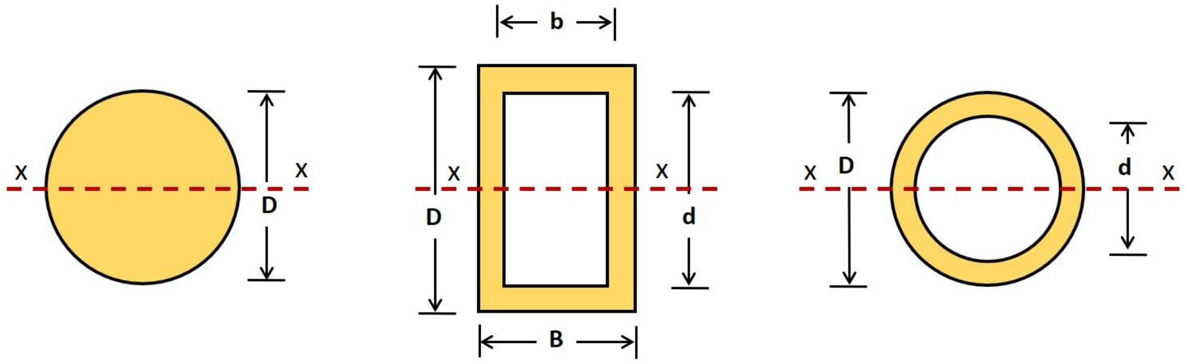


Figure 3: a rectangular section

It is always going to be depth cube and B you will remain same. Of course, the xx axis is the CG axis, without xx axis it is invalid, because as you know the CG of this will be at D/2. So, from that point of view, the moment of inertia about xx axis will be as shown in equation above. So similarly, for circular section, rectangular hollow section and for a ring is given in Figure 4.



$$I_{xx} = \frac{\pi}{64} D^4$$

$$I_{xx} = \frac{1}{12} BD^3 - \frac{1}{12} bd^3$$

$$I_{xx} = \frac{1}{12}\left(BD^3 - bd^3\right)$$

$$I_{xx} = \frac{\pi}{64} D^4 - \frac{1}{64} d^4$$

$$I_{xx} = \frac{\pi}{64}\left(D^4 - d^4\right)$$

Figure 4: moment of inertia about an axis I_{xx}, from left to right - for a circular section, hollow rectangular section and a ring section

In Figure 4, the red dotted line 'xx' at the centre is the axis of rotation, that is the CG axis. In circular section, 'D' is the depth. In the hollow rectangular section (the central image of Figure 4), 'D' is outer depth, 'd' is inner depth, 'B' is outer width and 'b' is inner width. In case of ring section, 'D' is outer width or diameter and 'd' is inner width or diameter.

In case of a ring or circular sections the D is raised to power 4; and instead of 1/12 it is PI/64. But for complex figures you have to actually see how can you do the change of the axis. Whatever we have discussed before, it is based on the xx axis, that is the CG axis. But what if it is not the CG axis? In complex figures the axis may not lie with the CG axis. Then, we have to go for the parallel axis theorem.

For parallel axis theorem you can refer any book of engineering mechanics; but very briefly I will explain this to you. So, as you know about the xx axis, of the CG axis, the I_{XX} is

$$I_{xx} = \frac{1}{12} BD^3$$

The parallel axis says that, if we want to find out the rotation, the I_{XX} , the amount of effort in the rotation in some other axis which is parallel to the CG axis, suppose the axis AA in right-hand side image of Figure 5, which is parallel to xx axis and shown red in colour.

