Structural System in Architecture
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Lecture – 12
Bending Stress in Beam – II

Welcome to the NPTEL online certification course on Structural Systems in Architecture. Today we are in the module 3, the third week. The topic of the module 3 is Structural Mechanics and today is the lecture number 12, the second lecture in this week and this is on bending stress in beam part-II.

So, in this particular lecture, we intend to cover the following concepts:

- Numerical Examples of Stress-Curvature Relationship
- Moment-Curvature Relationship
- Bending Equation
- Numerical Examples of Bending Equation
- Concept of Section Modulus
- Advantages of Hollow Sections

If you remember, we have already developed the stress curvature relationship in our first lecture of this week, and we will try to solve some small numerical examples from that and then we will again try to find out another relationship in the bending theory. That is the moment curvature relationship, which is one of the very important relationship in this bending theory and that particular moment curvature will take you to till the last stage, that is designing.

We will do some bending equations, and based on that we will try to solve some numerical examples. We will also look at some of the concept like section modulus and advantages of the hollow sections.

The main learning objectives of this lecture are:

- To Develop the Bending Equation.
- To understand the Application of Theory of Bending in basic Structural Engineering.

So, the learning objective of this thing is very clear-cut. Here we have to apply the theory of bending stress in the structural engineering. So, you will be able to see, how the basic things in structural engineering can be mobilized by this equation of bending.

So, if you remember these are the few things that I have taken out from the previous lecture, lecture no 11.

Bending Stress: When a beam is subjected to an external loading system, then the beam deforms. In simple terms, this axial deformation is called as bending of a beam. Due to the shear force and bending moment, the beam undergoes deformation. These normal stresses due to bending are called flexure stresses.

So, depending upon the bending it is going to rotate each and every section which we are going to rotate and that rotation gives you a kind of radius of curvature at a certain point of the above or maybe the below.

Assumptions in Bending Stress:

1. The material of the beam is homogeneous and isotropic.
2. The value of Young's Modulus of Elasticity is same in tension and compression.
3. The transverse sections which were plane before bending, remains plane after bending also.
4. The beam is initially straight and all longitudinal filaments bend into circular arcs with a common center of curvature.
5. The radius of curvature is large as compared to the dimensions of the cross-section.
6. Each layer of the beam is free to expand or contract, independently of the layer, above or below it.

If you remember, we have also derived the radius of curvature versus the stress equations, that is:

$$\frac{\sigma}{y} = \frac{E}{R}$$

Now we will spend some time here, the E is the young modulus of elasticity; that the material property. The R is the radius of curvature of the beam, that is the distance from the neutral axis or the neutral plane to the centre of the arc.

So, sometimes it is difficult to find out these values; or sometimes we may not require the value of this R. But, the value of σ and y are very important because they represent the property of each individual sections of the beam.

When I say individual section, that means, it is a part of the beam; which may be a rectangular section, a T section or an I section; and then we can find out the values of I_{XX}. So, when we take a section, may be a rectangular section then, the stress is zero at the neutral axis, and maximum at the top and the bottom. So, here ‘y’ is the depth of each individual layers. Therefore, when you try to find out the stress, you must know the depth of that particular layer. We can find out the value of ‘y’, with the help of CG, therefore, CG is important.

Now let us see a small numerical problem, “A straight steel scale is bend into an arc having 120^0 at the centre of curvature. Estimate the Maximum stress in the steel scale.”