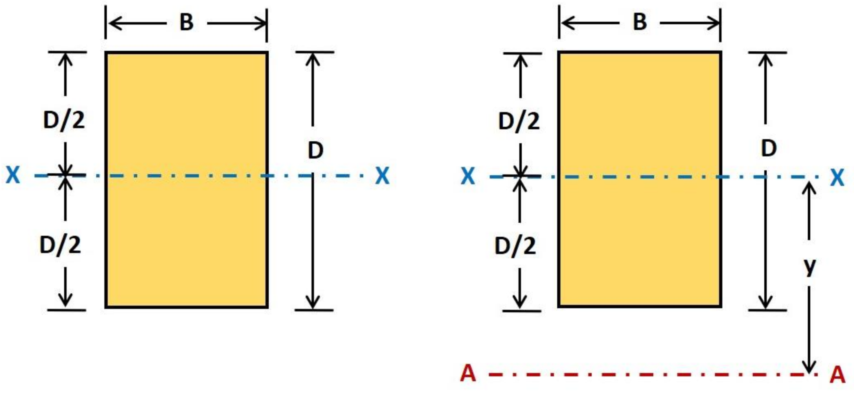


Figure 5: parallel axis theorem

Then:

Moment of Inertia about a new Axis (Parallel to CG Axis) =

Moment of Inertia about CG Axis of the element

+

Area of the Element X Square of the distance between the New axis and CG axis

So, new axis and CG, new axis is this AA and CG is the central axis, computing the distance between them is ‘y’. So, it is:

$$I_{AA} = \frac{1}{12}\, BD^3 + (BD)\ \times y^2$$

The equation above will be your equation of parallel axis theorem. Here, 1/12 BD^3 is about the CG axis and BD is the area and y^2 is the the distance square. So, this is the parallel axis theorem. You have to remember this parallel axis theorem for finding out all the I_{XX} values.

Now, let’s consider a rectangular plane, with a central axis of rotation, as shown in Figure 6 (a). Now we need to find out the rotation of the plane about a new axis, the axis at the bottom, in Figure 6 (b). The distance from bottom edge of the plane to the new axis is 2, and half of the depth (that is distance from CG to bottom of the plane) of the plane is given as 2. So, the distance to the CG from the new axis of rotation is 5.

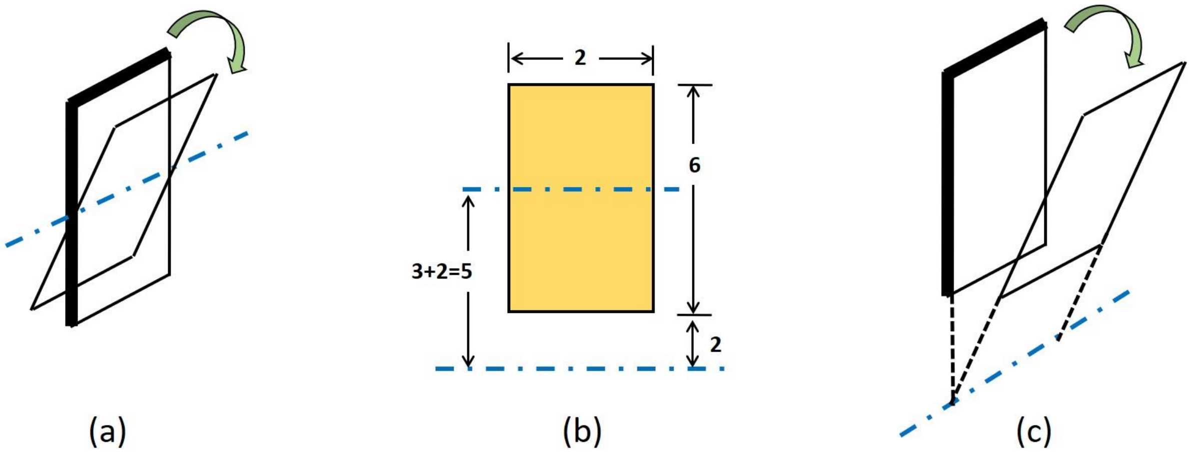


Figure 6: example of parallel axis theorem

So, applying the formulas, we can find out the moment of inertia for both the cases (a) and (b)

$I_{xx} = \frac{1}{12}\ BD^3 = \frac{1}{12} \times 6^3 = 36$

$I_{AA} = \frac{1}{12}\ BD^3 + (BD)\ \times y^2 = \frac{1}{12} \times 2\ \times 6^3 + (2\ \times 6) \times 5^2 = 36 + 300 = 336$

So now, I_{XX} will be the effort to rotate this plane about CG axis, and I_{AA} will be the effort to rotate bout the new axis on the bottom, which is parallel to CG. We found I_{XX} = 36, whereas I_{AA} = 336, which is huge amount of change. So, the rotation about any axis, other than the CG axis, the effort required for the rotation is going to be changed. Because of the bending, there is the rotation comes in the picture, and that is why we need to know about what is the amount of effort required for the rotation. So, based on the moment available in a beam, and the typical section, where the CG is lying of that particular beam, we can find out how much is the effort and how much is the I_{XX} value.

Now, let us see some of the examples. You can just go through this particular PPT and solve this; but here I'll briefly explain it. Considered here is a T section, the vertical portion is called web, and this web is 50/10; and consider it as part (i). The topmost horizontal part is called as flange, and consider it as part (ii). So, what I did is, I separated out these two parts as shown in Figure 7.

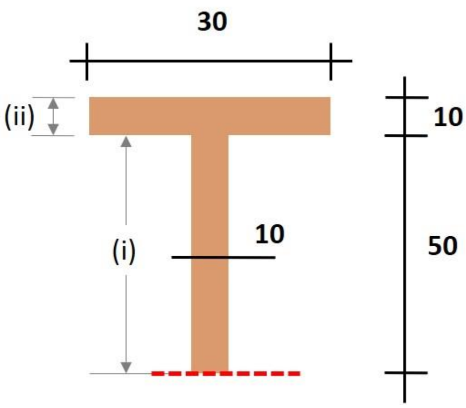


Figure 7: example with a T section

So, area of the part (i) 500, 50 into 10. The red dotted line is considered as the base line or the bottom line. The CG of part (i) is half of 50, that means 25 away from the bottom line; and plus is the top the area, to the top area is 300, 30 into 10 and the CG of part (ii) will be 50 plus half of 10 so this is 55, 55 away.

$$\bar{y} = \frac{\{(50 \times 10 \times 25\ ) + (30 \times 10 \times 55)\}}{\{(50 \times 10\ ) + (30 \times 10)\}} = \frac{29000}{800} = 36.25$$

So, you may say that, this $\bar{y}$ is:

$$\bar{y} = \frac{\sum A\bar{y}}{\sum A}$$

This will give you the $\bar{y}$, the exact $\bar{y}$ or the CG of the particular section. So now, I have got it as 36.25. So that means, the CG of the whole section lies at about 36.25 from the bottom or base line.

So now, I have tried to compute the I value, that is the moment of inertia of the T section.

We will take the web and the flange individually, and then sum it. First let us see the web part. Here the width (B) is 10, depth (D) is 50, distance of CG (d) from ground is 25, and difference in location of CG axis will be the difference between location of combined CG axis and the CG of the web; that is (36.25 – 25). Then second is the flange part. Here the width (B) is 30, depth (D) is 10, location of its own CG (d) from ground if 55, and difference between combined CG and own CG will be (36.25-55). For details refer Figure 8.

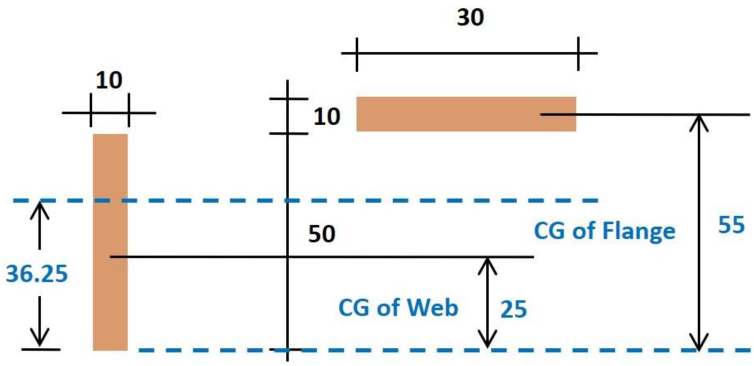


Figure 8: a T section details for calculation of IXX values

Now, putting these in the formula, we can find out the I_{XX} value for this T section.