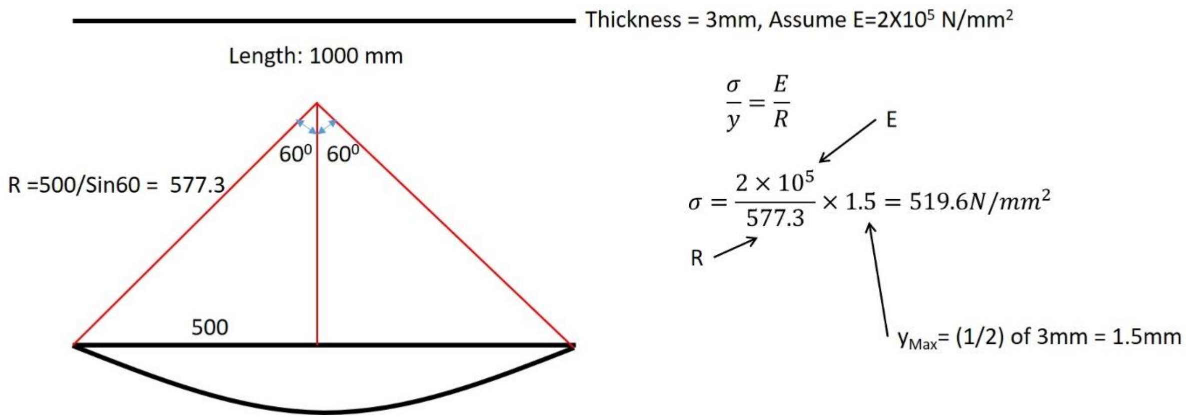


Figure 1: bending of a straight steel scale

Here, I have found out the R value. The angle is given 120º. So, half of this is 60º. So, with Pythagoras theorem or trigonometry I can find out the value of R. Please note that E value considered is $2x10^5$

Then, as I know the value of R and E, and σ is the stress value at 1.5 mm; because I want to find out the stress in this particular steel at the topmost part and the 3 millimetre is the thickness. Moreover, the CG will lie at the centre, so, half of 3 millimetre is 1.5, that is the top most part, and that is taken into account to find out the maximum stress at the top.

Now, let us go to another important topic, that is moment curvature relationship. As we know that a particular section will have a neutral axis NA and above the neutral axis, there will be some compressive stress which is shown here color green arrows, and below is the tensile stress which is in color blue arrows, in Figure 2.

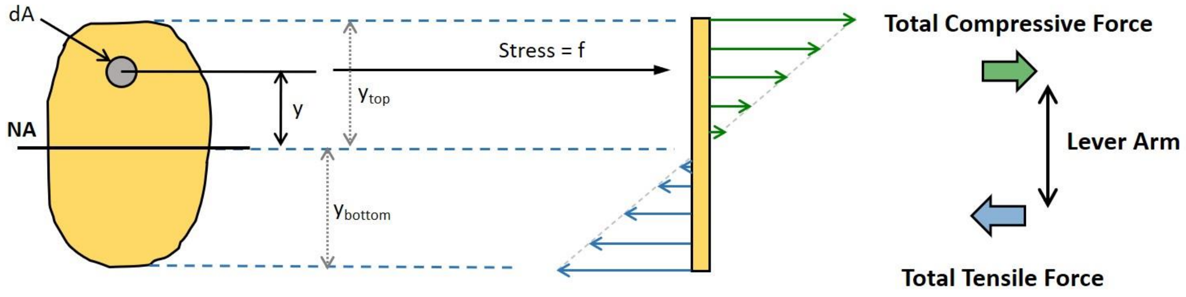


Figure 2: moment-curvature relationship

So, this total compressive stress is applied over the area above neutral axis, maybe it is varying from 0 to some value, but in totality it will give some kind of a compressive force, it is having an area of application and there is a stress value. Similarly, the total tensile force is also going to be evaluated because it is below the neutral axis and you know the area below the neutral axis you know the variation. So, we can find out the what is the tensile force. This will be separated by a distance called the Lever Arm, and as these two are the equal and opposite force it will create a moment (Refer Figure 2). So somehow, I can find out this particular moment which is actually generated because of the applied force and the moment; so, that is called the moment of resistance of the section.

So now, I will assume that there is a small area dA which is at a distance 'y' from the neutral axis; and this 'y' is very to calculate what will be the stress over there, and suppose the stress over there is 'f'. So, I can just project it in the stress distribution curve and find out the stress. So, I may say that,

The force at elementary area (dA) = f.dA

Moment of the force about NA = (f.dA) y

So, as I have assumed that at this particular 'y' the stress is your 'f' and this is the area which is almost at a distance 'y'. So, this is the small force elementary force and the moment created by this elementary force about the neutral axis is the force into the distance. So, this force and distance is the 'y' so it is fdA into y. So now if accumulate all the sum of these forces you can find out the total moment of resistance of the sections.

$$M = \sum_{y=y_{Bottom}}^{y=y_{Top}} f \times dA \times y$$

So that is why I have written this sum of from y from 'y' bottom, and 'y' top. So please remember, this y which is layer distance has to be calculated from the CG axis. So throughout, from bottom to top f.dA.y is nothing but the moment.