$$I_{XX} = \left\{\frac{1}{12} \times (10 \times 50^3) + (50 \times 10) \times (36.25 - 25)^2\right\} + \left\{\frac{1}{12} \times (30 \times 10^3) + (30 \times 10) \times (36.25 - 55)^2\right\} = 275416.67$$

Therefore, I_{XX} value of the whole section will be 275416.67

Similarly, we can compute the I_{XX} value of an I section. As I sections are symmetrical in nature, it is always very interesting to work. Here, as shown in Figure 9, I can find out the respective CGs for all three parts, that is the bottom flange, web and the top flange. The depth of bottom flange is 2, so the CG will be half of it; that means 1. The depth of web (the central vertical part) is 40, its half is 20, and CG will be 2+20=22. Here, as the location whole section is very uniform, the location of CG axis for the complete section will half of 44 (2+40+2=44) that is 22, that is the distance from the ground line. Similarly, for the top flange, the depth is 2; so, the CG axis point from the base will be 2+40+1=43.

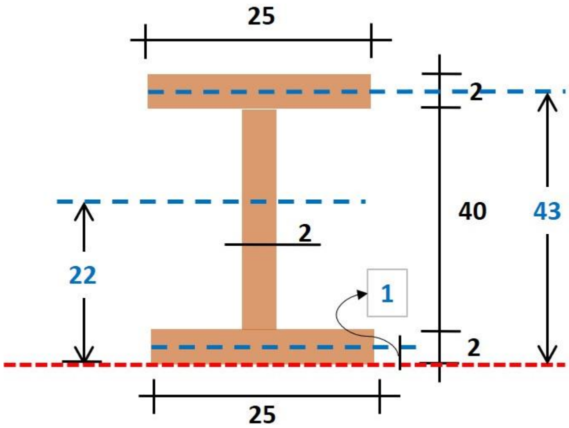


Figure 9: details of an I section and its I_{XX} value

Now, applying the formula, we can calculate the I_{XX} value for this I section.

$$I_{XX} = \left\{\frac{1}{12} \times (25 \times 2^3) + (25 \times 2) \times (22 - 1)^2\right\}$$

$$+ \left\{\frac{1}{12} \times (2 \times 40^3) + (2 \times 40) \times (22 - 22)^2\right\}$$

$$+ \left\{\frac{1}{12} \times (25 \times 2^3) + (25 \times 2) \times (22 - 43)^2\right\} = 54800$$

Therefore, I_{XX} value of the whole section will be 54800

So, now we will see an example with a square hollow section. I have given this problem in such a way that, the value of x has to be computed. The value should be such that the CG of the square hollow section lies 18 centimetre above the bottom line. It is given that the dimension of bigger rectangle, the depth (D) is 40 and width (B) is 20; and location of CG will be half of 40, that means at 20. The dimension of hollow part is, depth (D) is 30, width (B) is 10 and location of CG of hollow part will be at X+15. Refer Figure 10.

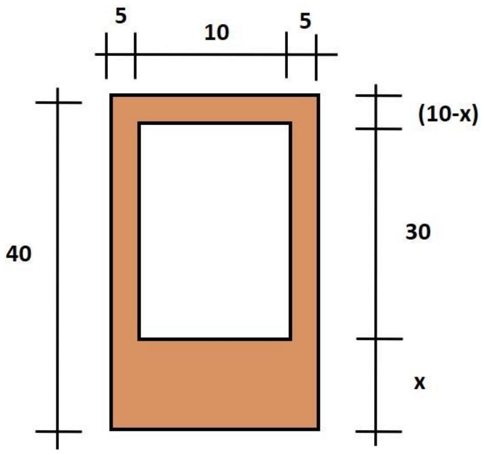


Figure 10: example of a hollow section

Now applying the formula, satisfying the given condition, it will be:

Given condition is: $\bar{y} = 18$

Then

$$\bar{y} = \frac{(40 \times 20 \times 20) - (30 \times 10) \times (X + 15)}{(40 \times 20) - (30 \times 10)} = 18$$

$$\frac{16000 - 300 \times (15 - X)}{(800 - 300)} = 18$$

$$16000 - 4500 - 300X = 9000$$

$$300X = 2500$$

$$X = 8.33$$

Therefore, to satisfy the condition, the value of X should be 8.33; then only the location of CG of the whole section will lie 18 above the ground line.

Now let us go to the theory of bending. Let us consider a beam; when it is subjected to any kind of the external load the beam is going to deform. For example, if you take a scale and just put some external load by your thumb, it will bend, and it will have some kind of axial deformation of the layers. This axial deformation is called as bending of a beam. There are two type of bending; one is called the hogging and another is sagging.

So, if we just put the load over the beam then it will bend as shown in Figure 11, the red dotted line; and this kind of bending is called the sagging. Here, the tension will be at bottom and the compression will be on the top. As a result of this deformation there some stresses will be developed and due to these stresses, the layer will go one after another or it will try to slide one after other. Due to shear force and bending moment, the beam undergoes deformation. These normal stresses due to bending are called flexure stresses.

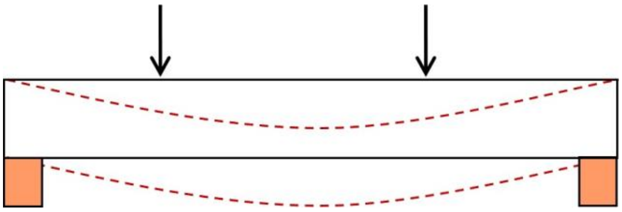


Figure 11: a beam and its bending

So, here we have some assumptions in theory of bending.

1. The material of the beam is homogeneous and isotropic.
2. The value of Young's Modulus of Elasticity is same in tension and compression.
3. The transverse sections which were plane before bending, remains plane after bending also.
4. The beam is initially straight and all longitudinal filaments bend into circular arcs with a common center of curvature.
5. The radius of curvature is large as compared to the dimensions of the cross-section.
6. Each layer of the beam is free to expand or contract, independently of the layer, above or below it.

When we talk about the material, homogeneous means the material is of same kind throughout; and isotropic means that the elastic property in all directions are equal.

The young modulus of elasticity is same in the tension and compression. This is truly one of the suitable assumptions for the theory of bending. Then, the transverse sections which was plane before bending remain plane. So, this is also one of the assumptions the plane sections remain plane after bending also. Then, the beam is initially straight and longitudinal filaments of the bend in circular. So finally, it will go with a circular arch. Initially the beam was straight, but it will go into a circular arc form, what we have shown in Figure 11. The radius of curvature is very large with compared to the dimension of the cross sections; and each layer of the beam is free to expand.

So now, let us discuss about pure bending. See, if you load a beam like it is shown in Figure 12, the portion in between the two loads will be no shear zone; but the bending moment exists there.

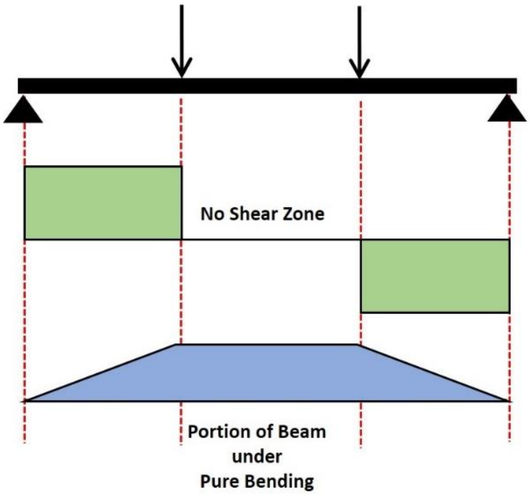


Figure 12: beam with loads and showing pure bending

Similarly, if we see a overhanging kind of a beam as shown in Figure 13, then in between the support there will be no shear force bending moment will exist.