Now, this has to be again thought and derive. So, 'f' I have changed because I know that:

$$\frac{\sigma}{y} = \frac{E}{R}$$

Please remember, this is the stress curvature relationship. So, from this relationship, replacing the value of 'f' with $E/R \times y$ we can derive the following:

$$M = \sum_{y=y\,Bottom}^{y=y\,Top} f \times dA \times y$$

$$M = \sum_{y=y\,Bottom}^{y=y\,Top} \left(\frac{E}{R}y\right) \times dA \times y$$

$$M = \frac{E}{R} \sum_{y=y\,Bottom}^{y=y\,Top} dA \times y^2$$

$$M = \frac{E}{R} \times I$$

Where, $I = \sum Ay^2$ (this is from the last lecture)

So, by rearranging we can get the final moment-curvature relationship as follows:

$$\frac{M}{I} = \frac{E}{R}$$

We can also say that this the relation of moment that created or the external moment created by the external loading by the sectional property, I = E by the radius of curvature.

So, the final bending equation is:

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

And, this is one of the very important equations.

This bending equitation, specially the first part $\frac{M}{I} = \frac{E}{R}$ is widely used in structural engineering.

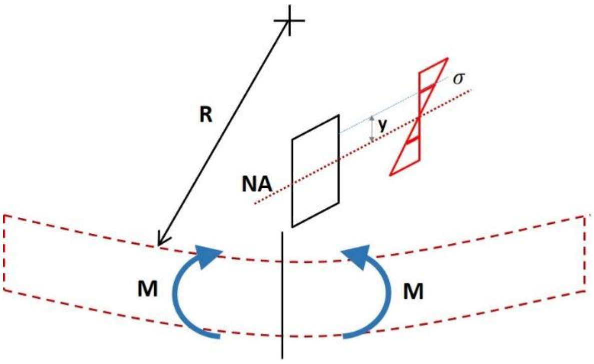


Figure 3: components of bending equation

M = External Moment or Moment of Resistance of the given beam section

I = Moment of Inertia of the Beam Cross section of the interest

R = Radius of Curvature of beam due to bending.

E = Young's Modulus of Elasticity of the beam material.

σ = Distribution of bending stress over a depth ‘y’ from neutral axis.

Now, let us suppose a simply supported beam, as shown in Figure 4, and the bending moment diagram along with. Then consider a rectangular cross section of the beam, as shown in yellow color. The bending moment will be very high the centre of the span and it will be very low towards the supports.

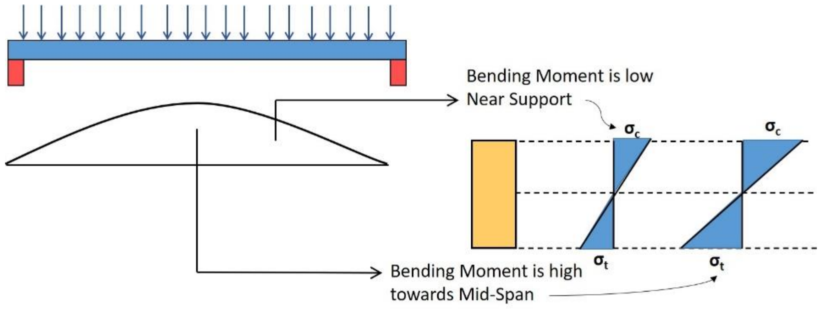


Figure 4: bending moment diagram of a simply supported beam

Similarly, for a simply supported beam, if we make the central portion of the beam thicker as shown in Figure 5, then the value of I will increase, so we can create a better way to support the particular stress.

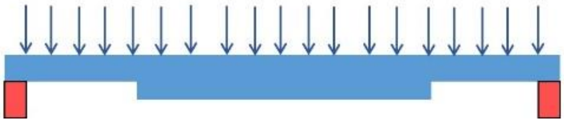


Figure 5: simply supported beam with a thicker central portion

Similarly, in case of the cantilever beam, the bending moments will be as shown in Figure 6. In this case the bending moment will be very high near to the fixed end and gradually decrease towards the end.