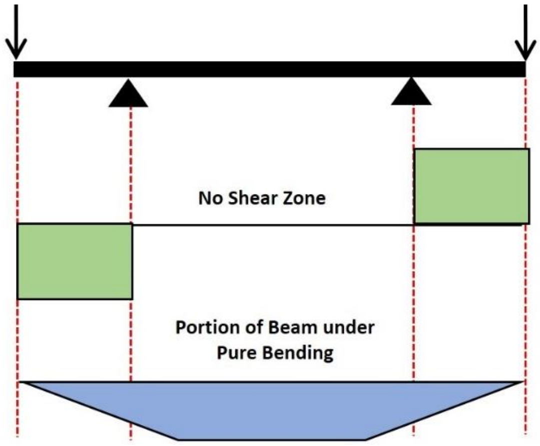


Figure 13: shear force and bending moment in an overhanging beam

So, the portions of the beams in between the loads will act as a pure bending. It may not be the whole portion of the beam, but a certain portion of the beam. Hence, pure bending zone will experience no shear but only bending moment.

Now let us see about stress curvature relationship. Let us consider a beam with loads, as shown in Figure 14 (a).

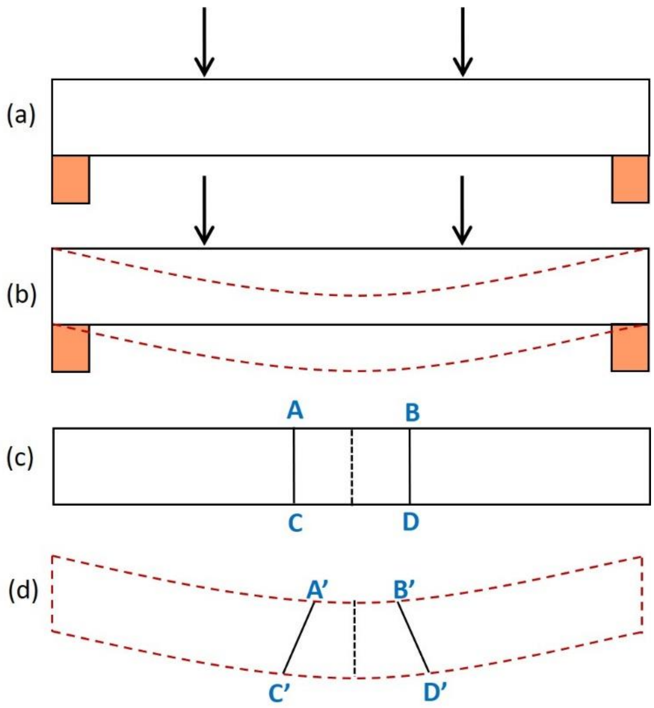


Figure 14: stress curvature relationship in a beam

When an external load is applied, it will bend as shown in Figure 14 (b), the red dotted line. Now let us say that, I have put two lines as shown in Figure 14 (c); AC and BD, equidistant from the centre of the beam; let us consider the distance as 10cm. Then when load is applied, after bending what will be the scenario? What will happen to this beam? The points A and B will come little closer like A', B' and C and D will go little away that is C', D' as shown in Figure 14 (d). It will happen this way because of the bending and because of the curvilinear nature.

So, the top portion of the layer will go little inward and the bottom portion will go outward. Hence, in case of the AB it will get shortened to A', B' and in case of CD it will be elongated to C', D'. Because as we all know, it is a sagging kind of a moment, therefore the tension will be created in the bottom and it will become elongated, whereas at top, there will be compression and it will result into shortening.

Now, if we go like this, then we can say that that A'B' < AB, so, compressive stress will be induced in the topmost layer; and similarly, C'D' > CD, so, tensile forces will be induced in the bottom most layer. The stress will decrease towards the centre. Therefore, there will be some layer in between AB and CD, suppose it is the EF, as shown in Figure 15, there with no change in the length.