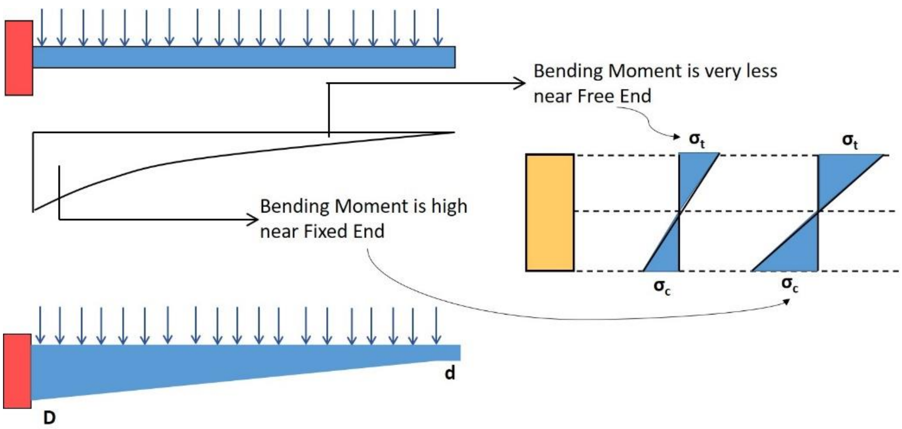


Figure 6: bending moment in a cantilever beam

As the stress distribution decreases towards the free end, therefore we need to have the beam thicker towards the fixed end where stress distribution is more, say ‘D’, and thinner, say ‘d’ towards the free end where the stress distribution is lesser, as shown in bottom image of Figure 5.

Here the σ_C that means the compression will be at bottom and the σ_T that means the tension will be at top.

So, now to understand this bending equation let us see a small numerical example. Let us consider a beam of 6 meters span, and I have placed 2 buckets of water and each bucket contains 1200 litres of water. These buckets are placed 2 meters apart. Now 1200 litres of water is approximately equal to 12 kilo Newton. So, I replace these 2 buckets of water by two-pointed load of 12 kilo Newton each, as shown in Figure 7. So, it will generate the maximum bending moment at the centre.

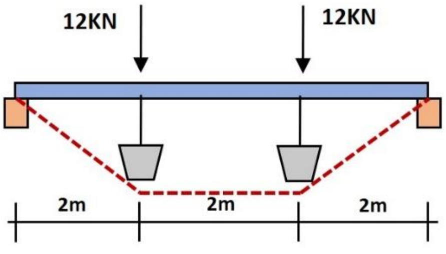


Figure 7: simply supported beam with pointed loads

Here the bending moment will look like trapezoidal distribution, as shown in Figure 7.

Then the M maximum will be:

$$M_{max} = 12 \times 2 = 24\ KNm$$

Now, let us suppose the beam cross-section is 300 depth and 120 wide.

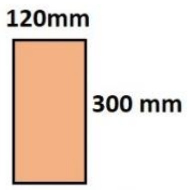


Figure 8: beam cross section

Then the I value will be:

$$I = \frac{1}{12} \times 120 \times 300^3 = 270 \times 10^6\ mm^4$$

Here we are calculating the values about the CG, the CG axis is the centre and therefore the ‘y’ or the y max is 150 and that gives me the maximum stress value.

So, now to find out the stress at topmost fibre:

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\sigma = \frac{M}{I} \times y = \frac{24 \times 10^6}{270 \times 10^6} \times 150 = 13.33 N/mm^2$$

On the other hand, let us suppose that the permissible bending stress of the beam material is only 10. So, it can only bear 10 N/mm^2. If we put those loads, that is 13.33, it is more than 10, that means more than permissible load; so, it will to crack. Hence, at this present condition of beam under bending stress 13.33 is going to fail. Then what can be the solutions for this? Because we have to make the structure safe. Therefore, we have to check the bending equation. So, what I have to do is, I have to reduce the value of σ, may be from 13.33 to 10 because 10 N/mm^2 is the permissible load. So, to gain that, now I can increase the value of I, that will definitely decrease the value of sigma; or I can reduce the value of M.

So, there are two way out, and beyond that we have some solutions available. One is to reduce the load, yes instead of 1200 litre you can put less amount of water to decrease the load. Second is, you can rearrange the positions of the bucket, sometimes that may also work. Third is, you can increase the depth of the beam section. If you increase the depth of the beams main sections, then the value of I will be increased, because we will consider depth cube, D^3 in the formula, which will result into decrease of σ value.

Now, let us go step by step to solve this.

At first, we will reduce the load by decreasing the amount of water in the bucket. Here the σ permissible is now 10.

$$I = \frac{1}{12} \times 120 \times 300^3 = 270 \times 10^6 \, mm^4$$

y = 150mm

$\sigma_{permissible}$ = 10 N/mm^2

Now, we will find out the M,

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$M = \frac{\sigma}{y} \times I = \frac{10}{150} \times 270 \times 10^6 = 18 \times 10^6 Nmm$$

$$M = 18 \times 10^6 Nmm = 18 KNm$$

Here, I value and y value is same, but we have reduced the σ value.

From this, we can find out the maximum bending moment.

So, maximum value of each bucket should not be more than 18/2 = 9KN

9 KN = 900 Kg = 900 Litres

Now, in case of the first option, we can conclude that,

- Reduce the load by decreasing the water in each bucket
- Each bucket can have maximum 900 Litres of water.

Second option to solve this is, rearranging the positions of the buckets. In this case we will keep the loads as 12KN, but we will only reposition the buckets. We also know that maximum permissible bending moment M is 18KNm.

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$M_{max} = 18 \times 10^6 Nmm = 18KNm$$

So now, let us assume that the bucket is x meter away from the support, as shown in Figure 9. Now, I can find out the moment equations and find out the value of x.

$$M = 12x = 18$$

$$x = \frac{18}{12} = 1.5$$

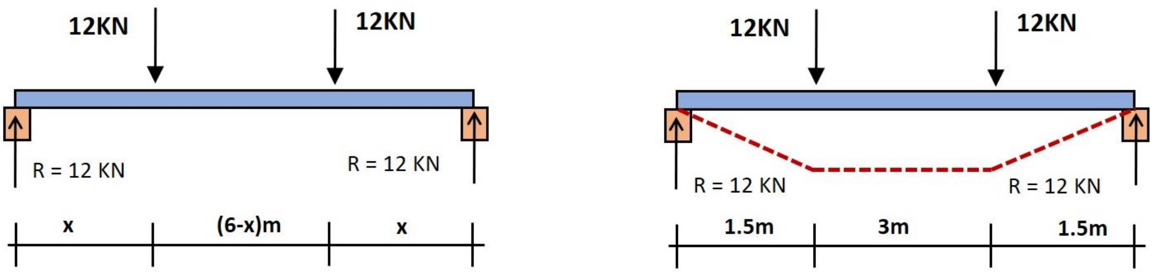


Figure 9: rearranging the position of buckets

Therefore, what I have written here is to move both the buckets 0.5 meter towards the support. If you remember, before the distance was 2, 2, 2; now it is 1.5, 3 and 1.5. The bending moment will be again as shown in Figure 9, right-hand side image; but this bending moment instead of the 24, now it will be 18.

Then the third way to solve this is by increasing the depth of the beam. Here we will keep the other variables unchanged. Let us assume that the depth of the beam is D. Here 120 is the width of the beam. Also, we have considered I value and the distance from the CG, as follows

$$I = \frac{1}{12} \times 120 \times D^3$$

$$y = \frac{D}{2}$$

Then solving this with the equation:

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\frac{\frac{1}{12} \times 120 \times D^3}{\frac{D}{2}} = \frac{24 \times 10^6}{10}$$

$$\frac{1}{6} \times 120 \times D^2 = 2.4 \times 10^6$$

$$D^2 = 120000$$

$$D = 347\ mm$$

So, the third option will be to increase the depth of the beam from 300 to 347 millimetre.

Now, let us go to the concept of section modulus.

If you see the given equation below you can see that I_{XX} and y are the properties of the section.

$$\frac{M}{I_{xx}} = \frac{\sigma}{y}$$

You also know that, maximum sigma will be encountered at the topmost layer of the section, as shown in Figure 10.

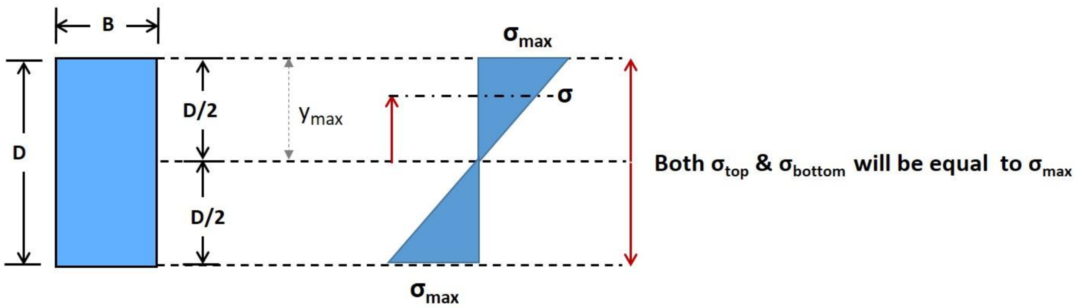


Figure 10: section modulus

Then, by rearranging we can write it as:

$$M = \frac{I_{xx}}{y_{max}} \times \sigma_{max}$$

$$M = Z_{xx} \times \sigma_{max}$$

This $\frac{I_{xx}}{y_{max}}$, we can say as Z_{xx}, or the section modulus.