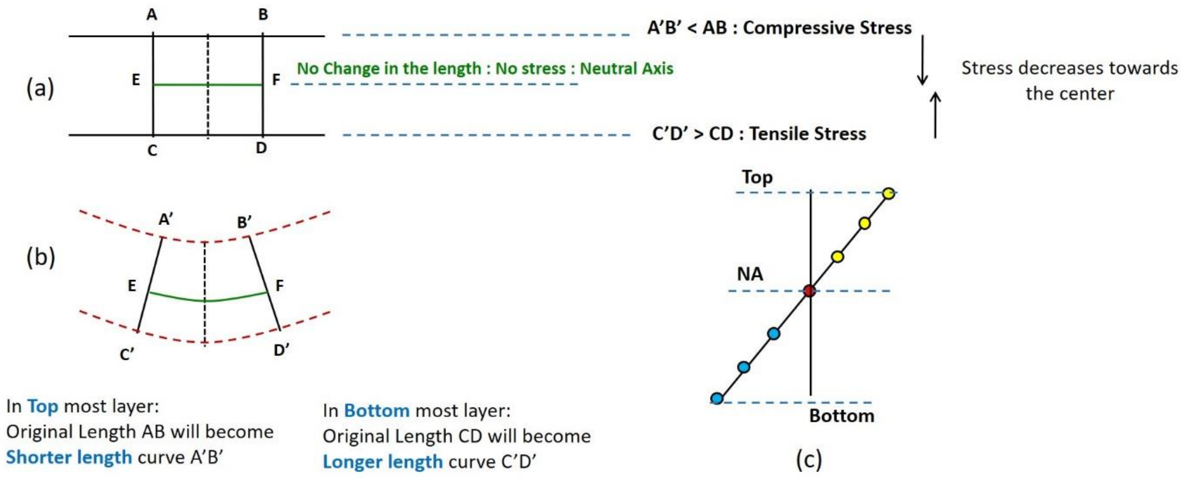


Figure 15: stress curvature relationship

In the beginning we have assumed the length between A and B (and C and D) before application of the load to be of 10-centimetre. But, after application of loads, when stresses will be developed, the distance between A and C will be reduced, may be say 9 centimetre. On the other hand, even though the distance between C and D was also 10 centimetre, after application of loads it will increase, let us say it will become 11 centimetre.

But there will be some portion or some layer, let us say it is EF, as shown in Figure 15, which will which remain same. So, there will be no stress at all.

I the topmost portion if you see there are stresses, and it is shown in yellow colour points in Figure 15 (c), they are the compressive stresses. So, the maximum compressive stresses will occur in the topmost layer and maximum possibility shortening will apply here, and gradually it will decrease. So, the layer next to it will elongate slightly, let us say it may be 9.1 or little more. In the neutral axis as shown in Figure 15, no change will occur. So, the axis where there no change in length occur is called the neutral plane in the beam and in the section it is called the neutral axis. Similarly, in the bottom plane also, as shown in blue dots of Figure 15 (c), are depicting the tensile stress. So that is why it is drawn in the other side of the yellow dots. So, in that way the stress will be distributed. Therefore, the Figure 15 (c) is called as the stress distribution diagram.

Now, if you see the radius of curvature, as shown in Figure 16, assuming it is a circular arc.

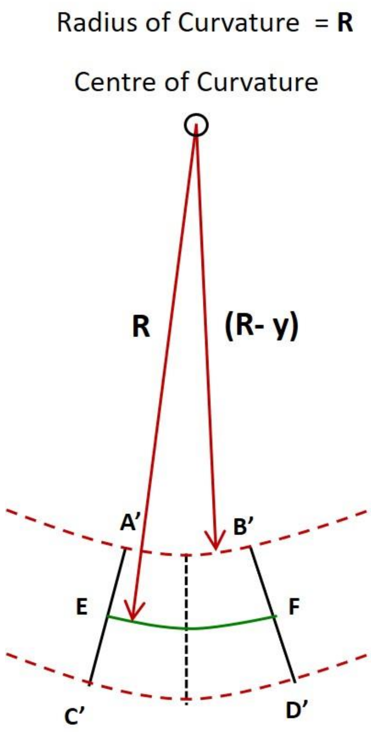


Figure 16: radius of curvature

So, I can say that the strain in the top layer is:

$$\frac{EF - A'B'}{EF} = \varepsilon = \frac{\sigma}{E}$$

Because, EF remains same, there is no change, and A' and B' is a shrink length. So, $'\varepsilon'$ is your strain; and strain to stress I can use this as a Hooke's law ($\frac{\sigma}{E}$). So now EF and A dash B dash can be related with the R and R – y.

$$\frac{EF - A'B'}{EF} = \frac{R\theta - (R - y)\theta}{R\theta} = \frac{R\theta - R\theta + y\theta}{R\theta} = \frac{y}{R}$$

$$\frac{\sigma}{E} = \frac{y}{R}$$

$$\frac{\sigma}{y} = \frac{E}{R}$$

Why it is R- y? I have kept the distance between B' and F as 'y'; because it is the half of the depth or the topmost layer portion is the 'y' and R is the centre of curvature till the EF.

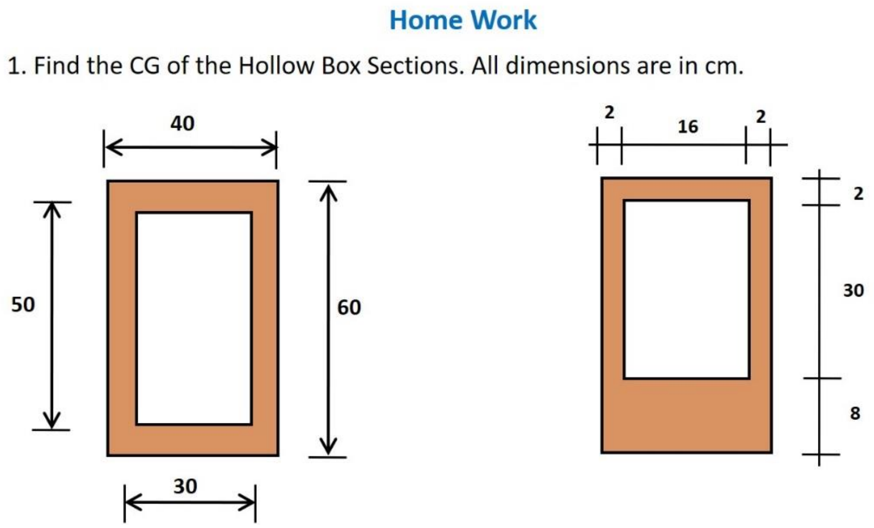
Therefore, the equation σ/E must be equal to y/R; and finally, σ/y = E/R is called the stress curvature relationship.

For this particular lecture I have taken these references:

- **Structure as Architecture** by Andrew W. Charleson, Elsevier Publication.
- **Basic Structures for Engineers and Architects** by Philip Garrison, Blackwell Publisher.
- **Structure and Architecture** by Meta Angus J. Macdonald, Elsevier Publication.
- **Examples of Structural Analysis** by William M.C. McKenzie.
- **Engineering Mechanics** by Timishenko and Young McGraw-Hill Publication.
- **Strength of Materials** by B.C. Punmia, Ashok K.Jain & Arun K.Jain Laxmi Publication.
- **Understanding Structures:** ***An Introduction to Structural Analysis*** by Meta A. Sozen & T. Ichinose, CRC Press.

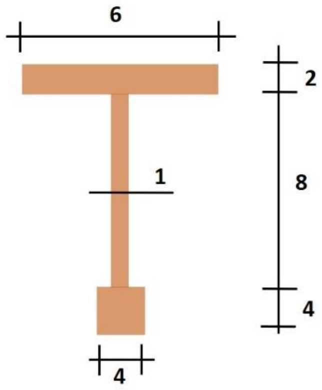
Finally, we can conclude that, the bending stress varies in the different layers depending upon its depth, and we have verified that in the stress curvature relationship.

Now, I have some homework for you, where you have to find out the CG of the given sections.



Also try to find out the moment of inertia of the 'I' section shown in Figure below. All the dimensions are given in centimetre.

2. Find the Moment of Inertia of the 'I-Section' shown in the figure below. All the dimensions are in cm.



Thank you very much.