We know that:

$$I = \frac{1}{12} BD^3$$

$$y_{max} = \frac{D}{2}$$

Because, CG will be at the centre.

$$Z_{xx} = \frac{1}{6} BD^2$$

Now, let us see few examples of some symmetrical sections where your y_{top} and y_{bottom} will be equal.

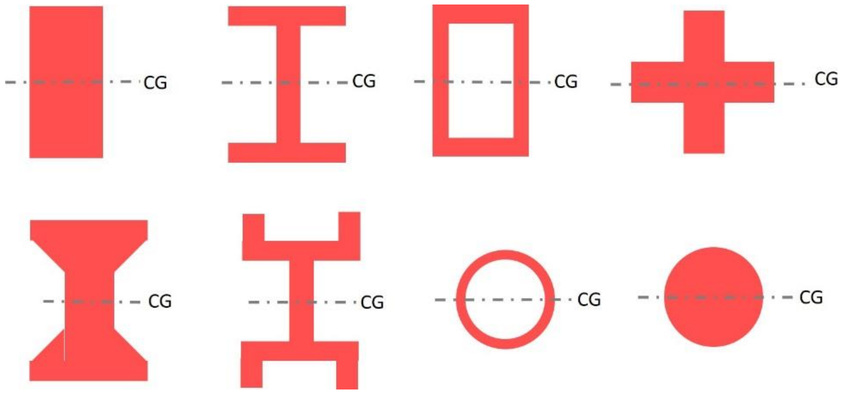


Figure 11: examples of some symmetrical sections

In such cases as the CG is at the centre, dividing the section into two equal parts, the y_{top} and y_{bottom} both the values will be same. But if both the values are different, then that we will see in the next lecture.

Now we will see some numerical examples; to find out section modulus.

First, let us consider a circular section with diameter D, as shown in Figure 12.

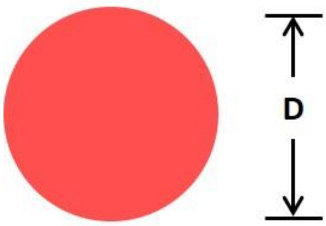


Figure 12: a circular section

We know that

$$Z_{xx} = \frac{I_{xx}}{y_{\max}}$$

Using the formula for circular sections, we can compute it as:

$$Z_{Circular} = \frac{\frac{\pi}{64} \times D^4}{\frac{D}{2}} = \frac{\pi}{32} D^3$$

Similarly, if we see a square section, as shown in Figure 13,

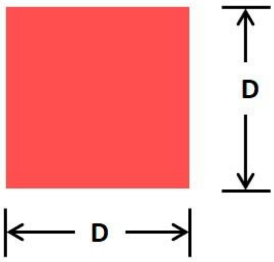


Figure 13: a square section

Then,

$$Z_{Square} = \frac{\frac{1}{12} \times D^4}{\frac{D}{2}} = \frac{1}{6} D^3$$

So now, these two may be compared and then we can find out the ratios of the moment. The equation of moment is:

$$M = Z_{xx} \times \sigma_{max}$$

So, comparing the square and the circular

$$\frac{M_{Square}}{M_{Circular}} = \frac{Z_{Square}}{Z_{Circular}} = \frac{\frac{1}{6} D^3}{\frac{\pi}{32} D^3} = \frac{32}{6\pi} = 1.69$$

Now we will see the example of an I section. This is a symmetrical I section as shown in Figure 14.

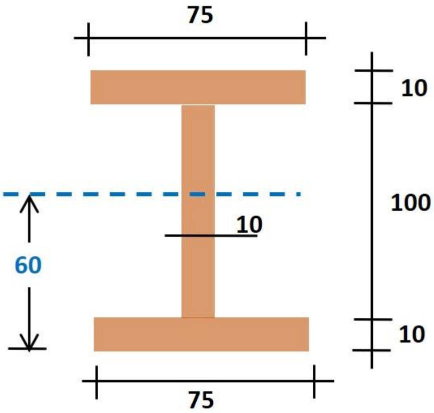


Figure 14: a symmetrical I section

Here, the $y_{max} = 60$, that is half of total depth and $\sigma_{permissible} = 250\ N/mm^2$

The cross-section area will be:

$$Cross\ section\ Area = (75 \times 10) + (100 \times 10) + (75 \times 10) = 2500\ mm^2$$

Moment of inertia of the beam section is given by:

$$I_{xx} = \left(\frac{1}{12} \times 75 \times 10^3\right) + \{75 \times 10\ (60 - 5)^2\} + \left(\frac{1}{12} \times 75 \times 100^3\right) + \left(\frac{1}{12} \times 75 \times 10^3\right)$$
$$+ \{75 \times 10\ (60 - 115)^2\}$$

$$I_{xx} = 2 \times \frac{1}{12} \times 75 \times 10^3 + 2 \times 75 \times 10(55)^2 + \frac{1}{12} \times 10 \times 100^3$$

$$I_{xx} = 5.388 \times 10^6\ mm^4$$

Now

$$Z_{xx} = \frac{I_{xx}}{y_{\max}}$$

$$Z_{xx} = \frac{5.388 \times 10^6}{60} = 89800\ mm^3$$

Now the moment of resistance of beam is:

$$M = \frac{I_{xx}}{y_{\max}} \times \sigma_{permissible}$$

$$M = Z_{xx} \times \sigma_{permissible}$$

$$M = 89800 \times 250 = 22.45 \times 10^6 N\ mm = 22.45\ KNm$$

So, 22.45KNm, this much amount of moment the beam can resist.

So now, if that beam, with this particular section is applicable over a span of 6 meter, and the beam is simply supported, as shown in Figure 15 (a), then the maximum bending moment at the centre will be:

Maximum bending moment at the centre of the span = $\frac{wL^2}{8} = 22.45$

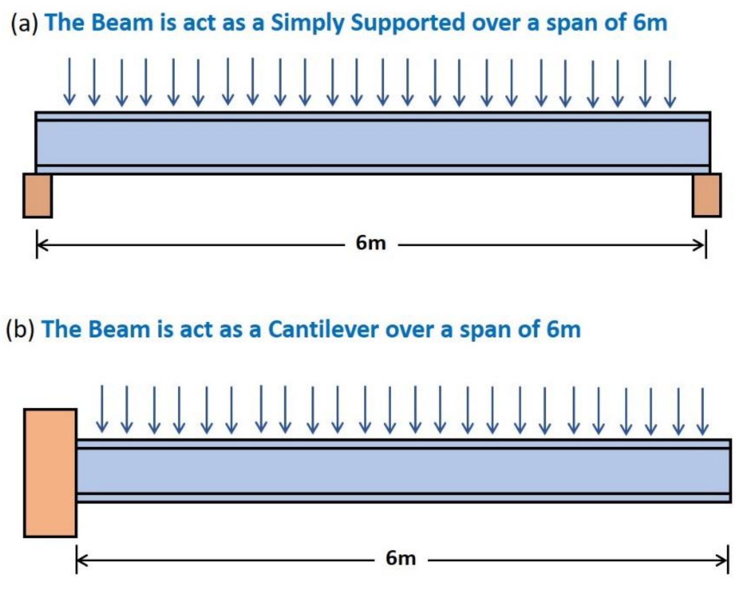


Figure 15: a simply supported beam and a cantilever beam

Therefore,

$$w = \frac{22.45 \times 8}{36} = 5KN/m$$

So, this $5KN/m$ is the maximum in UDL, the uniform distributed load we can put on the beam. Now, if we consider a cantilever beam of span 6 meter, as shown in Figure 15 (b) then,

Maximum bending moment at the centre of the span = $\frac{wL^2}{2} = 22.45$

$$w = \frac{22.45 \times 2}{36} = 1.25KN/m$$

So, the cantilever beam can resist only load of $1.25KN/m$

Now, let us see the advantage of hollow sections. Suppose, I have a square section of dimension 40mm by 40mm, so;

$$Cross\ section\ area = 40 \times 40 = 1600\ mm^2$$

$$Moment\ of\ inertia\ of\ the\ section = \frac{1}{12} \times 40 \times 40^3 = 213.3 \times 10^3\ mm^4$$

$$Section\ modulus\ of\ the\ beam\ section = \frac{213.3}{20} \times 10^3 = 10.67 \times 10^3\ mm^3$$

Now let us say there is a hollow section, which is having the outer dimension is 85mm by 85mm, and the inner dimension is 75mm by 75mm. Now, if you find out the area, then:

$$Cross\ section\ area = (85^2 \times 75^2) = 1600\ mm^2$$

This is same as the cross-sectional area of the solid section with dimension 40mm by 40mm. So, you we can say that, same amount of material has been used for both solid and the hollow sections. Now, if we see the moment of inertia of the section, then;

$$Moment\ of\ inertia\ of\ the\ section = \frac{1}{12} \times (85^4 - 75^4) = 1713.3 \times 10^3\ mm^4$$

But, here the moment of inertia has increased a lot.

Now let us see the section modulus for the section.

$$Section\ modulus\ of\ the\ beam\ section = \frac{173.3}{42.5} \times 10^3 = 40.31 \times 10^3\ mm^3$$

Now, if you try to compare and validate that which section is better for strength purpose, then, we have seen that both are having same area of cross section, there is no question of that; but If we see the Z values then, the Z value of hollow section is almost four times more than the solid section. The Z value is important because the product of Z value and the sigma permissible is the moment value.

Figure 16: a solid and a hollow section

So, we can write it as,

$$A_{Solid} = A_{Hollow} = 1600mm^2$$
$$Z_{Solid} = 10.67 \times 10^3 mm^3$$
$$Z_{Hollow} = 40.31 \times 10^3 mm^3$$

We know that;

$$M = \frac{I_{xx}}{y_{\max}} \times \sigma_{permissible}$$
$$M = Z_{xx} \times \sigma_{permissible}$$

Now, if we compare the resistance capacity of hollow section to the solid, then;

$$\frac{M_{Hollow}}{M_{Solid}} = \frac{Z_{Hollow}}{Z_{Solid}} = \frac{40.31 \times 10^3}{10.67 \times 10^3} = 3.77$$

Here, we have not considered the $\sigma_{permissible}$,because, we assume that both the sections have same material properties and it will nullify each other.

So, even though both the sections are consuming same amount of material, I must say that from the strength point of view, the hollow is 3.77 times better than the solid section. So, we can say that, the same cross-section area, same weight, same and material, but the hollow section can take 3.77 times higher moment than the solid. Therefore, the beam with hollow sections is far more effective than a solid section.

Following are the references I have taken for this particular presentation.

- **Structure as Architecture** by Andrew W. Charleson, Elsevier Publication
- **Basic Structures for Engineers and Architects** by Philip Garrison, Blackwell Publisher
- **Structure and Architecture** by Meta Angus J. Macdonald, Elsevier Publication
- **Examples of Structural Analysis** by William M.C. McKenzie
- **Engineering Mechanics** by Timishenko and Young McGraw-Hill Publication

- **Strength of Materials** by B.C. Punmia, Ashok K.Jain & Arun K.Jain Laxmi Publication
- **Understanding Structures:** ***An Introduction to Structural Analysis*** by Meta A. Sozen & T. Ichinose, CRC Press

In the conclusion I must say that the bending theory and bending equation plays a vital role in the structural engineering applications; and the beam sections needs to be chosen suitably to take the effective use of material.

And I have some homework for you;

1. A beam having 30 cm depth and I = 8000 cm^4 is simply supported over a span of 8 meter. Calculate:

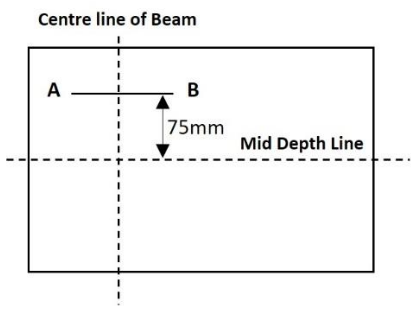
(i) Maximum UDL it may carry
(ii) The single concentrated load it may carry at the centre

The permissible bending stress is limited to 110 N/mm^2

2. A 4-meter long simply supported beam is having uniform rectangular section of 60mm X 200mm (200 mm as depth). The value of Young's Modulus of Elasticity of the beam material is 1.5 X 105 N/mm^2. A line AB of length 30 mm was marked symmetrically over the centre line of the beam. The line AB is 75 mm above the mid depth line of the beam. The line AB was straight before loading. After a UDL is imposed over the beam the length of AB decrease by 0.03 mm. Find the following:

(i) The stress at the top most layer of the beam and its nature.

(ii) The Magnitude of impose UDL.



Thank you; and this is the end of the lecture number 12.